Bol. Soc. Paran. Mat. (3s.) **v. 32** 1 (2014): 99–108.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v32i1.16178

Existence of solution for a class of biharmonic equations

Najib Tsouli, Omar Chakrone, Omar Darhouche and Mostafa Rahmani

Abstract: In this paper, We prove the solvability of the biharmonic problem

$$\begin{cases} \Delta^2 u = f(x,u) + h & in \;\; \Omega, \\ u = \Delta u = 0 & on \;\; \partial\Omega, \end{cases}$$

for a given function $h \in L^2(\Omega)$, if the limits at infinity of the quotients $f(x,s)/s$ and $2F(x,s)/s^2$ for a.e. $x \in \Omega$ lie between two consecutive eigenvalues of the biharmonic operator Δ^2, where $F(x,s)$ denotes the primitive $F(x,s) = \int_0^s f(x,t)dt$.

Key Words: Biharmonic equation, fourth elliptic equation, nonresonance.

Contents

1. Introduction

In this paper, we study a class of biharmonic problem of the form

$$\begin{cases} \Delta^2 u = f(x,u) + h & in \;\; \Omega, \\ u = \Delta u = 0 & on \;\; \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^N (N > 4)$ is a bounded smooth domain, Δ^2 denotes the biharmonic operator defined by $\Delta^2 u = \Delta(\Delta u)$. Let further $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a carathéodory function such that

$$\textbf{(R)} \qquad m_r(x) := \max_{|s|\le r} |f(x,s)| \in L^2(\Omega) \;\; \text{for each} \;\; r > 0, \tag{1.2}$$

and $h \in L^2(\Omega)$. We will also assume the conditions :

$$\textbf{(f)} \qquad \lambda_i \le l(x) := \liminf_{|s|\to\infty} \frac{f(x,s)}{s} \le \limsup_{|s|\to\infty} \frac{f(x,s)}{s} := k(x) \le \lambda_{i+1}$$

uniformly for a.e. $x \in \Omega$, and

$$\textbf{(F)} \qquad \lambda_i \le L(x) := \liminf_{|s|\to\infty} \frac{2F(x,s)}{s^2} \le \limsup_{|s|\to\infty} \frac{2F(x,s)}{s^2} := K(x) \le \lambda_{i+1}$$

2000 *Mathematics Subject Classification:* 35J60, 35J65

uniformly for a.e. $x \in \Omega$, with strict inequalities $\lambda_i < L(x)$, $K(x) < \lambda_{i+1}$ holding on subsets of positive measure where $F(x,s) = \int_0^s f(x,t)dt$ and $\lambda_i < \lambda_{i+1}$ are two consecutive eigenvalues of the problem $\Delta^2 u = \lambda u$ in Ω, $u = \Delta u = 0$ on $\partial\Omega$.
The conditions imposed on f are usually classified as non-resonant or resonant, according as they yield the solvability of problem (1.1) for every h or not.
Many papers have been devoted to the obtention of resonant conditions of the second order problem

$$\begin{cases} -\Delta u = f(x,u) + h & in \quad \Omega, \\ u = 0 \qquad on \quad \partial\Omega, \end{cases} \tag{1.3}$$

where $h \in L^p(\Omega)$, for some suitable $p \geq 2$ is given.
See for instance [6], [1] and the references given there.
To the best of our knowledge, the solvability of boundary value problem (1.1) has not been studied till now. The main purpose of this paper is to extend some of the results known in [1], concerning the Dirichlet problem (1.3) to the biharmonic problem (1.1) with Navier boundary condition.
Our main result is the following :

Theorem 1.1. *Under hypothesis* **(f)** *and* **(F)***, problem (1.1) is solvable for any given function $h \in L^2(\Omega)$*

The proof is based on variational method, we will use the well-known Rabinowitz saddle point theorem [3].
The plan of this paper is the following : in section 2, we prove some preliminary lemmas. In section 3 we give the proof of our main result.

