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On Characterization Of $\mathcal{B}-$ Focal Curves In $\mathbb{E}^3$

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ABSTRACT: In this paper, we study $\mathcal{B}-$focal curves in the Euclidean 3-space $\mathbb{E}^3$. We characterize $\mathcal{B}-$focal curves in terms of their focal curvatures.

Key Words: Bishop frame, Euclidean 3-space, Focal curve.

Contents

1. Preliminaries

The Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. γ is called a unit speed curve if velocity vector v of γ satisfies $\|a\| = 1$.

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet–Serret frame along the curve γ in the space $\mathbb{E}^3$. For an arbitrary curve γ with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet–Serret formulae is given

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N}, \\ \mathbf{N}' &= -\kappa \mathbf{T} + \tau \mathbf{B}, \\ \mathbf{B}' &= -\tau \mathbf{N}, \end{aligned}$$

where

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T}\rangle &= \langle \mathbf{N}, \mathbf{N}\rangle = \langle \mathbf{B}, \mathbf{B}\rangle = 1, \\ \langle \mathbf{T}, \mathbf{N}\rangle &= \langle \mathbf{T}, \mathbf{B}\rangle = \langle \mathbf{N}, \mathbf{B}\rangle = 0. \end{aligned}$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \|\mathbf{T}(s)\|$ and $\tau(s) = -\left\langle \mathbf{N}, \mathbf{B}'\right\rangle$.

Torsion of the curve γ is given by the aid of the mixed product

$$\tau(s) = \frac{[\gamma', \gamma'', \gamma''']}{\kappa^2}.$$

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In the rest of the paper, we suppose everywhere $\kappa(s) \neq 0$ and $\tau(s) \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. The Bishop frame is expressed as

$$\begin{aligned} \mathbf{T}' &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \mathbf{M}_1' &= -k_1\mathbf{T}, \\ \mathbf{M}_2' &= -k_2\mathbf{T}. \end{aligned} \tag{1.1}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_1\}$ as Bishop trihedra and k_1 and k_2 as Bishop curvatures. The relation matrix may be expressed as

$$\begin{aligned} &\mathbf{T} = \mathbf{T}, \\ &\mathbf{N} = \cos\theta(s)\,\mathbf{M}_1 + \sin\theta(s)\,\mathbf{M}_2, \\ &\mathbf{B} = -\sin\theta(s)\,\mathbf{M}_1 + \cos\theta(s)\,\mathbf{M}_2, \end{aligned}$$

where $\theta(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$. Here, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\cos\theta(s), \\ k_2 &= \kappa(s)\sin\theta(s). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} \mathbf{T} &= \mathbf{T}, \\ \mathbf{M}_1 &= \cos\theta(s)\,\mathbf{N} - \sin\theta(s)\,\mathbf{B}, \\ \mathbf{M}_2 &= \sin\theta(s)\,\mathbf{N} + \cos\theta(s)\,\mathbf{B}. \end{aligned}$$

In this paper, we study $\mathcal{B}-$focal curves in the Euclidean 3-space $\mathbb{E}^3$. We characterize $\mathcal{B}-$focal curves in terms of their focal curvatures.

2. $\mathcal{B}-$Focal Curves According To Bishop Frame In $\mathbb{E}^3$

Denoting the focal curve by $\mathfrak{F}_\gamma^{\mathcal{B}}$, we can write

$$\mathfrak{F}_\gamma^{\mathcal{B}}(s) = (\gamma + \mathfrak{f}_1^{\mathcal{B}}\mathbf{M}_1 + \mathfrak{f}_2^{\mathcal{B}}\mathbf{M}_2)(s), \tag{2.1}$$

where the coefficients $\mathfrak{f}_1^{\mathcal{B}}$, $\mathfrak{f}_2^{\mathcal{B}}$ are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively.

To separate a focal curve according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the focal curve defined above as $\mathcal{B}$-focal curve.

Theorem 2.1. *Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit speed curve and $\mathfrak{F}_\gamma^{\mathcal{B}}$ its focal curve on $\mathbb{E}^3$. Then,*

$$\mathfrak{F}_\gamma^{B}(s) = (\gamma + \mathfrak{p}\mathbf{M}_1 + \frac{1-\mathfrak{p}k_1}{k_2}\mathbf{M}_2)(s), \tag{2.2}$$

where $\mathfrak{p}$ is a constant.

Proof: Assume that γ is a unit speed curve and $\mathfrak{F}_\gamma^{\mathcal{B}}$ its focal curve on $\mathbb{E}^3$.
So, by differentiating of the formula (2.1), we get

$$\mathfrak{F}_\gamma^{\mathcal{B}}(s)' = (1 - \mathfrak{f}_1^{\mathcal{B}}k_1 - \mathfrak{f}_2^{\mathcal{B}}k_2)\mathbf{T} + \left(\mathfrak{f}_1^{\mathcal{B}}\right)'\mathbf{M}_1 + \left(\mathfrak{f}_2^{\mathcal{B}}\right)'\mathbf{M}_2. \tag{2.3}$$

Using above equation, the first 2 components vanish, we get

$$\begin{aligned} \mathfrak{f}_1^{\mathcal{B}}k_1 + \mathfrak{f}_2^{\mathcal{B}}k_2 &= 1, \\ \left(\mathfrak{f}_1^{\mathcal{B}}\right)' &= 0. \end{aligned}$$

Considering second equation above system, we chose

$$\mathfrak{f}_1^{\mathcal{B}} = \mathfrak{p} = \text{constant} \neq 0. \tag{2.4}$$

Since, we immediately arrive at

$$\mathfrak{f}_2^{\mathcal{B}} = \frac{1-\mathfrak{p}k_1}{k_2}. \tag{2.5}$$

By means of obtained equations, we express (2.2). This completes the proof. □

Corollary 2.2. *Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit speed curve and $\mathfrak{F}_\gamma^{\mathcal{B}}$ its focal curve on $\mathbb{E}^3$. Then, the focal curvatures of $\mathfrak{F}_\gamma^{\mathcal{B}}$ are*

$$\mathfrak{f}_1^{\mathcal{B}} = \textit{constant} \neq 0,$$

$$\mathfrak{f}_2^{\mathcal{B}} = \frac{1-\mathfrak{f}_1^{\mathcal{B}}k_1}{k_2}.$$

Proof: Combining (2.4) and (2.5), we have above system, which completes the proof. □

In the light of Theorem 2.1, we express the following corollary without proof:

Corollary 2.3. *Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit speed curve and $\mathfrak{F}_\gamma^{\mathcal{B}}$ its focal curve on $\mathbb{E}^3$. If k_1 and k_2 are constant then, the focal curvatures of $\mathfrak{F}_\gamma^{\mathcal{B}}$ are*

$$\mathfrak{f}_1^{\mathcal{B}} = \textit{constant} \neq 0,$$

$$\mathfrak{f}_2^{\mathcal{B}} = \textit{constant} \neq 0.$$

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