



An Example of Self–Acceleration for Incompressible Flows

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ABSTRACT: In this paper we consider the Cauchy problem for the unforced Euler and Navier–Stokes equations for incompressible flows. We give an example of a smooth initial velocity field of finite energy which is self–accelerating, i.e., the maximal speed increases for some time. The self–acceleration is due to the non–Bernoulli part of the pressure generated by the velocity field.

Key Words: incompressible flows, Euler equation, Navier–Stokes equation, self–acceleration

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1. Introduction

Consider the Cauchy problem for the Euler or the Navier–Stokes ¹ equations:

$$v_t + v \cdot \nabla v + \nabla p = \nu \Delta v, \quad \nabla \cdot v = 0, \quad (1.1)$$

$$v(x, 0) = u(x), \quad x \in \mathbb{R}^3. \quad (1.2)$$

Here $\nu = 0$ for the Euler equation and $\nu > 0$ for the Navier–Stokes equation. We assume that the initial vector field, $v(x, 0) = u(x)$, is given by a smooth, bounded, divergence–free function $u(x)$ of finite energy. Then a smooth solution $v(x, t)$ is known to exist in some finite time interval $0 \leq t < T$. It is an open problem, for both the Euler and the Navier–Stokes equations, if blow–up can occur in finite time or if a smooth solution $v(x, t)$ always exists for $0 \leq t < \infty$.

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¹ We consider the equations in nondimensional form with density $\rho = 1$.

Denote the spatial maximum norm by ²

$$|v(\cdot, t)|_\infty = \sup_x |v(x, t)| .$$

We will use the following terminology:

Definition 1.1 The initial velocity field $u = u(x)$ in (1.2) is *self-accelerating* for equation (1.1) if there is a time $t_0 > 0$ so that

$$|v(\cdot, t)|_\infty > |u|_\infty \quad \text{for } 0 < t \leq t_0 . \quad (1.3)$$

We use the term *self-accelerating* since no forcing terms or inhomogeneous boundary conditions are present in the problem. As we will see in the next section, self-acceleration at time $t = 0$, if it occurs, is due to the pressure term ∇p in (1.1). Here $p = p(x) = p(x, 0)$ is determined in terms of u by the Poisson equation ³

$$-\Delta p = \sum_{i,j=1}^3 D_i D_j (u_i u_j), \quad p(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty . \quad (1.4)$$

Interest in self-accelerating flows is motivated by the (open) blow-up problem for the Navier–Stokes equation. ⁴

For the Navier–Stokes equations, it is known (see, for example, [1] or [2]) that if the maximal interval of existence of a smooth solution $v(x, t)$ is $0 \leq t < t^*$ then

$$\lim_{t \rightarrow t^* -} |v(\cdot, t)|_\infty = \infty .$$

Therefore, if blow-up occurs, self-acceleration must occur at some time t less than t^* . What properties do self-accelerating flow-fields have? In Section 2 we give some simple properties.

We could not find any example of a self-accelerating flow field with finite energy in the literature. In Section 3 we give an explicit example for the Euler equation. A simple scaling argument then shows how one can obtain an example for the Navier–Stokes equation.

2. The Acceleration Term

Without loss of generality, we will assume that the maximal speed at time $t = 0$ occurs at the point $x = 0$:

² With $\langle a, b \rangle = \sum_{j=1}^3 a_j b_j$ and $|a| = \langle a, a \rangle^{1/2}$ we denote the Euclidean inner product and norm on $\mathbb{R}^3$.

³ We denote $D_j = \partial/\partial x_j$.

⁴ This blow-up problem is one of the Millennium Problems chosen by the Clay Mathematics Institute. The official description by Charles Fefferman can be found at www.claymath.org/millennium.

$$|u(0)| = |u|_\infty .$$

Then, since the function $\phi(x) = \frac{1}{2}|u(x)|^2$ is maximal at $x = 0$, we have

$$0 = D_j \phi(0) = \sum_{i=1}^3 \left(u_i D_j u_i \right) (0) \quad (2.5)$$

and

$$0 \geq D_j^2 \phi(0) = \sum_{i=1}^3 \left(u_i D_j^2 u_i \right) (0) + \sum_{i=1}^3 (D_j u_i)^2(0) \quad (2.6)$$

for $j = 1, 2, 3$. The vector field $u(x)$ is self-accelerating if

$$A := \frac{1}{2} \frac{d}{dt} |v(0, t)|^2 \Big|_{t=0} > 0 . \quad (2.7)$$

We have, at $x = 0, t = 0$:

$$\begin{aligned} A &= \langle v, v_t \rangle \\ &= -\langle v, v \cdot \nabla v \rangle - \langle v, \nabla p \rangle + \nu \langle v, \Delta v \rangle \\ &=: C + P + \nu V \end{aligned} \quad (2.8)$$