2. Preliminary lemmas

From the conditions **(R)** and **(f)**, it follows that there exist constants a, $A > 0$ and functions $b \in L^2(\Omega)$, $B \in L^1(\Omega)$ such that :

$$(1) \qquad |f(x,s)| \leq a|s| + b(x), \qquad x \in \Omega, \qquad s \in \mathbb{R},$$

$$(2) \qquad |F(x,s)| \leq As^2 + B(x), \qquad x \in \Omega, \qquad s \in \mathbb{R},$$

hence the functional

$$I(u) = \frac{1}{2}\int_\Omega (\Delta u)^2 dx - \int_\Omega F(x,u)dx - \int_\Omega hu\ dx$$

is well-defined and of class C^1 on the space $H := H_0^1(\Omega) \cap H^2(\Omega)$ with norm

$$\|u\| = \|\Delta u\|_{L^2} = \Big(\int_\Omega |\Delta u|^2 dx\Big)^{1/2},$$

where $\|.\|_{L^2}$ denote the usual norm in $L^2(\Omega)$.
The derivative $I'(u) \in H^*$ is given by

$$< I'(u), w >= \int_\Omega \Delta u \Delta w\ dx - \int_\Omega f(x,u)w\ dx - \int_\Omega hw\ dx$$

for all $u, w \in H$.
Thus the critical points of I are precisely the weak solutions $u \in H$ of (1.1).
Let $(u_n) \subset H$ be an unbounded sequence. Then, defining v_n by

$$v_n := \frac{u_n}{\|u_n\|},$$

we have $\|v_n\| = 1$ and , passing if necessary to a subsequence (still denoted by (v_n)), we may assume

$$\begin{aligned} v_n &\to v \text{ weakly in H}, \\ v_n &\to v \text{ strongly in } L^2(\Omega), \\ v_n(x) &\to v(x) \text{ a.e. in } \Omega, \end{aligned} \tag{2.1}$$

and $|v_n(x)| \le z(x)$ a.e., where $z \in L^2(\Omega)$.
Now, assuming **(f)**, we obtain that the sequence $(\frac{f(.,u_n)}{\|u_n\|})$ is bounded in $L^2(\Omega)$. Thus for a subsequence

$$\frac{f(.,u_n)}{\|u_n\|} \rightharpoonup \tilde{f} \text{ weakly in } L^2(\Omega) \tag{2.2}$$

Lemma 2.1. *The function $\tilde{f}$ above satisfies*

$$l(x) \le \frac{\tilde{f}(x)}{v(x)} \le k(x) \quad \textit{if} \quad v(x) \neq 0, \tag{2.3}$$

$$\tilde{f}(x) = 0 \quad \textit{if} \quad v(x) = 0, \tag{2.4}$$

*where v and l, k are given in (2.1) and **(f)**, respectively.*

Proof: see [2, Lemma 4]. □

Lemma 2.2. *Let $\Psi_n(x) = \frac{2F(x,u_n(x))}{\|u_n\|^2}$. If $\|u_n\| \to +\infty$ then*

$$L(x)v(x)^2 \le \liminf \Psi_n(x) \le \limsup \Psi_n(x) \le K(x)v(x)^2 \tag{2.5}$$

*for a.e. $x \in \Omega$, where v and K, L are given in (2.1) and **(F)**, respectively.*

Proof: see [1, Lemma 2]. □

The next result is a consequence of the equivalence between the unique continuation and the strict monotonicity of the biharmonic operator.

Lemma 2.3. *(see [4]) Let $m : \Omega \to \mathbb{R}$ be a $L^\infty(\Omega)$ function satisfying $\lambda_i \le m(x) \le \lambda_{i+1}$, with $\lambda_i < m(x)$ and $m(x) < \lambda_{i+1}$ on subsets of positive measure. If $v \in H$ is a weak solution of*

$$\begin{cases} \Delta^2 v = m(x)v & \textit{in } \Omega, \\ v = \Delta v = 0 & \textit{on } \partial\Omega, \end{cases} \tag{2.6}$$

then $v \equiv 0$.

3. Proof of the theorem 1.1

First we need to study the functional $I : H \to \mathbb{R}$ defined in the introduction. Throughout this section we will assume that conditions **(f)** and **(F)** hold.

