Bol. Soc. Paran. Mat. (3s.) **v. 32** 1 (2014): 109–122.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v32i1.15762

Commutativity theorems on prime and semiprime rings with generalized (σ, τ)-derivations

Basudeb Dhara, Sukhendu Kar and Sachhidananda Mondal

abstract: Let R be an associative ring, I a nonzero ideal of R and σ, τ two epimorphisms of R. An additive mapping $F : R \to R$ is called a generalized (σ, τ)-derivation of R if there exists a (σ, τ)-derivation $d : R \to R$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$.

The objective of the present paper is to study the following situations in prime and semiprime rings: (i) $[F(x), x]_{\sigma,\tau} = 0$, (ii) $F([x, y]) = 0$, (iii) $F(x \circ y) = 0$, (iv) $F([x, y]) = [x, y]_{\sigma,\tau}$, (v) $F(x \circ y) = (x \circ y)_{\sigma,\tau}$, (vi) $F(xy) - \sigma(xy) \in Z(R)$, (vii) $F(x)F(y) - \sigma(xy) \in Z(R)$ for all $x, y \in I$, when F is a generalized (σ, τ)-derivation of R.

Key Words: semiprime ring, epimorphism, (σ, τ)-derivation, generalized (σ, τ)-derivation.

Contents

1. Introduction

Throughout the present paper, R will denote an associative ring with center $Z(R)$. For any $x, y \in R$ the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ stands for the anti-commutator $xy + yx$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = 0$ implies either $a = 0$ or $b = 0$ and semiprime if for any $a \in R$, $aRa = 0$ implies $a = 0$. Let σ, τ be any two endomorphisms of R. For any $x, y \in R$, we set $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ and $(x \circ y)_{\sigma,\tau} = x\sigma(y) + \tau(y)x$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $d : R \to R$ is called a (σ, τ)-derivation if $d(xy) = d(x)\sigma(y)+\tau(x)d(y)$ holds for all $x, y \in R$. Of course every $(1, 1)$-derivation is a derivation of R, where 1 denotes the identity map of R.

Let S be a nonempty subset of R. A mapping $f : R \to R$ is called commuting (resp. centralizing) on S, if $[f(x), x] = 0$ for all $x \in S$ (resp. $[f(x), x] \in Z(R)$ for all $x \in S$). Over last few decades, several authors have investigated the relationship between the commutativity of the ring R and some specific types of derivations of R. The first result in this view is due to Posner [22] who proved that if a prime ring R admits a nonzero centralizing derivation d, then R must be commutative. A

2000 *Mathematics Subject Classification:* 16W25, 16W80, 16N60.

number of authors have proved commutativity theorems for prime and semiprime rings admitting automorphisms, derivations or (σ,τ)-derivations (we refer to [6], [7], [8], [10], [12], [21], [22], [27]; where further references can be found) which are commuting or centralizing in some subsets of R.

An additive mapping $F : R \to R$ is called a generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Then every derivation is a generalized derivation of R. When $d = 0$, then $F(xy) = F(x)y$ for all $x, y \in R$, which is called a left multiplier map of R. Thus generalized derivation covers the concept of derivation as well as the concept of left multiplier map. It is natural to extend the results concerning derivations to generalized derivations of R. In this view we refer to [4], [5], [9], [13], [14], [18], [23], [25], [26]; where further references can be found.

Being inspired by the definition of (σ,τ)-derivation, the notion of generalized (σ,τ)-derivation was extended as follows: An additive mapping $F : R \to R$ is said to be a generalized (σ,τ)-derivation of R, if there exists a (σ,τ)-derivation $d : R \to R$ such that

$$F(xy) = F(x)\sigma(y) + \tau(x)d(y) \text{ holds for all } x, y \in R.$$

Of course every generalized $(1, 1)$-derivation of R is a generalized derivation of R, where 1 means an identity map of R. If $d = 0$, we have $F(xy) = F(x)\sigma(y)$ for all $x, y \in R$, which is called a left σ-multiplier mapping of R. Thus, generalized (σ,τ)-derivation generalizes both the concepts, (σ,τ)-derivation as well as left σ-multiplier mapping of R. Recently, the authors (see [1], [2], [3], [15], [16], [19], [20], [24]) have extended the above results to generalized (σ,τ)-derivation. In this line of investigation, recently Marubayashi et al. [20] have extended many known results concerning derivations, (σ,τ)-derivation and generalized derivations to generalized (σ,τ)-derivation of R. More precisely, the authors study the commutativity of prime ring R admitting a generalized (σ,τ)-derivation F satisfying any one of the following situations: (i) $[F(x), x]_{\sigma,\tau} = 0$, (ii) $F[x, y] = 0$, (iii) $F(x \circ y) = 0$, (iv) $F([x, y]) = [x, y]_{\sigma,\tau}$, (v) $F(x \circ y) = (x \circ y)_{\sigma,\tau}$, (vi) $F(xy) - \sigma(xy) \in Z(R)$, (vii) $F(x)F(y) - \sigma(xy) \in Z(R)$, for all x, y in an appropriate subset of R, where σ, τ are automorphisms of R. In the present paper, we shall study all the above cases in semiprime ring, where σ and τ are considered as epimorphisms of R.

2. Preliminaries

Throughout the present paper, we shall use without explicit mention the following basic identities:

$$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y,$$

$$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z),$$

$$(x \circ (yz))_{\sigma,\tau} = (x \circ y)_{\sigma,\tau}\sigma(z) - \tau(y)[x, z]_{\sigma,\tau} = \tau(y)(x \circ z)_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z),$$

$$((xy) \circ z)_{\sigma,\tau} = x(y \circ z)_{\sigma,\tau} - [x, \tau(z)]y = (x \circ z)_{\sigma,\tau}y + x[y, \sigma(z)].$$

We need the following facts which will be used to prove our Theorems.

