



Existence of solutions for a Steklov problem involving the $p(x)$ -Laplacian

Aomar Anane, Omar Chakrone, Abdellah Zerouali and Belhadj Karim

ABSTRACT: By applying two versions of Mountain Pass Theorem, we prove two different situations of the existence of solutions for the following Steklov problem $\Delta_{p(x)}u = |u|^{p(x)-2}u$ in Ω , $|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q(x)-2}u$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$ and $p(\cdot), q(\cdot) : \bar{\Omega} \rightarrow (1, +\infty)$ are continuous functions.

Key Words: $p(x)$ -Laplacian; Variable exponent Sobolev trace embedding; Steklov problem; Mountain Pass Theorem.

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1. Introduction and Main Results

The purpose of this paper is to study the existence of solutions for the following nonlinear boundary value problem involving the $p(x)$ -Laplacian

$$\begin{cases} \Delta_{p(x)}u = |u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q(x)-2}u & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, ν is the unit outward normal to $\partial\Omega$, λ is a positive number and $p(\cdot), q(\cdot) \in C_+(\bar{\Omega}) := \left\{ h \in C(\bar{\Omega}); \min_{x \in \bar{\Omega}} h(x) > 1 \right\}$. The operator $\Delta_{p(x)} := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ is the $p(x)$ -Laplacian, which becomes p -Laplacian when $p(x) \equiv p$ (a constant).

Nonlinear boundary value problems with variable exponent has been received considerable attention in recent years. This is partly due to their frequent appearance in applications such as the modeling of electro-rheological fluids [1,16] and image processing [3], but these problems are very interesting from a purely mathematical point of view as well. Many results have been obtained on this kind of problems; see for example [4,5,6,17,18]. In [4], the authors have studied the

2000 *Mathematics Subject Classification:* 35J60, 35B33

case $p(x) = q(x)$ for all $x \in \bar{\Omega}$, they proved that the existence of infinitely many eigenvalue sequences. Unlike the p -Laplacian case, for a variable exponent $p(x)$ (\neq constant), there does not exist a principal eigenvalue and the set of all eigenvalues is not closed under some assumptions. Finally, they presented some sufficient conditions for the infimum of all eigenvalues is zero and positive, respectively.

Throughout this paper, we denote by $h^+ := \max_{x \in \Omega} h(x)$, $h^- := \min_{x \in \Omega} h(x)$ for any $h \in C_+(\bar{\Omega})$ and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N \\ \infty, & \text{if } p(x) \geq N \end{cases}$$

$$p_{\partial}^*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N \\ \infty, & \text{if } p(x) \geq N \end{cases}$$

Our main results in this paper are the proofs of the two following theorems, which are based on the Mountain Pass Theorem.

Theorem 1.1. *Let $p, q \in C_+(\bar{\Omega})$, such that $q^+ < p^-$. Then for any $\lambda > 0$ there exists a sequence (u_k) of non trivial weak solutions for the problem (1.1). Moreover $u_k \rightarrow 0$, as $k \rightarrow \infty$.*

Theorem 1.2. *Let $p, q \in C_+(\bar{\Omega})$, such that $p^+ < q^- \leq q^+ < p_{\partial}^*(x)$ for all $x \in \bar{\Omega}$, where $p_{\partial}^*(x)$ is defined above. Then for any $\lambda > 0$ the problem (1.1) possesses a nontrivial weak solutions.*

This paper consists of four sections. Section 1 contains an introduction and the main results. In Section 2, we state some elementary properties concerning the generalized Lebesgue-Sobolev spaces and an embedding results. The proofs of Theorem 1.1 and Theorem 1.2 are given respectively in Section 3 and Section 4.

2. Preliminaries

We first recall some basic facts about the variable exponent Lebesgue-Sobolev.

For $p \in C_+(\bar{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{ u; u : \Omega \rightarrow \mathbb{R} \text{ is a measurable and } \int_{\Omega} |u|^{p(x)} dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{p(x)} := \inf \left\{ \alpha > 0; \int_{\Omega} \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\},$$

which is separable and reflexive Banach space (see [15]).