Proposition 3.1. *The functional I satisfies the Palais-Smale condition (PS).*

Proof: : Let $(u_n) \in H$ be such that

$$|I(u_n)| \leq C \tag{3.1}$$

$$| < I'(u_n), v > | = \left| \int_\Omega \Delta u_n \Delta v \; dx - \int_\Omega f(x, u_n) v \; dx - \int_\Omega hv \; dx \right| \leq \varepsilon_n \|v\| \tag{3.2}$$

for all $v \in H$, where C is a constant and $\varepsilon_n \to 0$ as $n \to +\infty$. In order to show that (u_n) has a convergent subsequence, it suffices to show that (u_n) remains bounded in H.
Suppose by contradiction that $\|u_n\| \to +\infty$ as $n \to +\infty$. Then , as we observed in the previous section, (a subsequence of) $v_n = \frac{u_n}{\|u_n\|}$ is such that

$$\begin{aligned} v_n &\to v \;\text{ weakly in H}, \\ v_n &\to v \;\text{ strongly in }\; L^2(\Omega), \\ v_n(x) &\to v(x) \;\text{ a.e. in }\; \Omega, \end{aligned}$$

and $|v_n(x)| \leq z(x)$ a.e , where $z \in L^2(\Omega)$.
Moreover, we have

$$\frac{f(., u_n)}{\|u_n\|} \rightharpoonup \tilde{f} \;\text{ weakly in } L^2(\Omega) \tag{3.3}$$

where $\tilde{f}$ satisfies

$$l(x) \leq \frac{\tilde{f}(x)}{v(x)} \leq k(x) \;\text{ if }\; v(x) \neq 0, \tag{3.4}$$

$$\tilde{f}(x) = 0 \;\text{ if }\; v(x) = 0 \tag{3.5}$$

Let us define

$$m(x) = \begin{cases} \frac{\tilde{f}(x)}{v(x)} \;, & \text{if } \; v(x) \neq 0, \\ \\ \overline{\lambda} = \frac{1}{2}(\lambda_i + \lambda_{i+1}) \;, & \text{if } \; v(x) = 0. \end{cases}$$

Then $\tilde{f} = m(x)v(x)$ and, by (3.4) and (3.5) we have

$$l(x) \leq m(x) \leq k(x) \;\text{ if }\; v(x) \neq 0 \tag{3.6}$$

$$m(x) = \overline{\lambda} \;\text{ if }\; v(x) = 0, \tag{3.7}$$

so that $\lambda_i \leq m(x) \leq \lambda_{i+1}$ in view of **(f)**.
Now, we use (3.2) with $v = u_n$ and we divide by $\|u_n\|^2$ to obtain at the limit

$$\int_\Omega \frac{f(x,u_n)}{\|u_n\|} v_n \, dx \to 1$$

Consequently

$$\int_\Omega \tilde{f} v \, dx = 1 \tag{3.8}$$

so that $v \not\equiv 0$, necessarily.
On the other hand, for any $w \in H$, we have that

$$\frac{|< I'(u_n), w >|}{\|u_n\|} = \Big| \int_\Omega \Delta v_n \Delta w - \int_\Omega \frac{f(x,u_n)}{\|u_n\|} w - \frac{1}{\|u_n\|} \int_\Omega hw \Big| \leq \varepsilon_n \frac{\|w\|}{\|u_n\|} \to 0$$

from which using (3.3) and the fact that $v_n \rightharpoonup v$ weakly in H, we obtain

$$\int_\Omega \Delta v \Delta w = \int_\Omega \tilde{f} w, \qquad \text{for all} \quad w \in H.$$

Using Lemma 2.1, we see that $v \in H$ is a weak solution of the problem

$$\begin{cases} \Delta^2 v = \tilde{f} = mv & \text{in} \quad \Omega, \\ v = \Delta v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.9}$$

Now, we will distinguish three cases : (i) $m(x) \equiv \lambda_i$ (ii) $m(x) \equiv \lambda_{i+1}$ and (iii) $\lambda_i \leq m(x) \leq \lambda_{i+1}$ with $\lambda_i < m(x)$ and $m(x) < \lambda_{i+1}$ on subsets of positive measure. We will see that each case leads to a contradiction.

