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M^2-Type Sharp Estimates and Weighted Boundedness for Commutators Related to Singular Integral Operators Satisfying a Variant of Hörmander's Condition

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abstract: In this paper, we prove the M^k-type sharp maximal function estimates for the commutators related to some singular integral operators satisfying a variant of Hörmander's condition. As an application, we obtain the weighted boundedness of the commutators on Lebesgue and Morrey spaces.

Key Words: Singular integral operator; Commutator; Sharp maximal function; Morrey space; BMO.

Contents

1. Introduction

As the development of singular integral operators(see [14,15]), their commutators have been well studied(see [4]). In [13], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [1]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [8], some singular integral operators satisfying a variant of Hörmander's condition are introduced, and the boundedness for the operators are obtained(see [8,16]). The purpose of this paper is to prove the sharp maximal function inequalities for the the commutators related to some singular integral operators satisfying a variant of Hörmander's condition. As an application, we obtain the weighted boundedness of the commutator on Lebesgue and Morrey space.

2. Preliminaries

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f,

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the sharp maximal function of f is defined by

$$f^{\#}(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)dx$. We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and define $||f||_{BMO}=||f^{\#}||_{L^\infty}$.

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|dy.$$

For $\eta>0$, let $M_\eta(f)=M(|f|^\eta)^{1/\eta}$. For $k\in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)=M(f)$ and

$$M^k(f)=M(M^{k-1}(f))\text{ when }\ k\geq 2.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote that the Φ-average by, for a function f,

$$||f||_{\Phi,Q}=\inf\left\{\lambda>0:\frac{1}{|Q|}\int_Q\Phi\left(\frac{|f(y)|}{\lambda}\right)dy\leq 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x)=\sup_{x\in Q}||f||_{\Phi,Q}.$$

The Young functions to be using in this paper are $\Phi(t)=t(1+logt)$ and $\tilde{\Phi}(t)=exp(t)$, the corresponding average and maximal functions denoted by $||\cdot||_{L(logL),Q}$, $M_{L(logL)}$ and $||\cdot||_{expL,Q}$, M_{expL}. Following [13], we know the generalized Hölder's inequality and the following inequalities hold:

$$\frac{1}{|Q|}\int_Q|f(y)g(y)|dy\leq||f||_{\Phi,Q}||g||_{\tilde{\Phi},Q},$$

$$||f||_{L(logL),Q}\leq M_{L(logL)}(f)\leq CM^2(f),$$
$$||f-f_Q||_{expL,Q}\leq C||f||_{BMO}$$

and

$$||f-f_Q||_{expL,2^kQ}\leq Ck||f||_{BMO}.$$

The A_p weight is defined by (see [7])

$$A_p=\left\{w\in L^1_{loc}(R^n):\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty\right\},$$

$1<p<\infty$,

$$A_1=\{w\in L^p_{loc}(R^n):M(w)(x)\leq Cw(x),a.e.\}$$

and

$$A_\infty = \bigcup_{p\geq 1} A_p.$$

Given a weight function w. For $1 \leq p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$||f||_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x)dx\right)^{1/p} < \infty.$$

Definition 2.1. *Let $\Phi = \{\phi_1, ..., \phi_m\}$ be a finite family of bounded functions in R^n. For any locally integrable function f, the Φ sharp maximal function of f is defined by*

$$M_\Phi^{\#}(f)(x) = \sup_{Q\ni x} \inf_{\{c_1,\ldots,c_m\}} \frac{1}{|Q|}\int_Q |f(y) - \sum_{j=1}^m c_j\phi_j(x_Q - y)|dy,$$

where the infimum is taken over all m-tuples $\{c_1, ..., c_m\}$ of complex numbers and x_Q is the center of Q. For $\eta > 0$, let

$$M_{\Phi,\eta}^{\#}(f)(x) = \sup_{Q\ni x} \inf_{\{c_1,\ldots,c_m\}} \left(\frac{1}{|Q|}\int_Q |f(y) - \sum_{j=1}^m c_j\phi_j(x_Q - y)|^\eta dy\right)^{1/\eta}.$$

Remark 2.2. *We note that $M_\Phi^{\#} \approx f^{\#}$ if $m = 1$ and $\phi_1 = 1$.*

Definition 2.3. *Given a positive and locally integrable function f in R^n, we say that f satisfies the reverse Hölder's condition (write this as $f \in RH_\infty(R^n)$), if for any cube Q centered at the origin we have*

$$0 < \sup_{x\in Q} f(x) \leq C\frac{1}{|Q|}\int_Q f(y)dy.$$

In this paper, we will study some singular integral operators as following(see [16]).

