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Averaging in optimal control problems for systems of difference equations

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abstract: We apply the averaging method to the optimal control problem for system of difference equations in the standard Bogolyubov form. We construct the ε -optimal control.

Key Words: Optimal control; Averaging method; Averaged system; System of difference equations.

Contents

1. Introduction

In this paper we consider optimal control problem for systems of difference equations in the standard Bogolyubov form. The presence of small parameter allow us to average such systems and reduce the initial nonautonomous problem to more simple autonomous one.

The averaging method was studied by many authors. N. Bogolyubov [3] developed a general averaging approach for system of ordinary differential equations. Further this method was applied to systems of functional-differential equations, difference equations, stochastic systems [2,9].

Optimal control problems for systems of differential and difference equations are particularly important for applied goals. Methods have been developed to investigate them. For more details see [4,5,6,7,10,11]. Fundamental results in application of averaging method to optimal control problems were obtained by V. Plotnikov [11]. In work [10] authors propose a new scheme of averaging for optimal control problem.

This work is devoted to the application of averaging method to optimal control problems for systems of difference equations. We investigate relationship between optimal controls of the averaged and the original systems and prove that the optimal control for the averaged system is ε-optimal for the original problem.

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Statement of the problem.

Let us consider the optimal control problem for system of difference equations:

$$\Delta x_n = x_{n+1} - x_n = \varepsilon f_n(x_n, u_n), \tag{1}$$

with a given initial condition $x_0 \in D$.

Here $\varepsilon > 0$ is a small parameter, $x_n \in D$ is a phase vector, D is a domain in R^d, $u_n \in U \subset R^m$ is a control vector, f_n is a continuous vector-function on the domain, $n \in Z$.

Controls u_n are called admissible if the following conditions are satisfied:

1) $u_n \in U$ for all $n \in Z$.
2) for every u_n there exists a constant $u_0 \in D$ such that $\mid u_n - u_0 \mid \leq \varphi_n$, where φ_n does not depend on u_n and $\sum\limits_{n=1}^{\infty} \varphi_n < \infty$.

We introduce F to denote the set of all admissible controls. For every admissible control u_n we denote the solution of system (1) by $x_n(u_n)$.

Our aim is to find an admissible control $u = u_n$ which minimizes the functional

$$J_\varepsilon(u) = \Phi(x_{\left[\frac{T}{\varepsilon}\right]}(u)),$$

where $\Phi(x)$ is a given function, $T > 0$ is a certain constant, and [.] is an integer part of a number.

Denote

$$J_\varepsilon = \inf_{u_n \in F} J_\varepsilon(u_n).$$

We associate the system (1) on $\left[0, \frac{T}{\varepsilon}\right]$ with averaged system

$$\Delta y_n = y_{n+1} - y_n = \varepsilon f_0(y_n, u_n), \tag{2}$$

where $y_0 = x_0$,

$$f_0(y, u) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f_n(y, u), \tag{3}$$

and

$$\bar{J}_\varepsilon(u) = \Phi(y_{\left[\frac{T}{\varepsilon}\right]}(u)). \tag{4}$$

Let $\bar{u}_n^*(\varepsilon)$ be an optimal control for averaged problem (2).

In this work we prove that the control $\bar{u}_n^*(\varepsilon)$ is $\eta-$optimal for system (1), i.e., for any $\eta > 0$ there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$ the inequality

$$\mid J_\varepsilon(\bar{u}_n^*(\varepsilon)) - J_\varepsilon \mid \leq \eta$$

is true.

2. Preliminaries

To obtain the main result we need two lemmas. The first one is a discrete version of the known Gronwall-Bellman inequality [1,8], the second one is a generalization of averaging method for difference equations in the case where right sides depend on functional parameters.

Lemma 2.1. *(Discrete version of Gronwal -Bellman inequality). Let* $\{y_n\}$ *and* $\{a_n\}$ *be non-negative sequences and* $C > 0$ *is a constant. If* $y_n \leq C + \sum_{k=0}^{n} a_k y_k$, *then*

$$y_n \leq C \exp(\sum_{k=0}^{n} a_k).$$

Lemma 2.2. *Suppose that the following conditions are satisfied in the domain* $Q = \{x \in D \subset R^d,\ n \in Z,\ u \in U \subset R^m\}$*:*

1) $f_n(x,u)$ *is bounded and satisfies the Lipschitz condition with respect to* x *and* u *with a constant* M*;*
2) *a solution* $y = y_n(u_n)$, $y_0(u_0) = x_0$ *of the averaged system is defined for all admissible* u_n *and belongs to the domain* D *together with a some* $\rho-$*neighborhood;*
3) *the limit* (3) *exists uniformly in* $x \in D$ *and* $u \in U$*.*

Then for any $\eta > 0$ *and* $T > 0$ *there exists* $\varepsilon_0(\eta, T) > 0$*, such that for any* $0 < \varepsilon < \varepsilon_0$ *and integer* $n \in [0, [\frac{T}{\varepsilon}]]$ *the estimate*

$$| x_n(u_n) - y_n(u_n) | \leq \eta \tag{5}$$

holds for every admissible control.