Here the convection term is (at $x = 0, t = 0$):

$$C = - \sum_{i,j} v_j v_i D_j v_i = 0$$

because of (2.5). The viscous term is (at $x = 0, t = 0$):

$$\begin{aligned} V &= \sum_{i,j} v_i D_j^2 v_i \\ &= \sum_j D_j^2 \phi(0) - \sum_{i,j} (D_j u_i)^2(0) \\ &\leq 0 \end{aligned}$$

because of (2.6). Therefore, to obtain a vectorfield $u(x)$ with $A > 0$ (see (2.7)) it is necessary that the pressure term P in (2.8) satisfies

$$0 < P = -\langle u(0), \nabla p(0) \rangle . \quad (2.9)$$

The condition (2.9) is also sufficient for self-acceleration if $\nu = 0$, i.e., for the Euler equation. In the next section we will give an example of a velocity field $u(x)$ which has maximal speed at the origin and satisfies (2.9). Thus, this vector field is self-accelerating for the Euler equation.

3. Example of a Self-Accelerating Velocity Field

In this section we use the notation

$$\mathbf{x} = (x, y, z)$$

for the space variable. Define the velocity field $u := \nabla \times \psi$ where

$$\psi(x, y, z) = e^{-(x^2+y^2+z^2)} \begin{pmatrix} 4y \\ -x \\ xy \end{pmatrix}.$$

It is clear that u is divergence free and u as well as all its derivatives lie in L_2 .

Elementary calculations show that

$$u(x, y, z) = e^{-(x^2+y^2+z^2)} \begin{pmatrix} x - 2xy^2 - 2xz \\ -y - 8yz + 2x^2y \\ -5 + 2x^2 + 8y^2 \end{pmatrix} \quad (3.10)$$

which yields

$$u(0) = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}.$$

Furthermore, it is elementary to check that the function

$$\phi(x, y, z) = \frac{1}{2}|u(x, y, z)|^2$$

has a gradient which vanishes at the origin and has a negative definite Hessian at the origin:

$$H = \phi''(0) = \begin{pmatrix} -138 & 0 & 0 \\ 0 & -258 & 0 \\ 0 & 0 & -100 \end{pmatrix}.$$

This implies that the speed $|u(x, y, z)|$ has a local maximum at the origin. A field plot of the velocity is shown in figure 1.