Fact-1. If R is prime, I a nonzero ideal of R and $a, b \in R$ such that $aIb = 0$, then either $a = 0$ or $b = 0$.

Fact-2. (a) If R is a semiprime ring, the center of a nonzero one-sided ideal is contained in the center of R; in particular, any commutative one-sided ideal is contained in the center of R ([11, Lemma 2]).

(b) If R is a prime ring with a nonzero central ideal, then R must be commutative.

Fact-3. If R is any ring, I a nonzero ideal of R and σ an epimorphism of R, then $\sigma(I)$ is an ideal of R.

Fact-4. Let R be a prime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R such that $\sigma(I) \neq 0$ or $\tau(I) \neq 0$. If $d : R \to R$ is a (σ, τ)-derivation of R such that $d(I) = 0$, then $d(R) = 0$.

Proof. By assumption, we have $0 = d(rx) = d(r)\sigma(x) + \tau(r)d(x) = d(r)\sigma(x)$ for all $x \in I$ and $r \in R$, that is $d(R)\sigma(I) = 0$. If $\sigma(I) \neq 0$, this implies that $d(R) = 0$.

On the other hand, $0 = d(xr) = d(x)\sigma(r) + \tau(x)d(r) = \tau(x)d(r)$ for all $x \in I$ and $r \in R$, that is $\tau(I)d(R) = 0$. If $\tau(I) \neq 0$, this yields $d(R) = 0$.

Fact-5. If R is a semiprime ring and I is an ideal of R, then $I \cap ann_R(I) = 0$ (see [17, Corollary 2]).

Fact-6. Let R be a prime ring, $a \in R$ and $0 \neq z \in Z(R)$. If $az \in Z(R)$, then $a \in Z(R)$.

3. Main Results

Lemma 3.1. *Let R be a semiprime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R and d a (σ, τ)-derivation of R such that $\tau(I)d(I) \neq 0$. If for all $x \in I$, $[R, \tau(x)]\tau(I)d(x) = 0$, then R contains a nonzero central ideal.*

Proof: By our hypothesis we can write

$$[R, \tau(x)]R\tau(I)d(x) = 0 \tag{3.1}$$

for all $x \in I$.

Since R is semiprime, it must contain a family $\mathbf{P} = \{P_\alpha | \alpha \in \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$. If P is a typical member of $\mathbf{P}$ and $x \in I$, it follows that

$$[R, \tau(x)] \subseteq P \quad \texttt{or} \quad \tau(I)d(x) \subseteq P.$$

For fixed P, the set of $x \in I$ for which these two conditions hold are additive subgroups of I whose union is I; therefore,

$$[R, \tau(I)] \subseteq P \quad \texttt{or} \quad \tau(I)d(I) \subseteq P.$$

Thus both the cases together implies $[R, \tau(I)]\tau(I)d(I) \subseteq P$ for any $P \in \mathbf{P}$. Therefore, $[R, \tau(I)]\tau(I)d(I) \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha = 0$, that is $[R, \tau(I)]\tau(I)d(I) = 0$. Thus $0 = [R, \tau(RIR)]\tau(RI)d(I) = [R, R\tau(I)R]R\tau(I)d(I)$ and so $0 = [R, R\tau(I)d(I)R]R\tau(I)d(I)R$. This implies $0 = [R, J]RJ$, where $J = R\tau(I)d(I)R$ is a nonzero ideal of R, since $\tau(I)d(I) \neq 0$. Then $0 = [R, J]R[R, J]$. Since R is semiprime, it follows that $0 = [R, J]$ that is $J \subseteq Z(R)$. Hence the Lemma is proved. □

We begin with our first main result.

Theorem 3.2. *Let R be a semiprime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R. Suppose that F is a generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R such that $\tau(I)d(I) \neq 0$. If $[F(x), x]_{\sigma,\tau} = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof: By our assumption we have

$$[F(x), x]_{\sigma,\tau} = 0 \tag{3.2}$$

for all $x, y \in I$. Linearizing it yields

$$[F(x), y]_{\sigma,\tau} + [F(y), x]_{\sigma,\tau} = 0 \tag{3.3}$$

for all $x, y \in I$. Putting $y = yx$ we obtain

$$\begin{aligned}[F(x), y]_{\sigma,\tau}\sigma(x) + \tau(y)[F(x), x]_{\sigma,\tau} + [F(y), x]_{\sigma,\tau}\sigma(x)\\ +\tau(y)[d(x), x]_{\sigma,\tau} + [\tau(y), \tau(x)]d(x) = 0\end{aligned} \tag{3.4}$$

for all $x, y \in I$. Using (3.2) and (3.3), it gives

$$\tau(y)[d(x), x]_{\sigma,\tau} + [\tau(y), \tau(x)]d(x) = 0 \tag{3.5}$$

for all $x, y \in I$. Putting $y = ry$, $r \in R$ in (3.5), we get

$$\tau(r)\tau(y)[d(x), x]_{\sigma,\tau} + \tau(r)[\tau(y), \tau(x)]d(x) + [\tau(r), \tau(x)]\tau(y)d(x) = 0 \tag{3.6}$$

for all $x, y \in I$ and $r \in R$. Using (3.5), it reduces to

$$[\tau(r), \tau(x)]\tau(y)d(x) = 0 \tag{3.7}$$

for all $x, y \in I$ and $r \in R$. Since τ is an epimorphism of R, $[R, \tau(x)]\tau(y)d(x) = 0$ for all $x, y \in I$. Then by Lemma 3.1, the conclusion is obtained. □