Definition 2.4. *Let $K \in L^2(R^n)$ and satisfy*

$$||K||_{L^\infty} \leq C,$$

$$|K(x)| \leq C|x|^{-n},$$

there exist functions $B_1, ...B_m \in L^1_{loc}(R^n - \{0\})$ and $\Phi = \{\phi_1, ..., \phi_m\} \subset L^\infty(R^n)$ such that $|det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$, and for a fixed $\delta > 0$ and any $|x| > 2|y| > 0$,

$$|K(x - y) - \sum_{j=1}^m B_j(x)\phi_j(y)| \leq C\frac{|y|^\delta}{|x - y|^{n+\delta}}.$$

For $f \in C_0^\infty$, *we define the singular integral operator related to the kernel* K *by*

$$T(f)(x) = \int_{R^n} K(x-y)f(y)dy.$$

Let b *be a locally integrable function on* R^n. *The commutator related to* T *is defined by*

$$T^b(f)(x) = \int_{R^n} (b(x)-b(y))K(x-y)f(y)dy.$$

Remark 2.5. *Note that the classical Calderón-Zygmund singular integral operator satisfies* **Definition 2.4***(see [14,15]).*

Definition 2.6. *Let* φ *be a positive, increasing function on* R^+ *and there exists a constant* $D > 0$ *such that*

$$\varphi(2t) \le D\varphi(t) \quad for \quad t \ge 0.$$

Let w *be a weight function and* f *be a locally integrable function on* R^n. *Set, for* $1 \le p < \infty$,

$$||f||_{L^{p,\varphi}(w)} = \sup_{x\in R^n,\ d>0} \left(\frac{1}{\varphi(d)}\int_{Q(x,d)} |f(y)|^p w(y)dy\right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. *The generalized Morrey space is defined by*

$$L^{p,\varphi}(R^n, w) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\eta$, $\eta > 0$, *then* $L^{p,\varphi}(R^n, w) = L^{p,\eta}(R^n, w)$, *which is the classical weighted Morrey spaces (see [11,12]). If* $\varphi(d) = 1$, *then* $L^{p,\varphi}(R^n, w) = L^p(R^n, w)$, *which is the weighted Lebesgue spaces (see [7]).*

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [2,5,6,9,10]).

3. Theorems and Lemmas

We shall prove the following theorems.

Theorem 3.1. *Let* T *be the singular integral operator as* **Definition 2.4**, $0 < r < 1$ *and* $b \in BMO(R^n)$. *Then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M^{\#}_{\Phi,r}(T^b(f))(\tilde{x}) \le C||b||_{BMO}\left(M^2(f)(\tilde{x}) + M^2(T(f))(\tilde{x})\right).$$

Theorem 3.2. *Let* T *be the singular integral operator as* **Definition 2.4**, $1 < p < \infty$, $w \in A_1$ *and* $b \in BMO(R^n)$. *Then* T^b *is bounded on* $L^p(w)$.

Theorem 3.3. *Let T be the singular integral operator as* **Definition 2.4**, $0 < D < 2^n$, $1 < p < \infty$, $w \in A_1$ *and* $b \in BMO(R^n)$. *Then* T^b *is bounded on* $L^{p,\varphi}(R^n, w)$.

To prove the theorems, we need the following lemmas.

Lemma 3.4. *([7], p.485) Let* $0 < p < q < \infty$ *and for any function* $f \geq 0$. *We define that, for* $1/r = 1/p - 1/q$

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E *with* $0 < |E| < \infty$. *Then*

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} ||f||_{WL^q}.$$

Lemma 3.5. *(see [13]) We have*

$$\frac{1}{|Q|} \int_Q |f(x)g(x)|dx \leq ||f||_{expL,Q} ||g||_{L(logL),Q}.$$

Lemma 3.6. *(see [16]) Let* T *be the singular integral operator as* **Definition 2.3**. *Then* T *is bounded on* $L^p(w)$ *for* $1 < p < \infty$, $w \in A_1$ *and weak* (L^1, L^1) *bounded.*

Lemma 3.7. *(see [16]). Let* $1 < p < \infty$, $0 < \eta < \infty$, $w \in A_\infty$ *and* $\Phi = \{\phi_1, ..., \phi_m\} \subset L^\infty(R^n)$ *such that* $|det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$\int_{R^n} M_\eta(f)(x)^p w(x)dx \leq C \int_{R^n} M^{\#}_{\Phi,\eta}(f)(x)^p w(x)dx.$$

Lemma 3.8. *(see [2,5]) Let* $1 < p < \infty$, $w \in A_1$ *and* $0 < D < 2^n$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$||M(f)||_{L^{p,\varphi}(w)} \leq C||f||_{L^{p,\varphi}(w)}.$$

