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Integral Equations of Biharmonic Constant $\mathbf{\Pi}_1-$ Slope Curves according to Type-2 Bishop Frame in the Sol Space

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abstract: In this paper, we study biharmonic constant $\mathbf{\Pi}_1-$ slope curves according to type-2 Bishop frame in the $\mathfrak{SOL}^3$. We characterize the biharmonic constant $\mathbf{\Pi}_1-$ slope curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric integral equations in the $\mathfrak{SOL}^3$.

Key Words: type-2 Bishop frame, Sol Space, Curvatures.

Contents

1. Introduction

In the last decade there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations (see [8]), because biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE.

In this paper, we study biharmonic constant $\mathbf{\Pi}_1-$ slope curves according to type-2 Bishop in the $\mathfrak{SOL}^3$. We characterize the biharmonic curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric integral equations in the $\mathfrak{SOL}^3$.

2. Preliminaries

Sol space, one of Thurston's eight 3-dimensional geometries with group structure

$$(x, y, z)\cdot(\tilde{x}, \tilde{y}, \tilde{z}) = \left(x + e^{z}\tilde{x}, y + e^{-z}\tilde{y}, z + \tilde{z}\right)$$

and left invariant metric

$$g_{\mathfrak{SOL}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

2000 *Mathematics Subject Classification:* 53A04, 53A10 where (x, y, z) are the standard coordinates in $\mathbb{R}^3$.

The space Sol is realized as the following solvable matrix Lie group, [7]:

$$\mathfrak{SOL}^3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & z \\ 0 & e^z & 0 & x \\ 0 & 0 & e^{-z} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{sol}^3$ is given explicitly by

$$\mathfrak{sol}^3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & c & 0 & a \\ 0 & 0 & -c & b \\ 0 & 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Then we can take the following orthonormal basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ of $\mathfrak{sol}^3$

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{E}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{E}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Left-translating the basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, we obtain the following orthonormal frame field:

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.1}$$

Note that the Sol metric can also be written as:

$$g_{\mathfrak{SOL}^3} = \sum_{i=1}^{3} \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^i,$$

where

$$\boldsymbol{\omega}^1 = e^z dx, \quad \boldsymbol{\omega}^2 = e^{-z} dy, \quad \boldsymbol{\omega}^3 = dz.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\mathfrak{SOL}^3}$, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1,\mathbf{e}_2]=0,\quad [\mathbf{e}_2,\mathbf{e}_3]=-\mathbf{e}_2,\quad [\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_1.$$

The isometry group of $\mathfrak{SOL}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned}(x,y,z) &\rightarrow (x+c,y,z),\\ (x,y,z) &\rightarrow (x,y+c,z),\\ (x,y,z) &\rightarrow \left(e^{-c}x,e^{c}y,z+c\right).\end{aligned}$$

3. Biharmonic Constant $\mathbf{\Pi}_1-$Slope Curves according to New Type-2 Bishop Frame in Sol Space $\mathfrak{SOL}^3$

Assume that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T}+\tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N},\end{aligned}\tag{3.1}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned}g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{T}) &= 1,\ g_{\mathfrak{SOL}^3}(\mathbf{N},\mathbf{N})=1,\ g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{B})=1,\\ g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{N}) &= g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{B})=g_{\mathfrak{SOL}^3}(\mathbf{N},\mathbf{B})=0.\end{aligned}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1+k_2\mathbf{M}_2,\\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T},\\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T},\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{T}) &= 1,\ g_{\mathfrak{SOL}^3}(\mathbf{M}_1,\mathbf{M}_1)=1,\ g_{\mathfrak{SOL}^3}(\mathbf{M}_2,\mathbf{M}_2)=1,\\ g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{M}_1) &= g_{\mathfrak{SOL}^3}(\mathbf{T},\mathbf{M}_2)=g_{\mathfrak{SOL}^3}(\mathbf{M}_1,\mathbf{M}_2)=0.\end{aligned}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\mathfrak{U}(s)=\arctan\frac{k_2}{k_1}$, $\tau(s)=\mathfrak{U}'(s)$ and $\kappa(s)=\sqrt{k_1^2+k_2^2}$.