Corollary 3.3. *Let R be a prime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R such that $\tau(I) \neq 0$. Suppose that F is a generalized (σ, τ)-derivation of R associated with a nonzero (σ, τ)-derivation d of R. If $[F(x), x]_{\sigma,\tau} = 0$ for all $x, y \in I$, then R is commutative and $\sigma = \tau$.*

Proof: By Theorem 3.2, we conclude that if $d(I) \neq 0$ then R is commutative. Now if $d(I) = 0$, then by Fact-4, $d(R) = 0$, a contradiction. Hence R is commutative. In this case by our hypothesis, we have $(\sigma(x) - \tau(x))F(x) = 0$ for all $x \in I$. Linearizing, this yields

$$(\sigma(x) - \tau(x))F(y) + (\sigma(y) - \tau(y))F(x) = 0 \tag{3.8}$$

for all $x, y \in I$. Replacing y with yx, we have

$$(\sigma(x) - \tau(x))(F(y)\sigma(x) + \tau(y)d(x)) + (\sigma(y)\sigma(x) - \tau(y)\tau(x))F(x) = 0 \tag{3.9}$$

for all $x, y \in I$. Multiplying (3.8) by $\sigma(x)$ and then subtracting from (3.9), we have

$$(\sigma(x) - \tau(x))\tau(y)d(x) - \tau(y)(\tau(x) - \sigma(x))F(x) = 0 \tag{3.10}$$

for all $x, y \in I$. Since $(\sigma(x) - \tau(x))F(x) = 0$ for all $x \in I$, we have $(\sigma(x) - \tau(x))\tau(y)d(x) = 0$ for all $x, y \in I$. Since $\tau(I)$ is a nonzero ideal of R and R is prime, we have for $x \in I$, either $(\sigma(x) - \tau(x)) = 0$ or $d(x) = 0$. Since both of these two cases form additive subgroups of I whose union is I, we have either $\sigma(x) - \tau(x) = 0$ for all $x \in I$ or $d(I) = 0$. By Fact-4, $d(I) = 0$ leads $d(R) = 0$, a contradiction. Hence $\sigma(x) - \tau(x) = 0$ for all $x \in I$ and so $\sigma(rx) - \tau(rx) = 0$ for all $x \in I$ and $r \in R$. Thus $0 = \sigma(r)\sigma(x) - \tau(r)\tau(x) = (\sigma(r) - \tau(r))\tau(x)$ for all $x \in I$, since $(\sigma - \tau)(I) = 0$. Therefore, it follows that $\sigma(r) - \tau(r) = 0$ for all $r \in R$, that is $\sigma = \tau$. □

Theorem 3.4. *Let R be a semiprime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R. Suppose that F is a generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R such that $\tau(I)d(I) \neq 0$. If $F([x, y]) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof: By our hypothesis we have

$$F([x, y]) = 0 \tag{3.11}$$

for all $x, y \in I$. Replacing y with yx we obtain that

$$F([x, y])\sigma(x) + \tau([x, y])d(x) = 0 \tag{3.12}$$

which implies

$$\tau([x, y])d(x) = 0 \tag{3.13}$$

for all $x, y \in I$. We replace y with ry, $r \in R$, and obtain

$$(\tau(r)\tau([x, y]) + \tau([x, r])\tau(y))d(x) = 0 \tag{3.14}$$

which implies by using (3.13) that

$$\tau([x, r])\tau(y)d(x) = 0 \tag{3.15}$$

for all $x, y \in I$ and $r \in R$. Therefore, we have $[\tau(x), R]\tau(y)d(x) = 0$ for all $x, y \in I$. Then the result follows from Lemma 3.1. □

Corollary 3.5. *Let R be a prime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R such that $\sigma(I) \neq 0$ and $\tau(I) \neq 0$. Suppose that F is a nonzero generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R. If $F([x, y]) = 0$ for all $x, y \in I$, then R is commutative.*

Proof: By Theorem 3.4, we conclude that either $d(I) = 0$ or R is commutative. If R is commutative, we are done. So, assume that $d(I) = 0$. Then $d(R) = 0$ and F is left σ-multiplier map of R. Thus by our hypothesis,

$$0 = F(x)\sigma(y) - F(y)\sigma(x) \tag{3.16}$$

for all $x, y \in I$. Replacing y with yz, $z \in I$, we get

$$0 = F(x)\sigma(y)\sigma(z) - F(y)\sigma(z)\sigma(x) \tag{3.17}$$

for all $x, y \in I$. Right multiplying (3.16) by $\sigma(z)$, and then subtracting from (3.17), we have $0 = F(y)[\sigma(z), \sigma(x)]$ for all $x, y, z \in I$. Again replacing y with yr, $r \in R$, it yields $0 = F(I)\sigma(R)[\sigma(I), \sigma(I)]$. Since R is prime, either $F(I) = 0$ or $[\sigma(I), \sigma(I)] = 0$. Now $F(I) = 0$ implies $0 = F(RI) = F(R)\sigma(I)$ implying $F(R) = 0$, a contradiction. Hence $[\sigma(I), \sigma(I)] = 0$. This implies by Fact-2 that R is commutative. □