case(i) : If $m(x) \equiv \lambda_i$, then by multiplying (3.9) by v, integrating, and using (3.8) we obtain :

$$\int_\Omega (\Delta v)^2 = \lambda_i \int_\Omega v^2 = 1 \tag{3.10}$$

On the other hand, from (3.1) we obtain

$$\frac{2I(u_n)}{\|u_n\|^2} = 1 - \int_\Omega \frac{2F(x,u_n)}{\|u_n\|^2} - \frac{2}{\|u_n\|} \int_\Omega hv_n \to 0.$$

So that

$$\int_\Omega \frac{2F(x,u_n)}{\|u_n\|^2} \to 1 \tag{3.11}$$

Therefore, combining (3.10), (3.11) and Fatou's lemma yields

$$\lambda_i \int_\Omega v^2 = \lim \int_\Omega \frac{2F(x,u_n)}{\|u_n\|^2} \geq \int_\Omega \liminf \frac{2F(x,u_n)}{\|u_n\|^2} \tag{3.12}$$

Using Lemma 2.2 we get

$$\lambda_i \int_\Omega v^2 \geq \int_\Omega L(x) v^2.$$

But then, since $L(x) \geq \lambda_i$, we obtain $L(x) = \lambda_i$ a.e. in Ω which contradicts **(F)**.

Case(ii) : Similarly to the case (i), if $m(x) \equiv \lambda_{i+1}$ we obtain

$$1 = \lambda_{i+1} \int_\Omega v^2$$

and

$$\lambda_{i+1} \int_\Omega v^2 = \lim \int_\Omega \frac{2F(x, u_n)}{\|u_n\|^2} \leq \int_\Omega \limsup \frac{2F(x, u_n)}{\|u_n\|^2}$$

so that

$$\lambda_{i+1} \int_\Omega v^2 \leq \int_\Omega K(x) v^2$$

by Lemma 2.2 , and as $K(x) \leq \lambda_{i+1}$, we conclude that $K(x) = \lambda_{i+1}$ a.e. in Ω which again contradicts **(F)**.

Case (iii) : Since $v \not\equiv 0$, this case can not occur in view of Lemma 2.3.

Since neither one of cases (i), (ii), (iii) can occur, this shows that any (PS) sequence must be bounded, so that the functional I satisfies the Palais-Smale condition. □

Now, let us consider the decomposition of the space H as $H = V \oplus W$ where V is the subspace spanned by the eigenfunctions corresponding to $\lambda_1, ..., \lambda_i$ and $W = V^\perp$.
We define the two functionals A and B as follow :

$$A(v) = \|v\|^2 - \int_\Omega L(x) v^2, \forall v \in V,$$

$$B(w) = \|w\|^2 - \int_\Omega K(x) w^2, \forall w \in W,$$

We recall the two useful inequalities [5]:

$$\int_\Omega (\Delta v)^2 \leq \lambda_i \int_\Omega v^2, \qquad \forall v \in V.$$

$$\int_\Omega (\Delta w)^2 \geq \lambda_{i+1} \int_\Omega w^2, \qquad \forall w \in W$$

and the characterization of the first eigenvalue λ_1 of Δ^2 on H defined by

$$\lambda_1 = \inf\Big\{ \int_\Omega |\Delta u|^2 dx \ : \ u \in H \ \text{ and } \ \int_\Omega |u|^2 dx = 1 \Big\}.$$

They will be used in the proof of the the next proposition following the same ideas as in [1].