Let us define the space

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\| = \inf \left\{ \alpha > 0; \int_{\Omega} \left| \frac{\nabla u(x)}{\alpha} \right|^{p(x)} dx + \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}; \quad \forall u \in W^{1,p(x)}(\Omega).$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1. [7,12,13,15]

- (1) $W_0^{1,p(x)}(\Omega)$ is separable reflexive Banach space;
- (2) If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \bar{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous;
- (3) If $q \in C_+(\bar{\Omega})$ and $q(x) < p_\delta^*(x)$ for any $x \in \bar{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous;
- (4) (Poincaré) There is a constant $C > 0$, such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping ρ defined by

$$\rho(u) := \int_{\Omega} \left[|\nabla u|^{p(x)} + |u|^{p(x)} \right] dx, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Proposition 2.2. [10] For $u, u_k \in W^{1,p(x)}(\Omega); k = 1, 2, \dots$, we have

- (1) $\|u\| \geq 1$ implies $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;
- (2) $\|u\| \leq 1$ implies $\|u\|^{p^-} \geq \rho(u) \geq \|u\|^{p^+}$;
- (3) $\|u_k\| \rightarrow 0$ if and only if $\rho(u_k) \rightarrow 0$;
- (4) $\|u_k\| \rightarrow \infty$ if and only if $\rho(u_k) \rightarrow \infty$;

3. Proof of Theorem 1.1

The key argument in the proof of Theorem 1.1 is the following version of the Symmetric Mountain Pass Theorem (see [14]).

Theorem 3.1. Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the following two assumptions

- (A₁). $I(u)$ is even, bounded below; $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS);

(A₁). For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$.

Then $I(u)$ admits a sequence of critical points u_k such that $I(u_k) < 0$; $u_k \neq 0$ and $u_k \rightarrow 0$, as $k \rightarrow \infty$.

Where Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$ with

$$\gamma(A) := \inf\{k \in \mathbb{N}; \exists h : A \rightarrow \mathbb{R}^k \setminus \{0\} \text{ such that } h \text{ is continuous and odd}\}$$

is the genus of A .

The energy functional corresponding to problem (1.1) is defined as $\phi_\lambda : W^{1,p(x)}(\Omega) \rightarrow \mathbb{R}$

$$\phi_\lambda(u) := \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx - \lambda \int_{\partial\Omega} \frac{1}{q(x)} |u|^{q(x)} d\sigma,$$

where $d\sigma$ is the $N - 1$ dimensional Hausdorff measure. Standard arguments imply that $\phi_\lambda \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ and

$$\langle \phi'_\lambda(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_\Omega |u|^{p(x)-2} uv dx - \lambda \int_{\partial\Omega} |u|^{q(x)-2} uv d\sigma,$$

for any $u, v \in W^{1,p(x)}(\Omega)$. Thus the weak solutions of problem (1.1) are exactly the critical points of ϕ_λ .

We show now that the Symmetric Mountain Pass Theorem can be applied in this case.

Lemma 3.2. *Let $p, q \in C_+(\bar{\Omega})$. Assume that $q^+ < p^-$. Then the functional ϕ_λ is even, bounded from below, satisfies the Palais-Smale (PS) condition and $\phi_\lambda(0) = 0$.*

Proof: It is clear that ϕ_λ is even and $\phi_\lambda(0) = 0$.

According to the fact that

$$|u(x)|^{q^+} + |u(x)|^{q^-} \geq |u(x)|^{q(x)}, \quad \forall x \in \bar{\Omega},$$

we deduce that for all $u \in W^{1,p(x)}(\Omega)$, we have

$$\phi_\lambda(u) \geq \frac{1}{p^+} \rho(u) - \frac{\lambda}{q^-} \left(\int_{\partial\Omega} |u|^{q^+} d\sigma + \int_{\partial\Omega} |u|^{q^-} d\sigma \right) \tag{3.1}$$