Theorem 3.6. *Let R be a semiprime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R. Suppose that F is a nonzero generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R such that $\tau(I)d(I) \neq 0$. If $F(x \circ y) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof: By assumption, we have

$$F(x \circ y) = 0 \tag{3.18}$$

for all $x, y \in I$. Replacing y with yx, above relation yields

$$F(x \circ y)\sigma(x) + \tau(x \circ y)d(x) = 0 \tag{3.19}$$

for all $x, y \in I$. By (3.18), it yields $\tau(x \circ y)d(x) = 0$ for all $x, y \in I$. Now we replace y with ry, where $r \in R$ and obtain $0 = \{\tau(r)\tau(x \circ y) - [\tau(r), \tau(x)]\tau(y)\}d(x) = -[\tau(r), \tau(x)]\tau(y)d(x)$ for all $x, y \in I$ and $r \in R$. Then by Lemma 3.1, conclusion is obtained. □

Corollary 3.7. *Let R be a prime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R such that $\sigma(I) \neq 0$ and $\tau(I) \neq 0$. Suppose that F is a nonzero generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R. If $F(x \circ y) = 0$ for all $x, y \in I$, then char $(R) = 2$ and R is commutative.*

Proof: By Theorem 3.6, we have either $d(I) = 0$ or R is commutative. First we assume that $d(I) \neq 0$. Then R is commutative. Then by hypothesis, we have $0 = F(x \circ y) = 2F(xy) = 2F(xyr) = 2\{F(xy)\sigma(r) + \tau(x)\tau(y)d(r)\} = 2\tau(x)\tau(y)d(r)$ for all $x, y \in I$ and $r \in R$. Thus $0 = 2\tau(I)\tau(I)d(R)$. Since R is prime and $d(R) \neq 0$, char $(R) = 2$.

Next assume $d(I) = 0$. In this case by Fact-4, $d(R) = 0$ and hence F is a left σ-multiplier map of R. Then $F(x \circ y) = 0$ implies

$$0 = F(xy + yx) = F(x)\sigma(y) + F(y)\sigma(x) \tag{3.20}$$

for all $x, y \in I$. Replacing y with yr, $r \in R$, in (3.20) we have

$$0 = F(x)\sigma(y)\sigma(r) + F(y)\sigma(r)\sigma(x) \tag{3.21}$$

for all $x, y \in I$. Right multiplying (3.20) by $\sigma(r)$ and then subtracting from (3.21), we get $0 = F(y)[\sigma(r), \sigma(x)]$ for all $x, y \in I$ and $r \in R$. Thus $0 = F(ys)[\sigma(r), \sigma(x)] = F(y)\sigma(s)[\sigma(r), \sigma(x)]$ for all $x, y \in I$ and $r, s \in R$. Since R is prime and $\sigma(R)$ is a nonzero ideal of R, it follows that $F(I) = 0$ or $[\sigma(R), \sigma(I)] = 0$. Now $F(I) = 0$ implies $F(R) = 0$, a contradiction and $[\sigma(R), \sigma(I)] = 0$ implies $\sigma(I) \subseteq Z(R)$ implying R is commutative by Fact-2(b). Thus by previous argument result follows. □

Theorem 3.8. *Let R be a semiprime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R. Suppose that F is a generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R such that $\tau(I)d(I) \neq 0$ and $I\tau(I) \neq 0$. If $F([x, y]) = [x, y]_{\sigma,\tau}$ holds for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof: First we consider that $F \neq 0$. Then by our assumption,

$$F([x, y]) = [x, y]_{\sigma,\tau} \tag{3.22}$$

for all $x, y \in I$. Replacing y with yx we obtain that

$$F([x, y])\sigma(x) + \tau([x, y])d(x) = [x, y]_{\sigma,\tau}\sigma(x) + \tau(y)[x, x]_{\sigma,\tau} \tag{3.23}$$

which implies

$$\tau([x, y])d(x) = \tau(y)[x, x]_{\sigma,\tau} \tag{3.24}$$

for all $x, y \in I$. We replace y with ry, $r \in R$, and then obtain

$$(\tau(r)\tau([x, y]) + \tau([x, r])\tau(y))d(x) = \tau(r)\tau(y)[x, x]_{\sigma,\tau} \tag{3.25}$$

which implies by using (3.24) that

$$\tau([x, r])\tau(y)d(x) = 0 \tag{3.26}$$

for all $x, y \in I$ and $r \in R$. Therefore, we have $([\tau(x), R])\tau(y)d(x) = 0$ for all $x, y \in I$. Then the result follows from Lemma 3.1.

Now we consider that $F = 0$. Then we get

$$[x, y]_{\sigma,\tau} = 0 \tag{3.27}$$

for all $x, y \in I$. Replacing x with rx, $r \in R$ we obtain that

$$r[x, y]_{\sigma,\tau} + [r, \tau(y)]x = 0. \tag{3.28}$$

Then by using (3.27) we get

$$[r, \tau(y)]x = 0 \tag{3.29}$$

for all $x, y \in I$ and $r \in R$. Thus we have $[R, \tau(y)]I = 0$. This yields $[I, \tau(I)]I = 0$. This implies $[I, \tau(I)] \subseteq I \cap ann_R(I) = 0$ by Fact-5. Thus $[I, \tau(I)] = 0$. Let $J = \tau(I)$. Since τ is an epimorphism of R, J must be an ideal of R. Therefore we have $[I, J] = 0$ and hence $[IJ, IJ] = 0$. Since IJ is an commutative ideal of R and R is semiprime ring, it follows that $IJ \subseteq Z(R)$ by Fact-2(a). Thus semiprime ring contains a nonzero central ideal, provided $IJ = I\tau(I) \neq 0$. □