Proposition 3.2. *There exists $\delta > 0$ such that :*

(a) $B(w) \geq \delta\|w\|^2, \quad \forall w \in W;$

(b) $I(w) \to +\infty$ *as* $\|w\| \to +\infty, \quad w \in W$*;*

(c) $A(v) \leq -\delta\|v\|^2, \quad \forall v \in V$*;*

(d) $I(v) \to -\infty$ *as* $\|v\| \to +\infty, \quad v \in V$*.*

Proof: **(a)** First, since $\|w\|^2 \geq \lambda_{i+1}\|w\|_{L^2}^2$, for all $w \in W$, we have

$$B(w) \geq \int_\Omega (\lambda_{i+1} - K(x))w^2 \geq 0, \quad \forall w \in W. \tag{3.13}$$

By contradiction if (a) does not hold, then there exists a sequence $w_n \in W$ such that $\|w_n\| = 1$, $B(w_n) \to 0$, and for further subsequence $w_n \rightharpoonup w \in W$ weakly and $w_n \to w$ strongly in $L^2(\Omega)$, so that

$$B(w) \leq \liminf B(w_n) = 0$$

by the weak lower semicontinuity of the convex functional B on W. Therefore, we get $B(w) = 0$. We claim that $w \equiv 0$. Indeed , since $B(w) = 0$, by (3.13) we get $w = 0$ on the set

$$\Omega_K = \{x \in \Omega \ : \ K(x) < \lambda_{i+1}\}.$$

On the other hand

$$0 = B(w) = \|w\|^2 - \int_\Omega K(x)w^2 \geq \|w\|^2 - \lambda_{i+1}\|w\|_{L^2}^2 \geq 0$$

hence $\|w\|^2 = \lambda_{i+1}\|w\|_{L^2}^2$ which shows that w is an eigenfunction associated to λ_{i+1}. Therefore, since $w = 0$ on the set Ω_K of positive measure, using the unique continuation principle we get $w \equiv 0$.
But, then we have $w_n \rightharpoonup w = 0$ in $L^2(\Omega)$, hence

$$B(w_n) = 1 - \int_\Omega K(x)w_n^2 \to 1$$

which contradict $B(w_n) \to 0$. □

(b) Let $0 < \varepsilon < \delta\lambda_{i+1}$ where δ is given above. By **(F)** there exists $b_\varepsilon \in L^1(\Omega)$ such that

$$2F(x,s) \leq (K(x) + \varepsilon)s^2 + b_\varepsilon(x)$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. Therefore, we obtain for all $w \in W$,

$$\begin{aligned}
2I(w) &= \|w\|^2 - 2\int_\Omega F(x,w) - 2\int_\Omega hw \\
&\geq \|w\|^2 - \int_\Omega (K(x)+\varepsilon)w^2 - 2\int_\Omega hw - \int_\Omega b_\varepsilon \\
&= B(w) - \varepsilon\int_\Omega w^2 - 2\int_\Omega hw - \int_\Omega b_\varepsilon \\
&\geq \delta\|w\|^2 - \varepsilon\|w\|^2_{L^2} - 2\|h\|_{L^2}\|w\|_{L^2} - \|b_\varepsilon\|_{L^1} \\
&\geq \delta\|w\|^2 - \frac{\varepsilon}{\lambda_{i+1}}\|w\|^2 - C_1\|w\| - C_2 \\
&\geq (\delta - \frac{\varepsilon}{\lambda_{i+1}})\|w\|^2 - C_1\|w\| - C_2
\end{aligned}$$

where $\delta - \frac{\varepsilon}{\lambda_{i+1}} > 0$ and $\|.\|_{L^1}$ denote the usual norm in $L^1(\Omega)$. It follows that $I(w) \to +\infty$ as $\|w\| \to +\infty, \quad w \in W$. □

(c) Since $\|v\|^2 \leq \lambda_i\|v\|^2_{L^2}$ for all $v \in V$, we have

$$A(v) \leq \int_\Omega (\lambda_i - L(x))v^2 \leq 0, \qquad \forall v \in V.$$

Similarly to (a), we prove that $A(v) = 0$ implies $v \equiv 0$, by showing first that $v = 0$ on the set $\Omega_L = \{x \in \Omega \ : \ \lambda_i < L(x)\}$ of positive measure and that v is an eigenfunction associated to λ_i. Then by contradiction, we obtain $v_n \in V$ such that $\|v_n\| = 1$ and $A(v_n) \to 0$, where we may assume that $v_n \to v \in V$ in H since V is of finite dimension. Therefore we obtain $A(v) = 0$ so that $v = 0$ and consequently $\int_\Omega L(x)v_n^2 \to 0$, but then , it follows that