Since $q^+ < p^- < p_\partial^*(x)$ for any $x \in \bar{\Omega}$, then by Proposition 2.1, $W^{1,p(x)}(\Omega)$ is continuously embedded in $L^{q^+}(\partial\Omega)$ and in $L^{q^-}(\partial\Omega)$. It follows that there exist two positive constants C_1 and C_2 such that

$$\int_{\partial\Omega} |u|^{q^+} d\sigma \leq C_1 \|u\|^{q^+}, \quad \int_{\partial\Omega} |u|^{q^-} d\sigma \leq C_2 \|u\|^{q^-}, \quad \forall u \in W^{1,p(x)}(\Omega). \tag{3.2}$$

Relations (3.1) and (3.2) imply

$$\phi_\lambda(u) \geq \frac{1}{p^+} \rho(u) - \frac{\lambda}{q^-} (C_1 \|u\|^{q^+} + C_2 \|u\|^{q^-}). \tag{3.3}$$

Using (3.3) and Proposition 2.2, we have

$$\phi_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda}{q^-} (C_1 \|u\|^{q^+} + C_2 \|u\|^{q^-}) \text{ if } \|u\| \geq 1$$

and

$$\phi_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} (C_1 \|u\|^{q^+} + C_2 \|u\|^{q^-}) \text{ if } \|u\| \leq 1.$$

As $q^+ < p^-$, ϕ_λ is bounded from below and coercive. It remains to show that the functional ϕ_λ satisfies the (PS) condition to complete the proof.

Let $(u_k) \subset W^{1,p(x)}(\Omega)$ be (PS) sequence of ϕ_λ in $W^{1,p(x)}(\Omega)$, that is $\phi_\lambda(u_k)$ is bounded and $\phi'_\lambda(u_k) \rightarrow 0$. By the coercivity of ϕ_λ , the sequence (u_k) is bounded in $W^{1,p(x)}(\Omega)$. As $W^{1,p(x)}(\Omega)$ is reflexive (Proposition 2.1), for a subsequence still denoted (u_k) , we have $u_k \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$, $u_k \rightarrow u$ in $L^{p(x)}(\Omega)$ and $u_k \rightarrow u$ in $L^{q(x)}(\partial\Omega)$. Therefore

$$\langle \phi'_\lambda(u_k), u_k - u \rangle \rightarrow 0 \text{ and } \int_{\partial\Omega} |u_k|^{q(x)-2} u_k (u_k - u) \rightarrow 0.$$

Thus $\langle A(u_k), u_k - u \rangle := \int_{\Omega} |\nabla u_k|^{p(x)-2} \nabla u_k (\nabla u_k - \nabla u) + \int_{\Omega} |u_k|^{p(x)-2} u_k (u_k - u) \rightarrow 0$. According to the fact that the operator A satisfies condition (S^+) (see [8,9,11]), we deduce that $u_k \rightarrow u$ in $W^{1,p(x)}(\Omega)$, this completes the proof. \square

Lemma 3.3. *Let $p, q \in C_+(\bar{\Omega})$. Assume that $q^+ < p^-$. Then for each $k \in \mathbb{N}^*$, there exists an $H_k \in \Gamma_k$ such that $\sup_{u \in H_k} \phi_\lambda(u) < 0$.*

Proof: Let $v_1, v_2, \dots, v_k \in C^\infty(\mathbb{R}^N)$ such that $\overline{\{x \in \partial\Omega; v_i(x) \neq 0\}} \cap \overline{\{x \in \partial\Omega; v_j(x) \neq 0\}} = \emptyset$ if $i \neq j$ and $|\{x \in \partial\Omega; v_i(x) \neq 0\}| > 0 \forall i, j \in \{1, 2, \dots, k\}$. Take $F_k = \text{span}\{v_1, v_2, \dots, v_k\}$; we have $\dim F_k = k$. Denote $S = \{v \in W^{1,p(x)}(\Omega); \|v\| = 1\}$ and for $0 < t \leq 1$, $H_k(t) = t(F_k \cap S)$. For all $t \in]0, 1]$, $\gamma(H_k(t)) = k$. We show now that for any $k \in \mathbb{N}^*$, there exists $t_k \in]0, 1]$ such that $\sup_{u \in H_k(t_k)} \phi_\lambda(u) < 0$.