Corollary 3.9. *Let R be a prime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R such that $\tau(I) \neq 0$. Suppose that F is a generalized (σ, τ)-derivation of R associated with a nonzero (σ, τ)-derivation d of R. If $F([x, y]) = [x, y]_{\sigma,\tau}$ holds for all $x, y \in I$, then R is commutative and $\sigma = \tau$.*

Proof: By Theorem 3.8, we have either $d(I) = 0$ or R is commutative. Now $d(I) = 0$ leads $d(R) = 0$ by Fact-4, a contradiction. Hence R is commutative. By our hypothesis, we have $0 = [x, y]_{\sigma,\tau}$ for all $x, y \in I$ which implies $0 = x(\sigma(y) - \tau(y))$ for all $x, y \in I$. Since R is prime, $\sigma(x) - \tau(x) = 0$ for all $x \in I$. Therefore, for all $r \in R$ and $x \in I$ we have $0 = \sigma(xr) - \tau(xr) = \sigma(x)\sigma(r) - \tau(x)\tau(r) = \tau(x)(\sigma(r) - \tau(r))$, since $\sigma(x) = \tau(x)$. This implies $\sigma(r) - \tau(r) = 0$ for all $r \in R$, that is $\sigma = \tau$. □

Theorem 3.10. *Let R be a semiprime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R. Suppose that F is a generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R such that $\tau(I)d(I) \neq 0$ and $I\tau(I) \neq 0$. If $F(x \circ y) = (x \circ y)_{\sigma,\tau}$ holds for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof: First we assume that $F \neq 0$. Then by our hypothesis, we have

$$F(x \circ y) = (x \circ y)_{\sigma,\tau} \tag{3.30}$$

for all $x, y \in I$. Replacing y with yx we obtain that

$$F(x \circ y)\sigma(x) + \tau(x \circ y)d(x) = (x \circ y)_{\sigma,\tau}\sigma(x) - \tau(y)[x, x]_{\sigma,\tau} \tag{3.31}$$

which implies

$$\tau(x \circ y)d(x) = -\tau(y)[x, x]_{\sigma,\tau} \tag{3.32}$$

for all $x, y \in I$. We replace y with ry, $r \in R$, we obtain

$$(\tau(r)\tau(x \circ y) + \tau([x, r])\tau(y))d(x) = -\tau(r)\tau(y)[x, x]_{\sigma,\tau} \tag{3.33}$$

which implies by using (3.32) that

$$\tau([x, r])\tau(y)d(x) = 0 \tag{3.34}$$

for all $x, y \in I$ and $r \in R$. Therefore, we have $[\tau(x), R])\tau(y)d(x) = 0$ for all $x, y \in I$. Then the result follows from Lemma 3.1.

Next we assume that $F = 0$. Then we get

$$(x \circ y)_{\sigma,\tau} = 0 \tag{3.35}$$

for all $x, y \in I$. Replacing x with rx, $r \in R$ we obtain that

$$r(x \circ y)_{\sigma,\tau} - [r, \tau(y)]x = 0. \tag{3.36}$$

Then by using (3.35) we get

$$[r, \tau(y)]x = 0 \tag{3.37}$$

for all $x, y \in I$ and $r \in R$. Then we get $[R, \tau(y)]I = 0$. Then by same argument of Theorem 3.8, we obtain our conclusion. □

Corollary 3.11. *Let R be a prime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R such that $\tau(I) \neq 0$. Suppose that F is a generalized (σ, τ)-derivation of R associated with a nonzero (σ, τ)-derivation d of R. If $F(x \circ y) = (x \circ y)_{\sigma,\tau}$ holds for all $x, y \in I$, then R is commutative.*

Proof: The result follows by Theorem 3.10. □

Theorem 3.12. *Let R be a semiprime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R such that $\sigma(I) \neq 0$. Suppose that F is a generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R such that $\tau(I)d(I) \neq 0$. If $F(xy) \pm \sigma(xy) \in Z(R)$ holds for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof: First we consider that $F \neq 0$. Then by our assumption we have,

$$F(xy) \pm \sigma(xy) \in Z(R) \tag{3.38}$$

for all $x, y \in I$. Putting $y = yr$, where $r \in R$, we get

$$\begin{aligned} F(xyr) \pm \sigma(xyr) &= (F(xy)\sigma(r) + \tau(xy)d(r)) \pm \sigma(xy)\sigma(r) \\ &= (F(xy) \pm \sigma(xy))\sigma(r) + \tau(xy)d(r) \in Z(R) \end{aligned} \tag{3.39}$$

for all $x, y \in I$. Now commuting both sided with $\sigma(r)$ and using (3.38), we get

$$[\tau(xy)d(r), \sigma(r)] = 0 \tag{3.40}$$

for all $x, y \in I$ and $r \in R$. Now replacing x with sx, where $s \in R$, above relation yields

$$\begin{aligned} 0 &= [\tau(s)\tau(xy)d(r), \sigma(r)] \\ &= \tau(s)[\tau(xy)d(r), \sigma(r)] + [\tau(s), \sigma(r)]\tau(xy)d(r) \\ &= [\tau(s), \sigma(r)]\tau(xy)d(r) \end{aligned} \tag{3.41}$$

for all $x, y \in I$ and $r, s \in R$. Again replacing x with tx, where $t \in R$, we obtain that

$$[\tau(s), \sigma(r)]\tau(t)\tau(x)\tau(y)d(r) = 0 \tag{3.42}$$

for all $x, y \in I$ and $r, s, t \in R$. Since τ is an epimorphism of R, above relation implies that

$$[R, \sigma(r)]R\tau(I)\tau(I)d(r) = 0 \tag{3.43}$$

for all $r \in R$.