Lemma 3.9. *Let* $1 < p < \infty$, $0 < \eta < \infty$, $w \in A_1$, $0 < D < 2^n$ *and* $\Phi = \{\phi_1, ..., \phi_m\} \subset L^\infty(R^n)$ *such that* $|det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$||M_\eta(f)||_{L^{p,\varphi}(w)} \leq C||M^{\#}_{\Phi,\eta}(f)||_{L^{p,\varphi}(w)}.$$

Proof: For any cube $Q = Q(x_0, d)$ in R^n, we know $M(w\chi_Q) \in A_1$ for any cube

$Q = Q(x,d)$ by [3]. If $x \in Q^c$, by Lemma 3.7, we have, for $f \in L^{p,\varphi}(R^n, w)$,

$$\begin{aligned}
&\int_Q |M_\eta(f)(y)|^p w(y)dy \\
= &\int_{R^n} |M_\eta(f)(y)|^p w(y)\chi_Q(y)dy \\
\le &\int_{R^n} |M_\eta(f)(y)|^p M(w\chi_Q)(y)dy \\
\le &C\int_{R^n} |M^\#_{\Phi,\eta}(f)(y)|^p M(w\chi_Q)(y)dy \\
= &C\left(\int_Q |M^\#_{\Phi,\eta}(f)(y)|^p M(w\chi_Q)(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |M^\#_{\Phi,\eta}(f)(y)|^p M(w\chi_Q)(y)dy\right) \\
\le &C\left(\int_Q |M^\#_{\Phi,\eta}(f)(y)|^p w(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |M^\#_{\Phi,\eta}(f)(y)|^p \frac{w(Q)}{|2^{k+1}Q|}dy\right) \\
\le &C\left(\int_Q |M^\#_{\Phi,\eta}(f)(y)|^p w(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q} |M^\#_{\Phi,\eta}(f)(y)|^p \frac{M(w)(y)}{2^{n(k+1)}}dy\right) \\
\le &C\left(\int_Q |M^\#_{\Phi,\eta}(f)(y)|^p w(y)dy + \sum_{k=0}^\infty \int_{2^{k+1}Q} |M^\#_{\Phi,\eta}(f)(y)|^p \frac{w(y)}{2^{nk}}dy\right) \\
\le &C||M^\#_{\Phi,\eta}(f)||^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty 2^{-nk}\varphi(2^{k+1}d) \\
\le &C||M^\#_{\Phi,\eta}(f)||^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty (2^{-n}D)^k\varphi(d) \\
\le &C||M^\#_{\Phi,\eta}(f)||^p_{L^{p,\varphi}(w)}\varphi(d),
\end{aligned}$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M_\eta(f)(x)^p w(x)dx\right)^{1/p} \le C\left(\frac{1}{\varphi(d)}\int_Q M^\#_{\Phi,\eta}(f)(x)^p w(x)dx\right)^{1/p}$$

and

$$||M_\eta(f)||_{L^{p,\varphi}(w)} \le C||M^\#_{\Phi,\eta}(f)||_{L^{p,\varphi}(w)}.$$

This finishes the proof. □

Lemma 3.10. *Let T be the singular integral operator as* **Definition 2.4**, $1 < p < \infty$, $w \in A_1$ *and* $0 < D < 2^n$. *Then*

$$||T(f)||_{L^{p,\varphi}(w)} \le C||f||_{L^{p,\varphi}(w)}.$$

The proof of the Lemma is similar to that of Lemma 3.9 by Lemma 3.6, we omit the details.

4. Proofs of Theorems

Proof of Theorem 3.1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^r dx\right)^{1/r} \le C||b||_{BMO}\left(M^2(f)(\tilde{x}) + M^2(T(f))(\tilde{x})\right),$$

where Q is any a cube centered at x_0, $C_0 = \sum_{j=1}^m g_j\phi_j(x_0 - x)$ and $g_j = \int_{R^n} B_j(x_0 - y)(b(y) - b_{2Q})f_2(y)dy$. Let $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$T^b(f)(x) = (b(x) - b_{2Q})T(f)(x) - T((b - b_{2Q})f_1)(x) - T((b - b_{2Q})f_2)(x).$$