Indeed, for $0 < t \leq 1$, we have

$$\begin{aligned} \sup_{u \in H_k(t)} \phi_\lambda(u) &\leq \sup_{v \in F_k \cap S} \phi_\lambda(tv) \\ &= \sup_{v \in F_k \cap S} \left\{ \int_{\Omega} \frac{t^{p(x)}}{p(x)} |\nabla v|^{p(x)} dx + \int_{\Omega} \frac{t^{p(x)}}{p(x)} |v|^{p(x)} dx - \lambda \int_{\partial\Omega} \frac{t^{q(x)}}{q(x)} |v|^{q(x)} d\sigma \right\} \\ &\leq \sup_{v \in F_k \cap S} \left\{ \frac{t^{p^-}}{p^-} \rho(v) - \lambda \frac{t^{q^+}}{q^+} \int_{\partial\Omega} |v|^{q(x)} d\sigma \right\} \\ &= \sup_{v \in F_k \cap S} \left\{ t^{p^-} \left(\frac{1}{p^-} - \lambda \frac{t^{q^+}}{q^+ t^{p^-}} \int_{\partial\Omega} |v|^{q(x)} d\sigma \right) \right\}. \end{aligned}$$

Let $c = \min_{v \in F_k \cap S} \int_{\partial\Omega} |v|^{q(x)} d\sigma > 0$, we may choose $t_k \in]0, 1]$ which is small enough such that

$$\frac{1}{p^-} - \frac{c\lambda}{q^+ t^{p^- - q^+}} < 0.$$

□

Proof: [Proof of Theorem 1.1] By Lemmas 3.2, 3.3 and Theorem 3.1 the problem (1.1) admits a sequence of non trivial weak solutions (u_k) , such that $\phi_\lambda(u_k) < 0$ and $\lim u_k = 0$. □

4. Proof of Theorem 1.2

For the proof of Theorem 1.2, we want to construct a mountain geometry.

Lemma 4.1. *Let $p, q \in C_+(\bar{\Omega})$. Assume that $p^+ < q^- \leq q^+ < p_\partial^*(x)$ for all $x \in \bar{\Omega}$. Then there exist $\eta, b > 0$ such that $\phi_\lambda(u) \geq b$ for $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = \eta$.*

Proof: Since $q^+ < p_\partial^*(x)$ for all $\bar{\Omega}$, similar arguments as those used in the proof of Lemma 3.2 give the following inequalities

$$\phi_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda}{q^-} (C_1 \|u\|^{q^+} + C_2 \|u\|^{q^-}) \text{ if } \|u\| \geq 1$$

and

$$\phi_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} (C_1 \|u\|^{q^+} + C_2 \|u\|^{q^-}) \text{ if } \|u\| \leq 1.$$

Thus

$$\phi_\lambda(u) \geq \|u\|^{p^-} \left(\frac{1}{p^+} - \frac{\lambda}{q^-} (C_1 \|u\|^{q^+ - p^-} + C_2 \|u\|^{q^- - p^-}) \right) \text{ if } \|u\| \geq 1$$

and

$$\phi_\lambda(u) \geq \|u\|^{p^+} \left(\frac{1}{p^+} - \frac{\lambda}{q^-} (C_1 \|u\|^{q^+ - p^+} + C_2 \|u\|^{q^- - p^+}) \right) \text{ if } \|u\| \leq 1.$$

As $p^+ < q^- \leq q^+$, the functional $h : [0, 1] \rightarrow \mathbb{R}$ defined by

$$h(t) = \frac{1}{p^+} - \frac{\lambda C_1}{q^-} t^{q^+ - p^+} - \frac{\lambda C_2}{q^-} t^{q^- - p^+}$$

is positive on neighborhood of the origin. So the Lemma 4.2 is proved. □

Lemma 4.2. *Let $p, q \in C_+(\bar{\Omega})$. Assume that $p^+ < q^- \leq q^+ < p_\partial^*(x)$ for all $x \in \bar{\Omega}$. Then there exists $e \in W^{1,p(x)}(\Omega)$ with $\|e\| > \eta$ such that $\phi_\lambda(e) < 0$; where η is given in Lemma 4.1.*