Since R is semiprime, it must contain a family $\mathbf{P} = \{P_\alpha | \alpha \in \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$. If P is a typical member of $\mathbf{P}$ and $r \in R$, (3.43) shows that

$$[R, \sigma(r)] \subseteq P \quad \texttt{or} \quad \tau(I)\tau(I)d(r) \subseteq P.$$

For fixed P, the set of $r \in R$ for which these two conditions hold are additive subgroups of R whose union is R; therefore,

$$[R, \sigma(R)] \subseteq P \quad \texttt{or} \quad \tau(I)\tau(I)d(R) \subseteq P$$

that is

$$[R, R] \subseteq P \quad \texttt{or} \quad \tau(I)\tau(I)d(R) \subseteq P.$$

Together of these two conditions imply that $[R, \tau(I)]\tau(I)d(R) \subseteq P$ for any $P \in \mathbf{P}$. Therefore, $[R, \tau(I)]\tau(I)d(R) \subseteq \cap_{\alpha \in \Lambda} P_\alpha = 0$, that is $[R, \tau(I)]\tau(I)d(R) = 0$. In particular $[R, \tau(I)]\tau(I)d(I) = 0$. Then by Lemma 3.1, we obtain our conclusion. Next, we take $F = 0$. Then we get $\sigma(xy) \in Z(R)$ holds for all $x, y \in I$, that is $\sigma(I)^2 \in Z(R)$. Since $\sigma(I)^2$ is a nonzero ideal of R, we obtain our conclusion. □

Corollary 3.13. *Let R be a prime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R such that $\sigma(I) \neq 0$ and $\tau(I) \neq 0$. Suppose that F is a generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R. If $F(xy) \pm \sigma(xy) \in Z(R)$ holds for all $x, y \in I$, then one of the following holds:*

(1) R is commutative;

(2) $F(x) = \mp\sigma(x) + \zeta(x)$ for all $x \in I$, where $\zeta : I \to Z(R)$ is an additive σ-multiplier map.

Proof: By Theorem 3.12, either $d(I) = 0$ or R is commutative. If R is commutative, we obtain our conclusion (1). Now assume that $d(I) = 0$. By Fact-4, $d(R) = 0$ and hence F is σ-multiplier map. Then by our hypothesis, we have $F(x)\sigma(y) \pm \sigma(x)\sigma(y) \in Z(R)$ for all $x, y \in I$. This yields

$$(F(x) \pm \sigma(x))\sigma(y) \in Z(R) \tag{3.44}$$

for all $x, y \in I$. Commuting both sides of (3.44) with $F(x) \pm \sigma(x)$ we have

$$(F(x) \pm \sigma(x))[F(x) \pm \sigma(x), \sigma(y)] = 0 \tag{3.45}$$

for all $x, y \in I$. Replacing y with yr, $r \in R$, we get

$$\begin{aligned} 0 &= (F(x) \pm \sigma(x))[F(x) \pm \sigma(x), \sigma(y)\sigma(r)] \\ &= (F(x) \pm \sigma(x))\{\sigma(y)[F(x) \pm \sigma(x), \sigma(r)] + [F(x) \pm \sigma(x), \sigma(y)]\sigma(r)\} \\ &= (F(x) \pm \sigma(x))\sigma(y)[F(x) \pm \sigma(x), \sigma(r)] \end{aligned} \tag{3.46}$$

for all $x, y \in I$ and $r \in R$. Since R is prime, for each $x \in I$, either $F(x) \pm \sigma(x) = 0$ or $[F(x) \pm \sigma(x), \sigma(r)] = 0$. Both cases implies that $[F(x) \pm \sigma(x), \sigma(r)] = 0$ for all $x \in I$ and $r \in R$. This yields that $F(x) \pm \sigma(x) = \zeta(x) \in Z(R)$ for all $x \in I$, that is $F(x) = \mp\sigma(x) + \zeta(x)$ for all $x \in I$, where $\zeta : I \to Z(R)$ is an additive map. Since F is σ-multiplier map, ζ is also σ-multiplier map, which is our conclusion (2). □

Theorem 3.14. *Let R be a semiprime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R such that $\sigma(I) \neq 0$. Suppose that F is a generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R such that $\tau(I)d(I) \neq 0$. If $F(x)F(y) \pm \sigma(xy) \in Z(R)$ holds for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof: First we consider that $F \neq 0$. Then by our hypothesis we have

$$F(x)F(y) \pm \sigma(xy) \in Z(R) \tag{3.47}$$

for all $x, y \in I$. Substituting yr for y in (3.47), where $r \in R$, we find that

$$\begin{aligned} F(x)F(yr) \pm \sigma(xyr) &= F(x)(F(y)\sigma(r) + \tau(y)d(r)) \pm \sigma(xyr) \\ &= (F(x)F(y) \pm \sigma(xy))\sigma(r) + F(x)\tau(y)d(r) \in Z(R) \end{aligned} \tag{3.48}$$

for all $x, y \in I$ and $r \in R$. Commuting both sided with $\sigma(r)$ and using (3.47) we get that

$$[F(x)\tau(y)d(r), \sigma(r)] = 0 \tag{3.49}$$

for all $x, y \in I$ and for all $r \in R$. Replace x by xs, $s \in R$ to get

$$\begin{aligned} 0 &= [F(xs)\tau(y)d(r), \sigma(r)] = [(F(x)\sigma(s) + \tau(x)d(s))\tau(y)d(r), \sigma(r)] \\ &= [F(x)\sigma(s)\tau(y)d(r), \sigma(r)] + [\tau(x)d(s)\tau(y)d(r), \sigma(r)] \end{aligned} \tag{3.50}$$