Then

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |T^b(f)(x) - C_0|^r dx\right)^{1/r} \\
\le\ & C\left(\frac{1}{|Q|}\int_Q |(b(x) - b_{2Q})T(f)(x)|^r dx\right)^{1/r} + C\left(\frac{1}{|Q|}\int_Q |T((b - b_{2Q})f_1)(x)|^r dx\right)^{1/r} \\
& + C\left(\frac{1}{|Q|}\int_Q |T((b - b_{2Q})f_2)(x) - C_0|^r dx\right)^{1/r} \\
=\ & I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, by Hölder's inequality and Lemma 3.6, we obtain

$$\begin{aligned}
I_1 &\le \frac{C}{|Q|}\int_{2Q} |b(x) - b_{2Q}||T(f)(x)|dx \\
&\le C||b - b_{2Q}||_{expL,2Q}||T(f)||_{L(logL),2Q} \\
&\le C||b||_{BMO}M^2(T(f))(\tilde{x}),
\end{aligned}$$

For I_2, by Lemma 3.4, 3.5 and 3.6, we obtain

$$\begin{aligned}
I_2 &\le C\left(\frac{1}{|Q|}\int_{R^n} |T((b - b_{2Q})f_1)(x)|^r \chi_Q(x)dx\right)^{1/r} \\
&\le C|Q|^{-1}\frac{||T((b - b_{2Q})f_1)\chi_Q||_{L^r}}{|Q|^{1/r-1}} \\
&\le C|Q|^{-1}||T((b - b_{2Q})f_1)||_{WL^1} \\
&\le C|Q|^{-1}||(b - b_{2Q})f_1||_{L^1} \\
&\le \frac{C}{|2Q|}\int_{2Q} |b(x) - b_{2Q}||f(x)|dx \\
&\le C||b - b_{2Q}||_{expL,2Q}||f||_{L(logL),2Q} \\
&\le C||b||_{BMO}M^2(f)(\tilde{x}),
\end{aligned}$$

For I_3, we have

$$\begin{aligned}
I_3 &\leq \frac{C}{|Q|}\int_Q |T((b-b_{2Q})f_2)(x)-C_0)|dx \\
&\leq \frac{C}{|Q|}\int_Q\int_{R^n}\left|(K(x-y)-\sum_{j=1}^{m}B_j(x_0-y)\phi_j(x_0-x))(b(y)-b_{2Q})f_2(y)\right|dydx \\
&\leq \frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}\left(\int_{2^kd\leq|y-x_0|<2^{k+1}d}\frac{|x-x_0|^\delta}{|y-x_0|^{n+\delta}}|b(y)-b_{2Q}||f(y)|dy\right)dx \\
&\leq C\sum_{k=1}^{\infty}\frac{d^\delta}{(2^kd)^{n+\delta}}\int_{2^{k+1}Q}|b(y)-b_{2Q}||f(y)|dy \\
&\leq C\sum_{k=1}^{\infty}\frac{d^\delta}{(2^kd)^{n+\delta}}(2^kd)^n||b-b_{2Q}||_{expL,2^{k+1}Q}||f||_{L(logL),2^{k+1}Q} \\
&\leq C||b||_{BMO}M^2(f)(\tilde{x})\sum_{k=1}^{\infty}k2^{-k\delta} \\
&\leq C||b||_{BMO}M^2(f)(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 3.1.

Proof of Theorem 3.2. By Theorem 3.1 and Lemma 3.6-3.7, we have

$$\begin{aligned}
&||T^b(f)||_{L^p(w)} \leq \|M_r(T^b(f))\|_{L^p(w)} \leq C\|M^{\#}_{\Phi,r}(T^b(f))\|_{L^p(w)} \\
\leq\ & C||b||_{BMO}\left(\|M^2(T(f))\|_{L^p(w)}+\|M^2(f)\|_{L^p(w)}\right) \\
\leq\ & C||b||_{BMO}(\|T(f)\|_{L^p(w)}+\|f\|_{L^p(w)}) \\
\leq\ & C||b||_{BMO}\|f\|_{L^p(w)}.
\end{aligned}$$

This completes the proof of the theorem.

Proof of Theorem 3.3. By Theorem 3.1 and Lemma 3.8-3.10, we have

$$\begin{aligned}
&||T^b(f)||_{L^{p,\varphi}(w)} \leq \|M_r(T^b(f))\|_{L^{p,\varphi}(w)} \leq C\|M^{\#}_{\Phi,r}(T^b(f))\|_{L^{p,\varphi}(w)} \\
\leq\ & C||b||_{BMO}\left(\|M^2(T(f))\|_{L^{p,\varphi}(w)}+\|M^2(f)\|_{L^{p,\varphi}(w)}\right) \\
\leq\ & C||b||_{BMO}(\|T(f)\|_{L^{p,\varphi}(w)}+\|f\|_{L^{p,\varphi}(w)}) \\
\leq\ & C||b||_{BMO}\|f\|_{L^{p,\varphi}(w)}.
\end{aligned}$$

This completes the proof of the theorem.

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