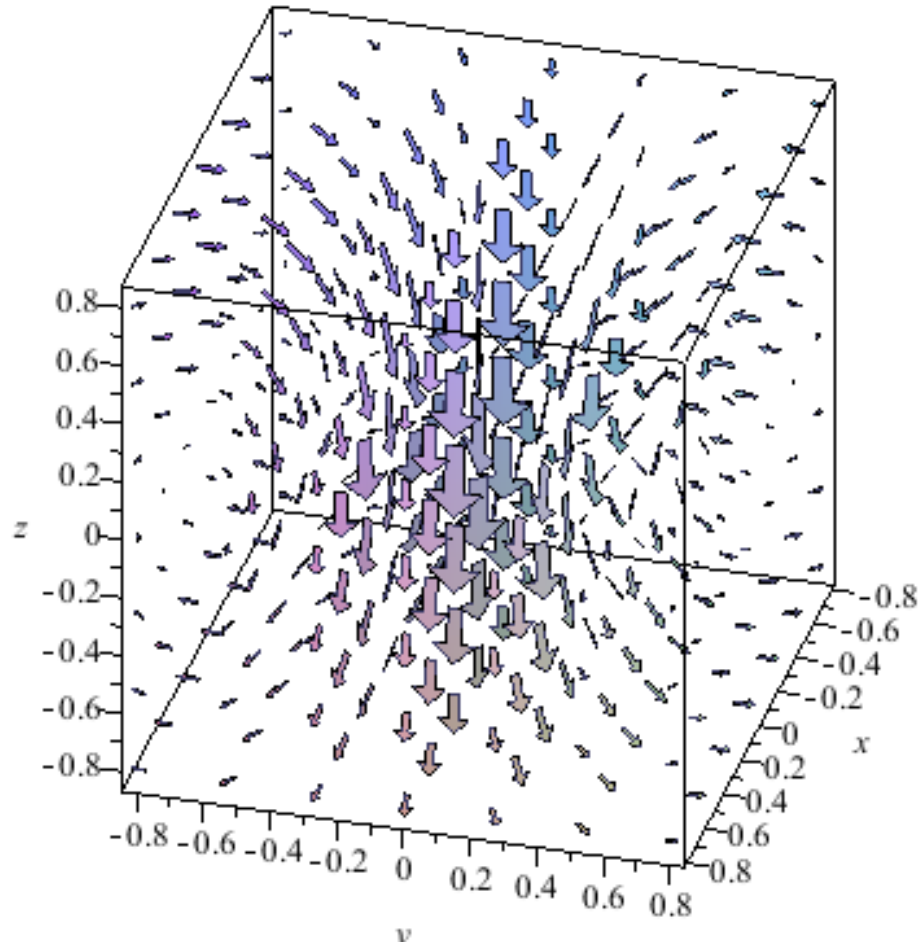


Figure 1: A Self-Accelerating Velocity Field

In fact, the local maximum is a global maximum as can be verified by numerical computations. See Figure 2.

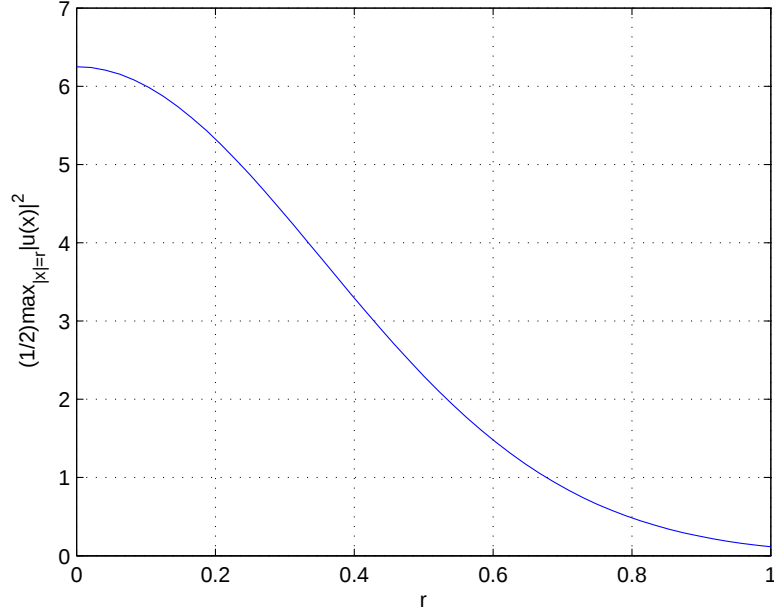


Figure 2: The function $\max_{|x|=r} \frac{1}{2} |u(x)|^2$

For the above example, the pressure term P (see (2.9)) becomes

$$P = 5 \frac{\partial}{\partial z} p(x, y, z)|_{x=y=z=0}$$

where the pressure p is determined by the Poisson equation (1.4).

Let $Q(x, y, z)$ denote the right-hand side of (1.4). Then we have

$$\begin{aligned}
 P &= 5 \frac{\partial}{\partial z} p(x, y, z)|_{x=y=z=0} \\
 &= \frac{5}{4\pi} \int_{\mathbb{R}^3} \frac{\frac{\partial}{\partial z} Q(x, y, z)}{(x^2 + y^2 + z^2)^{1/2}} dx dy dz \\
 &= -\frac{5}{4\pi} \int_{\mathbb{R}^3} \frac{z Q(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} dx dy dz \\
 &=: J
 \end{aligned}$$

Elementary computations, for which we applied Maple, show that the right-hand side $Q = \sum D_i D_j (u_i u_j)$ for the above example reads

$$Q(x, y, z) = q(x, y, z) e^{-2(x^2 + y^2 + z^2)}$$

where q is the polynomial

$$\begin{aligned} q(x, y, z) = & 2 \left(-48x^2y^2z + 6z + 24y^2z - 36x^2z \right. \\ & - 32x^2 - 212y^2 + 1 + 8x^2z^2 + 44x^2y^2 + 128y^2z^2 \\ & \left. + 12x^4 + 132y^4 + 84z^2 + 8x^2y^4 + 8x^4y^2 \right) \end{aligned}$$

The first four terms in the formula for $q(x, y, z)$ are antisymmetric in z whereas the remaining terms are symmetric in z . Therefore, noting the factor z in the integrand of the integral J , only the first four terms of $q(x, y, z)$ lead to a non-zero contribution in J .