Proof: First we find a sequence $\{\psi_N\}$ such that for all $x \in D$ and admissible control u_n the estimate

$$\frac{1}{N} | \sum_{n=1}^{N-1} [f_n(x,u_n) - f_0(x,u_0)] | \leq \psi_N \tag{6}$$

holds. Note that

$$\lim_{N\to\infty} \psi_N = 0.$$

Indeed, condition (3) implies the existence of sequence $\{a_N\}$, that converges to 0, such that for all $x \in D$, $u \in U$ it follows

$$\frac{1}{N} | \sum_{n=1}^{N-1} [f_n(x,u_n) - f_0(x,u_0)] | \leq a_N. \tag{7}$$

Thus we have

$$\begin{aligned}\frac{1}{N} \quad | \quad & \sum_{n=1}^{N-1}\left[f_n(x,u_n)-f_0(x,u_0)\right] |= \frac{1}{N} \mid \sum_{n=1}^{N-1}[f_n(x,u_n)-f_n(x,u_0) \\ & +f_n(x,u_0)-f_0(x,u_0)] \mid \\ \leq \quad & \frac{1}{N} \mid \sum_{n=1}^{N-1}\left[f_n(x,u_n)-f_n(x,u_0)\right] \mid + \frac{1}{N} \mid \sum_{n=1}^{N-1}\left[f_n(x,u_0)-f_0(x,u_0)\right] \mid\end{aligned}$$

$$\leq \frac{M}{N}\sum_{n=1}^{N-1} \mid u_n-u_0 \mid +a_N \leq \frac{M}{N}\sum_{n=1}^{\infty}\varphi_n + a_N.$$

It remains to denote

$$\psi_N = \frac{M}{N}\sum_{n=1}^{\infty}\varphi_n + a_N.$$

Hence for any admissible u_n in system (1), we can consider the next system

$$\Delta y_n = \varepsilon f_0(y_n,u_0), \; y_0 = x_0 \tag{8}$$

as the averaged one.

Now for solutions $x_n(u_n)$ and $y_n(u_0)$ of systems (1) and (8) we apply the analog of the first Bogolyubov theorem for difference equations.
From estimate (6) it follows that for any $\eta > 0$ and $T > 0$ there exists $\varepsilon_0 = \varepsilon_0(\eta,T) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $n \in \left[0, \left[\frac{T}{\varepsilon}\right]\right]$ the estimate

$$\mid x_n(u_n) - y_n(u_0) \mid \leq \frac{\eta}{2} \tag{9}$$

holds. Here ε_0 does not depend on u_n.

Next, we evaluate the norm of the difference between solutions of systems (8) and (2) for $n \in \left[0, \left[\frac{T}{\varepsilon}\right]\right]$. Represent systems (2) and (8) in the form

$$y_n(u_n) = x_0 + \varepsilon \sum_{k=0}^{n-1}\left[f_0(y_k(u_k),u_k)\right], \tag{10}$$

$$y_n(u_0) = x_0 + \varepsilon \sum_{k=0}^{n-1}\left[f_0(y_k(u_0),u_0)\right]. \tag{11}$$

Subtracting (11) from (10) and adding to and subtracting from the right side of the equality the function $f_0(y_k(u_0),u_k)$, we obtain:

$$y_n(u_n) - y_n(u_0) = \varepsilon \sum_{k=0}^{n-1} [f_0(y_k(u_k), u_k) - f_0(y_k(u_0), u_k)] + \varepsilon \sum_{k=0}^{n-1} [f_0(y_k(u_0), u_k) - f_0(y_k(u_0), u_0)] .$$

Hence, using Lipschitz condition, we get

$$| \quad y_n(u_n) - y_n(u_0) |= \varepsilon M \sum_{k=0}^{n-1} | y_k(u_k) - y_k(u_0) | + \varepsilon M \sum_{k=0}^{n-1} | u_k - u_0 |$$
$$\leq \quad \varepsilon M \sum_{k=0}^{n-1} | y_k(u_k) - y_k(u_0) | + \varepsilon M \sum_{k=0}^{\infty} \varphi_k.$$