Bishop curvatures are defined by

$$\begin{aligned}k_1 &= \kappa(s)\cos\mathfrak{U}(s),\\ k_2 &= \kappa(s)\sin\mathfrak{U}(s).\end{aligned}$$

Theorem 3.1. *$\gamma : I \longrightarrow \mathfrak{SOL}^3$ is a biharmonic curve according to Bishop frame if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= \text{constant} \neq 0, \\ k_1'' - \left[k_1^2 + k_2^2\right] k_1 &= -k_1 \left[2M_2^3 - 1\right] - 2k_2 M_1^3 M_2^3, \\ k_2'' - \left[k_1^2 + k_2^2\right] k_2 &= 2k_1 M_1^3 M_2^3 - k_2 \left[2M_1^3 - 1\right]. \end{aligned} \tag{3.3}$$

Let γ be a unit speed regular curve in $\mathfrak{SOL}^3$ and (3.1) be its Frenet–Serret frame. Let us express a relatively parallel adapted frame:

$$\begin{aligned} \nabla_{\mathbf{T}} \mathbf{\Pi}_1 &= -\epsilon_1 \mathbf{B}, \\ \nabla_{\mathbf{T}} \mathbf{\Pi}_2 &= -\epsilon_2 \mathbf{B}, \\ \nabla_{\mathbf{T}} \mathbf{B} &= \epsilon_1 \mathbf{\Pi}_1 + \epsilon_2 \mathbf{\Pi}_2, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{B}) &= 1,\ g_{\mathfrak{SOL}^3}(\mathbf{\Pi}_1,\mathbf{\Pi}_1) = 1,\ g_{\mathfrak{SOL}^3}(\mathbf{\Pi}_2,\mathbf{\Pi}_2) = 1, \\ g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{\Pi}_1) &= g_{\mathfrak{SOL}^3}(\mathbf{B},\mathbf{\Pi}_2) = g_{\mathfrak{SOL}^3}(\mathbf{\Pi}_1,\mathbf{\Pi}_2) = 0. \end{aligned}$$

We shall call this frame as Type-2 Bishop Frame. In order to investigate this new frame's relation with Frenet–Serret frame, first we write

$$\tau = \sqrt{\epsilon_1^2 + \epsilon_2^2}. \tag{3.5}$$

The relation matrix between Frenet–Serret and type-2 Bishop frames can be expressed

$$\begin{aligned} &\mathbf{T} = \sin \mathfrak{A}(s)\, \mathbf{\Pi}_1 - \cos \mathfrak{A}(s)\, \mathbf{\Pi}_2, \\ &\mathbf{N} = \cos \mathfrak{A}(s)\, \mathbf{\Pi}_1 + \sin \mathfrak{A}(s)\, \mathbf{\Pi}_2, \\ &\mathbf{B} = \mathbf{B}. \end{aligned}$$

So by (3.5), we may express

$$\begin{aligned} \epsilon_1 &= -\tau \cos \mathfrak{A}(s), \\ \epsilon_2 &= -\tau \sin \mathfrak{A}(s). \end{aligned}$$

By this way, we conclude

$$\mathfrak{A}(s) = \arctan \frac{\epsilon_2}{\epsilon_1}.$$

The frame $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}\}$ is properly oriented, and τ and $\mathfrak{A}(s) = \int_0^s \kappa(s) ds$ are polar coordinates for the curve γ. We shall call the set $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}, \boldsymbol{\epsilon}_1, \epsilon_2\}$ as type-2 Bishop invariants of the curve γ, [19].

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{\Pi}_1 &= \pi_1^1\mathbf{e}_1 + \pi_1^2\mathbf{e}_2 + \pi_1^3\mathbf{e}_3, \\ \mathbf{\Pi}_2 &= \pi_2^1\mathbf{e}_1 + \pi_2^2\mathbf{e}_2 + \pi_2^3\mathbf{e}_3. \\ \mathbf{B} &= B^1 e_1 + B^2 e_2 + B^3 e_3, \end{aligned} \tag{3.6}$$

Theorem 3.2. *Let $\gamma : I \longrightarrow \mathfrak{SOL}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_1-$ slope curves according to type-2 Bishop frame in the $\mathfrak{SOL}^3$. Then, the parametric equations of γ are*

$$\begin{aligned} \boldsymbol{x}(s) &= \int e^{\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E} - \frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E} - \mathcal{R}_3}[\sin[\kappa s]\sin\mathfrak{E}\cos[\mathcal{R}_1 s + \mathcal{R}_2] \\ &\quad - \cos[\kappa s]\cos\mathfrak{E}\cos[\mathcal{R}_1 s + \mathcal{R}_2]]ds, \\ \boldsymbol{y}(s) &= \int e^{-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E} + \frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E} + \mathcal{R}_3}[\sin[\kappa s]\sin\mathfrak{E}\sin[\mathcal{R}_1 s + \mathcal{R}_2] \\ &\quad - \cos[\kappa s]\cos\mathfrak{E}\sin[\mathcal{R}_1 s + \mathcal{R}_2]]ds, \\ \boldsymbol{z}(s) &= -\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E} + \frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E} + \mathcal{R}_3, \end{aligned} \tag{3.7}$$

where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are constants of integration.