One obtains

$$J = J_1 + J_2 + J_3 + J_4$$

where (see the Appendix)

$$\begin{aligned} J_1 &= -\frac{5 \cdot 96}{4\pi} \int_{\mathbb{R}^3} \frac{x^2 y^2 z^2 e^{-2(x^2+y^2+z^2)}}{(x^2 + y^2 + z^2)^{3/2}} dx dy dz \\ &= -\frac{5 \cdot 96}{4\pi} \int_0^\infty r^5 e^{-2r^2} dr \int_0^\pi \cos^2 \theta \sin^5 \theta d\theta \int_0^{2\pi} \sin^2 \phi \cos^2 \phi d\phi \\ &= -\frac{4}{7} \end{aligned}$$

Similarly,

$$\begin{aligned} J_2 &= \frac{5 \cdot 12}{4\pi} \int_{\mathbb{R}^3} \frac{z^2 e^{-2(x^2+y^2+z^2)}}{(x^2 + y^2 + z^2)^{3/2}} dx dy dz \\ &= 5 \\ J_3 &= \frac{5 \cdot 48}{4\pi} \int_{\mathbb{R}^3} \frac{y^2 z^2 e^{-2(x^2+y^2+z^2)}}{(x^2 + y^2 + z^2)^{3/2}} dx dy dz \\ &= 2 \\ J_4 &= -\frac{5 \cdot 72}{4\pi} \int_{\mathbb{R}^3} \frac{x^2 z^2 e^{-2(x^2+y^2+z^2)}}{(x^2 + y^2 + z^2)^{3/2}} dx dy dz \\ &= -3 \end{aligned}$$

Therefore,

$$J = -\frac{4}{7} + 5 + 2 - 3 = \frac{24}{7} > 0 .$$

Lemma 3.1. *Let u denote the velocity field (3.10). Then its speed $|u(\mathbf{x})|$ is maximal at $\mathbf{x} = 0$ and if p is the pressure corresponding to u (see (1.4)) then*

$$P = -\langle u(0), \nabla p(0) \rangle = \frac{24}{7} > 0 .$$

Therefore, u is self-accelerating for the Euler equation.

Scaling argument for Navier–Stokes. Let $\lambda > 0$ denote a scaling parameter and let u denote the velocity field (3.10). We set

$$u_\lambda(\mathbf{x}) = \lambda u(\mathbf{x}) .$$

Then the pressure determined by u_λ is $p_\lambda = \lambda^2 p$ and the term corresponding to P (see (2.9)) is $P_\lambda = \lambda^3 P$. If

$$V_\lambda = \langle u_\lambda(0), \Delta u_\lambda(0) \rangle$$

denotes the viscous term corresponding to u_λ (see (2.8)) then $V_\lambda = \lambda^2 V$. Therefore, the acceleration term (see (2.8)) corresponding to u_λ is

$$A_\lambda = \lambda^3 P + \nu \lambda^2 V .$$

Since $P > 0$ it follows that $A_\lambda > 0$ for sufficiently large λ .

Theorem 3.1. *Let u denote the velocity field (3.10) and let $u_\lambda(\mathbf{x}) = \lambda u(\mathbf{x})$. If $\lambda > 0$ is large enough, then u_λ is self-accelerating for the Navier–Stokes equation.*

4. Remarks on the Pressure Equation

If p is the pressure in the Navier–Stokes equation then

$$\begin{aligned} -\Delta p &= \nabla \cdot (u \cdot \nabla u) \\ &= \sum (D_i u_j)(D_j u_i) \\ &= \sum D_i D_j (u_i u_j) \\ &=: Q \end{aligned}$$

Using the vector identity

$$\nabla(F \cdot G) = F \cdot \nabla G + G \cdot \nabla F + F \times (\nabla \times G) + G \times (\nabla \times F)$$

with $F = G = u$ and taking the divergence yields

$$\nabla \cdot (u \cdot \nabla u) = \frac{1}{2} \Delta(|u|^2) + \nabla \cdot (\omega \times u) .$$

Therefore,

$$-\Delta p = Q = \frac{1}{2} \Delta(|u|^2) + \nabla \cdot (\omega \times u) ,$$

i.e.,

$$-\Delta(p + \frac{1}{2}|u|^2) = \nabla \cdot (\omega \times u) .$$

If one defines $q(x)$ by

$$-\Delta q = \nabla \cdot (\omega \times u), \quad q(x) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty ,$$

then

$$p = -\frac{1}{2}|u|^2 + q .$$

One may call

$$p_B = -\frac{1}{2}|u|^2$$

the Bernoulli-part of p and

$$q = p_{nB}$$

the non-Bernoulli-part of p . (Note that we have nondimensionalized the Navier-Stokes equations and assumed the density to be one. In dimensional variables one has $p_B = -\frac{\rho}{2}|u|^2$ where ρ is density.)