$$A(v_n) = 1 - \int_\Omega L(x)v_n^2 \to 1$$

which contradict $A(v_n) \to 0$. □

(d) Let $0 < \varepsilon < \delta\lambda_1$, fixed. By the assumption **(F)** we get

$$2F(x,s) \geq (L(x) - \varepsilon)s^2 - b_\varepsilon(x)$$

for all a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, which implies

$$\begin{aligned}
2I(v) &\leq A(v) + \varepsilon\|v\|^2_{L^2} + 2\|h\|_{L^2}\|v\|_{L^2} + \|b_\varepsilon\|_{L^1} \\
&\leq -(\delta - \frac{\varepsilon}{\lambda_1})\|v\|^2 + C_1\|v\| + C_2
\end{aligned}$$

for all $v \in V$. Since $\delta - \frac{\varepsilon}{\lambda_1} > 0$ it follows that $I(v) \to -\infty$ as $\|v\| \to +\infty, \ v \in V$.
□

Now we are in position to prove our main result.

Proof of Theorem 1.1.
We write the functional I as :

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega (F(x,u) + hu)dx \\ &= \frac{1}{2}\|u\|^2 + N(u), \quad u \in H, \end{aligned}$$

where $N(u) = -\int_\Omega (F(x,u) + hu)dx$

Note that I is weakly lower semicontinuous, as the sum of the weakly lower semicontinuous functional $\frac{1}{2}\|.\|^2$ and the weakly continuous functional N. Therefore, since I is coercive on W by Proposition 2(b), the infimum $\beta := \inf_W I > -\infty$ is attained. Now, let $\alpha < \beta$, by Proposition 2(d), there exists $R > 0$ such that $I(v) \leq \alpha$ for all $v \in V$ with $\|v\| \geq R$.
Finally, since I satisfies the Palais-Smale condition by Proposition 3.1, we can use the saddle point theorem of P. Rabinowitz [3] to conclude the existence of a critical point $\tilde{u} \in H$ of I with $I(\tilde{u}) \geq \beta$. □

References

1. D.G.COSTA, and A.S. OLIVEIRA Existence of solution for a class of semilinear elliptic problems at double resonance ,Bol. Soc. BRAS. Mat., vol 19, pp. 21-37, (1988).

2. H. BERESTYCKI ,D.G. DE FIGUEREIDO Double resonance in semilinear elliptic problems, Comm. Partial Diff. Equations, 6(1), 91-120 (1981).

3. P.H. RABINOWITZ Some minimax theorems and applications to nonlinear partial differential equations, in ed, . Academic Presss, 1978, 161-177.

4. N.TSOULI, O. CHAKRONE, O. DARHOUCHE and M.RAHMANI, Strict monotonicity and unique continuation of the biharmonic operator,Bol. Soc. Paran. Mat .,v.30 2(2012): 1-5.

5. D. G. De FIGUIEREDO Positive solutions of semilinear elliptic problems, Differential equations (SÃo Paulo, 1981), pp. 34-87, Lecture Notes in Math., 957, Springer, Berlin-New York, 1982.

6. F. Zanolin and P. OMARI, Resonance at two consecutive eigenvalues for semilinear elliptic equations, Annali di Mathematica Pure ed Applicata, volume 163, Number 1, 181-198, 1993.

Najib Tsouli
Department of Mathematics, University Mohamed 1^{ier}*,*
Oujda, Morocco.
E-mail address: `Ntsouli@hotmail.com`

and

Omar Chakrone
Department of Mathematics, University Mohamed 1^{ier}*,*
Oujda, Morocco.
E-mail address: `chakrone@yahoo.fr`

and

Omar Darhouche
Department of Mathematics, University Mohamed 1^{ier}*,*
Oujda, Morocco.
E-mail address: `omarda13@hotmail.com`

and

Mostafa Rahmani
Department of Mathematics, University Mohamed 1^{ier}*,*
Oujda, Morocco.
E-mail address: `rahmani.mostafa.63@hotmail.com`