According to the Lemma 2.1, we observe that for all integers $n \in \left[0, \left[\frac{T}{\varepsilon}\right]\right]$ the following estimate is true

$$| y_n(u_n) - y_n(u_0) | \leq \varepsilon M \sum_{k=0}^{\infty} \varphi_k \exp(2M[T]), \tag{12}$$

Now, we choose $\varepsilon_1 \leq \varepsilon_0$, such that for all $\varepsilon \leq \varepsilon_1$ as $n \in \left[0, \left[\frac{T}{\varepsilon}\right]\right]$ the inequality

$$| y_n(u_n) - y_n(u_0) | \leq \frac{\eta}{2} \tag{13}$$

holds.
Finally from (9) and (13), we obtain estimate (5) in the lemma. This completes the proof of the Lemma.

□

3. Main result

Let us now state and prove the main result of this work.

Theorem 3.1. *Assume that in the domain* $Q = \left\{x \in D \subset R^d, n \in Z, u \in U \subset R^m\right\}$

1) *the function $f_n(x,u)$ is bounded by a constant K. Furthermore, it is Lipschitz with respect to x and u with the constant M;*

2) *the solution $y = y_n(u_n)$, $y(u_0) = x_0$ of the averaged system* (2) *is defined for all admissible u_n and it belongs to the domain D together with some $\rho-$neighborhood;*

3) *the limit* (3) *exists uniformly in $x \in D$ and $u \in U$;*

4) the function $\Phi(x)$ satisfies Lipschitz condition with constant L in the domain D;

5) there exists an admissible control $\bar{u}_n^*(\varepsilon)$ for system (2).

Then for any $\eta > 0$ there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(\eta)$ such that the following statements hold

a) $J_\varepsilon > -\infty$ for any $0 < \varepsilon < \tilde{\varepsilon}$.

b) $\mid J_\varepsilon(\bar{u}_n^*(\varepsilon)) - J_\varepsilon \mid \leq \eta$.

Proof: We prove this theorem in two steps.
$i)$ We prove statement (a) by contradiction. If (a) is not true, then there exists a sequence $\{\varepsilon_p\}$, such that $\varepsilon_p \to 0$ as $p \to \infty$ and

$$J_{\varepsilon_p} \to -\infty. \tag{14}$$

According to the definition of infimum for each ε_p there exists a sequence of admissible controls $u_n^{p,m}$ such that $J_{\varepsilon_p}(u_n^{p,m}) \to -\infty$ as $m \to \infty$. The controls $u_n^{p,m}$ are admissible.
Hence there exist solutions $x_n^{p,m}$ and $y_n^{p,m}$ for systems (1) and (2) respectively. Note that

$$J_{\varepsilon_p}(u_n^{p,m}) = \Phi(x^{p,m}_{\left[\frac{T}{\varepsilon_p}\right]}).$$

Since an optimal control exists for system (2), $\bar{J}_{\varepsilon_p}(y^{p,m}) > \bar{J}_{\varepsilon_p} > -\infty$ there. Now, we fix some $0 < \eta_0 < \frac{\rho}{2}$. Thus, there exists natural number p_0 such that

$$\mid J_{\varepsilon_p}(u_n^{p,m}) - \ \bar{J}_{\varepsilon_p}(u_n^{p,m}) \mid = \mid \Phi(\ x^{p,m}_{\left[\frac{T}{\varepsilon_p}\right]}) - \Phi(\ y^{p,m}_{\left[\frac{T}{\varepsilon_p}\right]}) \mid \leq L \mid \ x^{p,m}_{\left[\frac{T}{\varepsilon_p}\right]} - y^{p,m}_{\left[\frac{T}{\varepsilon_p}\right]} \mid \leq L\eta_0.$$

for $\varepsilon_p < \varepsilon_{p_0}$.
Therefore

$$J_{\varepsilon_p}(u_n^{p,m}) = J_{\varepsilon_p}(u_n^{p,m}) + \bar{J}_{\varepsilon_p}(u_n^{p,m}) - \bar{J}_{\varepsilon_p}(u_n^{p,m}) > J_{\varepsilon_p}(u_n^{p,m}) - \bar{J}_{\varepsilon_p}(u_n^{p,m}) + \bar{J}_{\varepsilon_p} > \bar{J}_{\varepsilon_p} - L\eta_0.$$