The Bernoulli-part $p_B = -\frac{1}{2}|u|^2$ of p is irrelevant for determining the crucial quantity P (see (2.9)): If the speed $|u(x)|$ has a maximum at any point $x = x_0$, then the gradient of p_B is zero at $x = x_0$.

For the non-Bernoulli part $q = p_{nB}$ we have

$$\begin{aligned} -\Delta q &= \nabla \cdot (\omega \times u) \\ &= u \cdot (\nabla \times \omega) - |\omega|^2 \end{aligned}$$

showing the importance of vorticity for self-acceleration.

5. Appendix: Some Elementary Integrals

The evaluation of $\nabla p(0)$ leads to integrals of the form

$$J_{klm} = \int_{\mathbb{R}^3} x^{2k} y^{2l} z^{2m} r^{-3} e^{-2r^2} dx dy dz$$

where

$$r^2 = x^2 + y^2 + z^2$$

and where k, l, m are nonnegative integers with

$$N := k + l + m \geq 1 .$$

In spherical coordinates:

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \\ dV &= r^2 \sin \theta \, dr d\theta d\phi \end{aligned}$$

Therefore,

$$J_{klm} = \int_0^\infty r^{2N-1} e^{-r^2} \, dr \cdot \int_0^\pi \sin^{2k+2l+1} \theta \cos^{2m} \theta \, d\theta \cdot \int_0^{2\pi} \cos^{2k} \phi \sin^{2l} \phi \, d\phi .$$

Here

$$\begin{aligned} J_r(N) &= \int_0^\infty r^{2N-1} e^{-r^2} \, dr \\ &= \frac{1}{4} \int_0^\infty r^{2N-2} (4r e^{-2r^2}) \, dr \quad (2r^2 = s, 4r dr = ds) \\ &= \frac{1}{4} \int_0^\infty (s/2)^{N-1} e^{-s} \, ds \\ &= \frac{\Gamma(N)}{2^{N+1}} \end{aligned}$$

In particular,

$$J_r(1) = \frac{1}{4}, \quad J_r(2) = J_r(3) = \frac{1}{8} .$$

The angular integrals lead to Euler's Beta-function,

$$\begin{aligned} B(\alpha, \beta) &= 2 \int_0^{\pi/2} \sin^{2\alpha-1} s \cos^{2\beta-1} s \, ds \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \end{aligned}$$

One obtains

$$\begin{aligned} J_\theta(k, l, m) &= \int_0^\pi \sin^{2k+2l+1} \theta \cos^{2m} \theta \, d\theta \\ &= 2 \int_0^{\pi/2} \sin^{2k+2l+1} \theta \cos^{2m} \theta \, d\theta \\ &= B(k + l + 1, m + \frac{1}{2}) \end{aligned}$$

and

$$\begin{aligned}
 J_\phi(k, l) &= \int_0^{2\pi} \sin^{2l} \phi \cos^{2k} \phi \, d\phi \\
 &= 4 \int_0^{\pi/2} \sin^{2l} \phi \cos^{2k} \phi \, d\phi \\
 &= 2B(l + \frac{1}{2}, k + \frac{1}{2})
 \end{aligned}$$

Therefore,

$$J_{klm} = \frac{\Gamma(N)}{2^N} \cdot \frac{\Gamma(k + \frac{1}{2})\Gamma(l + \frac{1}{2})\Gamma(m + \frac{1}{2})}{\Gamma(N + \frac{3}{2})} \quad \text{where } N = k + l + m.$$

In particular,

$$\begin{aligned}
 J_{001} &= \frac{1}{2} \cdot \frac{\sqrt{\pi}\sqrt{\pi}\Gamma(\frac{3}{2})}{\Gamma(1 + \frac{3}{2})} \\
 &= \frac{\pi}{2} \cdot \frac{2}{3} \\
 &= \frac{\pi}{3} \\
 J_{011} &= \frac{1}{4} \cdot \frac{\sqrt{\pi}\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(2 + \frac{3}{2})} \\
 &= \frac{\pi}{30} \\
 J_{111} &= \frac{2}{8} \cdot \frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3 + \frac{3}{2})} \\
 &= \frac{\pi}{210}
 \end{aligned}$$

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