Proof: Choose $\varphi \in C_0^\infty(\bar{\Omega})$, $\varphi \geq 0$ and $\varphi \not\equiv 0$. For $t > 1$, we have

$$\phi_\lambda(t\varphi) \leq \frac{t^{p^+}}{p^-} \rho(\varphi) - \frac{\lambda t^{q^-}}{q^+} \int_{\partial\Omega} |\varphi|^{q(x)} d\sigma.$$

Since $p^+ < q^-$, we deduce that $\lim_{t \rightarrow +\infty} \phi_\lambda(t\varphi) = -\infty$. Therefore for all $\varepsilon > 0$ there exists $\alpha > 0$ such that $|t| > \alpha \phi_\lambda(t\varphi) < -\varepsilon < 0$. This completes the proof. \square

Lemma 4.3. *Let $p, q \in C_+(\bar{\Omega})$. Assume that $p^+ < q^-$. Then the functional ϕ_λ satisfies the Palais-Smale (PS) condition.*

Proof: Let $(u_k) \subset W^{1,p(x)}(\Omega)$ be a sequence such that $C = \sup_{k \in \mathbb{N}^*} \phi_\lambda(u_k)$ and $\phi'_\lambda(u_k) \rightarrow 0$. Suppose by contradiction that $\|u_k\| \rightarrow \infty$, there exists $k_0 \in \mathbb{N}^*$ such that $\|u_k\| > 1$ for any $k \geq k_0$. Thus

$$\begin{aligned} C + \|u_k\| &\geq \phi_\lambda(u_k) - \frac{1}{q^-} \langle \phi'_\lambda(u_k), u_k \rangle \\ &\geq \int_\Omega \frac{1}{p(x)} |\nabla u_k|^{p(x)} dx + \int_\Omega \frac{1}{p(x)} |u_k|^{p(x)} dx - \lambda \int_{\partial\Omega} \frac{1}{q(x)} |u_k|^{q(x)} d\sigma - \\ &\quad \frac{1}{q^-} \rho(u_k) + \frac{\lambda}{q^-} \int_{\partial\Omega} |u_k|^{q(x)} d\sigma \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \rho(u_k) + \lambda \int_{\partial\Omega} \left(\frac{1}{q^-} - \frac{1}{q(x)} \right) |u_k|^{q(x)} d\sigma \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \rho(u_k) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \|u_k\|^{p^-}. \end{aligned}$$

Since $p^+ < q^-$, this contradicts the fact that $p^- > 1$. So, the sequence (u_k) is bounded in $W^{1,p(x)}(\Omega)$ and similar arguments as those used in the proof of Lemma 3.2 completes the proof. \square

Proof: [Proof of Theorem 1.2] Using the Lemmas 4.1 and 4.2, we obtain

$$\max(\phi_\lambda(0), \phi_\lambda(e)) = \phi_\lambda(0) < \inf_{\|u\|=\eta} \phi_\lambda(u) =: \beta.$$

By Lemma 4.3 and the Mountain Pass Theorem [2], we deduce the existence of critical points of ϕ_λ associated of the critical value given by

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \phi_\lambda(\gamma(t)) \geq \beta,$$

where

$$\Gamma = \{ \gamma \in C([0, 1], W^{1,p(x)}(\Omega)); \gamma(0) = 0 \text{ and } \gamma(1) = e \}.$$

This completes the proof. \square

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Aomar Anane
Université Mohamed I, Faculté des Sciences,
Département de Mathématiques et Informatique,
Oujda, Maroc
E-mail address: ananeomar@yahoo.fr

and

Omar Chakrone
Université Mohamed I, Faculté des Sciences,
Département de Mathématiques et Informatique,
Oujda, Maroc
E-mail address: chakrone@yahoo.fr

and

Abdellah Zerouali
Centre Pédagogique Régional,
Fès, Maroc
E-mail address: abdellahzerouali@yahoo.fr

and

Belhadj Karim
Faculté des Sciences et Techniques,
Errachidia, Maroc
E-mail address: karembelf@hotmail.com