for all $x, y \in I$ and $r, s \in R$. Now in (3.49) replacing y with sy, where $s \in R$, we find that

$$[F(x)\tau(s)\tau(y)d(r), \sigma(r)] = 0 \tag{3.51}$$

for all $x, y \in I$ and for all $r, s \in R$. Since τ is an epimorphism of R, we have $[F(x)R\tau(y)d(r), \sigma(r)] = 0$ for all $x, y \in I$ and $r \in R$. In particular we can write $[F(x)\sigma(s)\tau(y)d(r), \sigma(r)] = 0$ for all $x, y \in I$ and $r, s \in R$. Using this fact, (3.50) gives

$$0 = [\tau(x)d(s)\tau(y)d(r), \sigma(r)] \tag{3.52}$$

for all $x, y \in I$ and $r, s \in R$. In above relation we put $x = tx$, $t \in R$, and obtain that

$$\begin{aligned} 0 &= [\tau(t)\tau(x)d(s)\tau(y)d(r), \sigma(r)] \\ &= \tau(t)[\tau(x)d(s)\tau(y)d(r), \sigma(r)] + [\tau(t), \sigma(r)]\tau(x)d(s)\tau(y)d(r) \\ &= [\tau(t), \sigma(r)]\tau(x)d(s)\tau(y)d(r) \end{aligned} \tag{3.53}$$

for all $x, y \in I$ and $r, s, t \in R$. Since τ is an epimorphism of R, from above we have $[R, \sigma(r)]\tau(I)d(R)\tau(I)d(r) = 0$ and hence $[R, \sigma(r)]R\tau(I)d(R)\tau(I)d(r) = 0$ for all $r \in R$.

Since R is semiprime, it must contain a family $\mathbf{P} = \{P_\alpha | \alpha \in \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$. If P is a typical member of $\mathbf{P}$ and $r \in R$, we have from above that

$$[R, \sigma(r)] \subseteq P \quad \texttt{or} \quad \tau(I)d(R)\tau(I)d(r) \subseteq P.$$

For fixed P, the set of $r \in R$ for which these two conditions hold are additive subgroups of R whose union is R; therefore,

$$[R, \sigma(R)] \subseteq P \quad \texttt{or} \quad \tau(I)d(R)\tau(I)d(R) \subseteq P$$

that is

$$[R, R] \subseteq P \quad \texttt{or} \quad \tau(I)d(R)\tau(I)d(R) \subseteq P.$$

Together of these two conditions imply that $[R, \tau(I)]\tau(I)d(R)\tau(I)d(R) \subseteq P$ for any $P \in \mathbf{P}$. Therefore, $[R, \tau(I)]\tau(I)d(R)\tau(I)d(R) \subseteq \cap_{\alpha \in \Lambda} P_\alpha = 0$. It follows that

$$\begin{aligned} 0 &= [R, \tau(I)]\tau(I)d(R)\tau(RI)d(R) \\ &= [R, \tau(I)]\tau(I)d(R)R\tau(I)d(R) \end{aligned} \tag{3.54}$$

and hence we can write $[R, \tau(I)]\tau(I)d(R)R[R, \tau(I)]\tau(I)d(R) = 0$. Since R is semiprime, it follows that $[R, \tau(I)]\tau(I)d(R) = 0$. Particularly, $[R, \tau(I)]\tau(I)d(I) = 0$. Then by Lemma 3.1, we obtain our conclusion. Next we take $F = 0$. Then we get $\sigma(xy) \in Z(R)$ holds for all $x, y \in I$. Then the result follows by the same argument of Theorem 3.12. □

Corollary 3.15. *Let R be a prime ring, I a nonzero ideal of R and σ, τ two epimorphisms of R such that $\sigma(I) \neq 0$ and $\tau(I) \neq 0$. Suppose that F is a generalized (σ, τ)-derivation of R associated with a (σ, τ)-derivation d of R. If $F(x)F(y) \pm \sigma(xy) \in Z(R)$ holds for all $x, y \in I$, then one of the following holds:*
(1) R is commutative;
(2) F is σ-multiplier map and $[F(x), \sigma(x)] = 0$ for all $x \in I$.

Proof: By Theorem 3.14, we have either $d(I) = 0$ that is $d(R) = 0$ or R is commutative. Let $d(R) = 0$. By assumption, we have

$$F(x)F(y) \pm \sigma(xy) \in Z(R) \tag{3.55}$$

for all $x, y \in I$. Replacing y with yz, $z \in I$, we get $F(x)F(y)\sigma(z) \pm \sigma(xy)\sigma(z) \in Z(R)$ that is $(F(x)F(y) \pm \sigma(xy))\sigma(z) \in Z(R)$ for all $x, y, z \in I$. Since $F(x)F(y) \pm \sigma(xy) \in Z(R)$, by Fact-6 either $F(x)F(y) \pm \sigma(xy) = 0$ for all $x, y \in I$ or $\sigma(I) \subseteq Z(R)$. Now $\sigma(I) \subseteq Z(R)$ implies R is commutative. Assume that $F(x)F(y) \pm \sigma(xy) = 0$ for all $x, y \in I$. Replacing x with xy and y with y^2 respectively, we get $F(x)\sigma(y)F(y) \pm \sigma(xy^2) = 0$ for all $x, y \in I$ and $F(x)F(y)\sigma(y) \pm \sigma(xy^2) = 0$ for all $x, y \in I$. Subtracting one from another yields $F(x)[F(y), \sigma(y)] = 0$ for all $x, y \in I$. Putting xz for x in the last expression, we have $F(x)\sigma(z)[F(y), \sigma(y)] = 0$ for all $x, y, z \in I$. This implies that $[F(x), \sigma(x)]\sigma(z)[F(y), \sigma(y)] = 0$ for all $x, y, z \in I$. Since R is prime ring, we conclude that $[F(x), \sigma(x)] = 0$ for all $x \in I$. □