This contradicts the hypothesis in (14).
$ii)$ Now let us prove statement (b). Note that

$$J_\varepsilon \leq J_\varepsilon(\bar{u}_n^*(\varepsilon)) = \bar{J}_\varepsilon + [J_\varepsilon(\bar{u}_n^*(\varepsilon)) - \bar{J}_\varepsilon(\bar{u}_n^*(\varepsilon))].$$

We estimate the difference

$$\mid J_\varepsilon(\bar{u}_n^*(\varepsilon)) - \bar{J}_\varepsilon(\bar{u}_n^*(\varepsilon)) \mid = \mid \Phi(\ x_{\left[\frac{T}{\varepsilon}\right]}(\bar{u}_n^*(\varepsilon))) - \Phi(\ y_{\left[\frac{T}{\varepsilon}\right]}(\bar{u}_n^*(\varepsilon))) \mid,$$

where $x_{\left[\frac{T}{\varepsilon}\right]}(\bar{u}_n^*(\varepsilon))$ is a solution of system (1) for the optimal control $\bar{u}_n^*(\varepsilon)$ of the averaged system, and $y_{\left[\frac{T}{\varepsilon}\right]}(\bar{u}_n^*(\varepsilon))$ is the optimal control of system (2).
Since the function Φ is Lipschitz, it follows that

$$\mid \Phi(\ x_{\left[\frac{T}{\varepsilon}\right]}(\bar{u}_n^*(\varepsilon))) - \Phi(\ y_{\left[\frac{T}{\varepsilon}\right]}(\bar{u}_n^*(\varepsilon))) \mid \leq L \mid \ x_{\left[\frac{T}{\varepsilon}\right]}(\bar{u}_n^*(\varepsilon)) - \ y_{\left[\frac{T}{\varepsilon}\right]}(\bar{u}_n^*(\varepsilon)) \mid .$$

Using Lemma 2.2, for an arbitrary $\eta_1 < \frac{\rho}{2}$ and all sufficiently small ε we obtain

$$J_\varepsilon \leq \bar{J}_\varepsilon + L\eta_1. \tag{15}$$

From the definition of infimum, for chosen $\eta_1 > 0$ there exists an admissible control $u_n^{\eta_1}(\varepsilon)$ such that

$$J_\varepsilon(u_n^{\eta_1}(\varepsilon)) \leq \bar{J}_\varepsilon + \eta_1.$$

Hence we obtain the estimate

$$\bar{J}_\varepsilon = \bar{J}_\varepsilon(\bar{u}_n^*(\varepsilon)) \leq \bar{J}_\varepsilon(u_n^{\eta_1}(\varepsilon)) \leq \bar{J}_\varepsilon(u_n^{\eta_1}(\varepsilon)) + J_\varepsilon + \eta_1 - J_\varepsilon(u_n^{\eta_1}(\varepsilon)).$$

Using Lipschitz condition for the function Φ we have

$$\mid \bar{J}_\varepsilon(u_n^{\eta_1}(\varepsilon)) - J_\varepsilon(u_n^{\eta_1}(\varepsilon)) \mid \leq L \mid \ y_{\left[\frac{T}{\varepsilon}\right]}(u_n^{\eta_1}(\varepsilon)) - \ x_{\left[\frac{T}{\varepsilon}\right]}(u_n^{\eta_1}(\varepsilon)) \mid \leq L\eta_1.$$

Thus

$$\bar{J}_\varepsilon \leq J_\varepsilon + \eta_1 + L\eta_1.$$

Hence, it follows from (15) that

$$\mid \bar{J}_\varepsilon - J_\varepsilon \mid \leq (L+1)\eta_1. \tag{16}$$

Now, we consider the difference

$$\mid J_\varepsilon(\bar{u}_n^*(\varepsilon)) - J_\varepsilon \mid = \mid J_\varepsilon(\bar{u}_n^*(\varepsilon)) - \bar{J}_\varepsilon + \bar{J}_\varepsilon - J_\varepsilon \mid \leq \mid J_\varepsilon(\bar{u}_n^*(\varepsilon)) - \bar{J}_\varepsilon \mid + \mid \bar{J}_\varepsilon - J_\varepsilon \mid .$$

It is easily seen from the optimality criterion that

$$\mid J_\varepsilon(\bar{u}_n^*(\varepsilon)) - \bar{J}_\varepsilon \mid \leq L\eta_1.$$

From the last estimate and inequality (16) we have

$$\mid J_\varepsilon(\bar{u}_n^*(\varepsilon)) - J_\varepsilon \mid \leq \eta,$$

where $\eta = \eta_1(2L+1)$. The theorem is proved.

□

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