References

1. F. Ali and M. A. Chaudhry, On generalized (α, β)-derivations of semiprime rings, *Turk. J. Math.* 34 (2010) 1-6.

2. N. Argac and E. Albas, On generalized (σ, τ)-derivations, *Siberian Math. J.* 43 (6) (2002) 977-984.

3. N. Argac, A. Kaya and A. Kisir, (σ, τ)-derivations in prime rings, *Math. J. Okayama Univ.* 29 (1987) 173-177.

4. M. Ashraf, N. Rehman, S. Ali and M. R. Mozumder, On semiprime rings with generalized derivations, *Bol. Soc. Paran. Mat.* 28 (2) (2010) 25-32.

5. M. Ashraf, A. Ali and S. Ali, Some commutativity theorems for rings with generalized derivations, *Southeast Asian Bull. Math.* 31 (3) (2007) 415-421.

6. N. Aydin and K. Kaya, Some generalizations in prime rings with (σ, τ)-derivation, *Turk. J. Math.* 16 (1992) 169-176.

7. H. E. Bell and M. N. Daif, On commutativity and strong commutativity-preserving maps, *Canad. Math. Bull.* 37 (4) (1994) 443-447.

8. H. E. Bell and W. S. Martindale III, Centralizing mappings of semiprime rings, *Canad. Math. Bull.* 30 (1) (1987) 92-101.

9. H. E. Bell and N. Rehman, Generalized derivations with commutativity and anti-commutativity conditions, *Math. J. Okayama Univ.* 49 (2007) 139-147.

10. M. Brešar, Centralizing mappings and derivations in prime rings, *J. Algebra* 156 (1993), 385-394.

11. M. N. Daif and H. E. Bell, Remarks on derivations on semiprime rings, *Internat. J. Math. Math. Sci.* 15 (1) (1992) 205-206.

12. Q. Deng and H. E. Bell, On derivations and commutativity in semiprime rings, *Comm. Algebra* 23 (10) (1995) 3705-3713.

13. B. Dhara, V. De Filippis and R. K. Sharma, Generalized derivations and left multipliers on Lie ideals, *Aequationes Math.* 81 (3) (2011) 251-261.

14. B. Dhara, Remarks on generalized derivations in prime and semiprime rings, *Internt. J. Math. & Math. Sci.*, Volume 2010, Article ID 646587, 6 pages.

15. O. Golbasi and E. Koc, Some commutativity theorems of prime rings with generalized (σ,τ)-derivation, *Commun. Korean Math. Soc.* 26 (3) (2011) 445-454.

16. O. Golbasi and E. Koc, Notes on generalized (σ,τ)-derivation, *Rend. Sem. Mat. Univ. Padova* 123 (2010) 131-139.

17. I. N. Herstein, *Rings with involution*, Univ. of Chicago Press, Chicago 1976.

18. B. Hvala, Generalized derivations in prime rings, *Comm. Algebra* 26 (4) (1998) 1147-1166.

19. Y. S. Jung and K. H. Park, On generalized (α,β)-derivations and commutativity in prime rings, *Bull. Korean Math. Soc.* 43 (1) (2006) 101-106.

20. H. Marubayashi, M. Ashraf, N. Rehman and S. Ali, On generalized (α,β)-derivations in prime rings, *Algebra Colloq.* 17 (Spec 1) (2010) 865-874.

21. J. Mayne, Centralizing mappings of prime rings, *Canad. Math. Bull.* 27 (1984) 122-126.

22. E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.* 8(1957), 1093-1100.

23. M. A. Quadri, M. S. Khan and N. Rehman, Generalized derivations and commutativity of prime rings, *Indian J. Pure Appl. Math.* 34(9) (2003)1393-1396.

24. N. Rehman, R. M. AL-Omary and C. Haetinger, On Lie structure of prime rings with generalized (α,β)-derivations, *Bol. Soc. Paran. Mat.* 27 (2) (2009) 43-52.

25. N. Rehman and V. De Filippis, On n-commuting and n-skew commuting maps with generalized derivations in prime and semiprime rings, *Siberian Math. J.* 52 (3) (2011) 516-523.

26. N. Rehman, On commutativity of rings with generalized derivations, *Math. J. Okayama Univ.* 44 (2002) 43-49.

27. J. Vukman, Commuting and centralizing mappings in prime rings, *Proc. Amer. Math. Soc.* 109 (1990) 47-52.

Basudeb Dhara
Department of Mathematics
Belda College, Belda,
Paschim Medinipur
721424(W.B.), (INDIA)
E-mail address: basu_dhara@yahoo.com

and

Sukhendu Kar
Department of Mathematics
Jadavpur University
Kolkata-700032, (INDIA)
E-mail address: karsukhendu@yahoo.co.in

and

Sachhidananda Mondal
Department of Mathematics
Jadavpur University
Kolkata-700032, (INDIA)
E-mail address: sachhidananda.mondal@gmail.com