Proof: We suppose that γ is a unit speed non-geodesic biharmonic $\mathbf{\Pi}_1-$slope curve. Since

$$g(\mathbf{\Pi}_1, \mathbf{e}_3) = \pi_1^3 = \cos\mathfrak{E}, \tag{3.8}$$

where $\mathfrak{E}$ is constant angle.

On the other hand, the vector $\mathbf{\Pi}_1$ is a unit vector, we have the following equation

$$\mathbf{\Pi}_1 = \sin\mathfrak{E}\cos[\mathcal{R}_1 s + \mathcal{R}_2]\,\mathbf{e}_1 + \sin\mathfrak{E}\sin[\mathcal{R}_1 s + \mathcal{R}_2]\,\mathbf{e}_2 + \cos\mathfrak{E}\mathbf{e}_3, \tag{3.9}$$

where $\mathcal{R}_1, \mathcal{R}_2 \in \mathbb{R}$.

Then by type-2 Bishop formulas (3.4) and (2.1), we have

$$\mathbf{\Pi}_2 = \cos\mathfrak{E}\cos[\mathcal{R}_1 s + \mathcal{R}_2]\,\mathbf{e}_1 + \cos\mathfrak{E}\sin[\mathcal{R}_1 s + \mathcal{R}_2]\,\mathbf{e}_2 - \sin\mathfrak{E}\mathbf{e}_3. \tag{3.10}$$

Applying above equation and (3.9), we get

$$\mathbf{B} = -\sin[\mathcal{R}_1 s + \mathcal{R}_2]\,\mathbf{e}_1 + \cos[\mathcal{R}_1 s + \mathcal{R}_2]\,\mathbf{e}_2. \tag{3.11}$$

Also, we obtain

$$\begin{aligned} \mathbf{T} = \; & [\sin[\kappa s]\sin\mathfrak{E}\cos[\mathcal{R}_1 s + \mathcal{R}_2] - \cos[\kappa s]\cos\mathfrak{E}\cos[\mathcal{R}_1 s + \mathcal{R}_2]]\mathbf{e}_1 \\ & + [\sin[\kappa s]\sin\mathfrak{E}\sin[\mathcal{R}_1 s + \mathcal{R}_2] - \cos[\kappa s]\cos\mathfrak{E}\sin[\mathcal{R}_1 s + \mathcal{R}_2]]\mathbf{e}_2 \\ & + [\sin[\kappa s]\cos\mathfrak{E} + \cos[\kappa s]\sin\mathfrak{E}]\mathbf{e}_3. \end{aligned}$$

Therefore,

$$\frac{dz}{ds} = \sin[\kappa s]\cos\mathfrak{E} + \cos[\kappa s]\sin\mathfrak{E}.$$

Integrating both sides, we have

$$z(s) = -\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E} + \frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E} + \mathcal{R}_3,$$

where $\mathcal{R}_3$ is constant of integration. This proves our assertion. Thus, the proof of theorem is completed. □

Corollary 3.3. *Let $\gamma : I \longrightarrow \mathfrak{SOL}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_1-$ slope curves according to type-2 Bishop frame in the $\mathfrak{SOL}^3$. Then, the position vector of γ is*

$$\begin{aligned}
\gamma(s) \quad = \quad & [e^{-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3}\int e^{\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}-\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}-\mathcal{R}_3} \\
& [\sin[\kappa s]\sin\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]-\cos[\kappa s]\cos\mathfrak{E}\cos[\mathcal{R}_1 s+\mathcal{R}_2]]ds]\mathbf{e}_1, \\
& +[e^{\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}-\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}-\mathcal{R}_3}\int e^{-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3} \\
& [\sin[\kappa s]\sin\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]-\cos[\kappa s]\cos\mathfrak{E}\sin[\mathcal{R}_1 s+\mathcal{R}_2]]ds]\mathbf{e}_2, \\
& +[-\frac{1}{\kappa}\cos[\kappa s]\cos\mathfrak{E}+\frac{1}{\kappa}\sin[\kappa s]\sin\mathfrak{E}+\mathcal{R}_3]\mathbf{e}_3,
\end{aligned}$$

where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are constants of integration.

We can use Mathematica in Theorem 3.2, yields

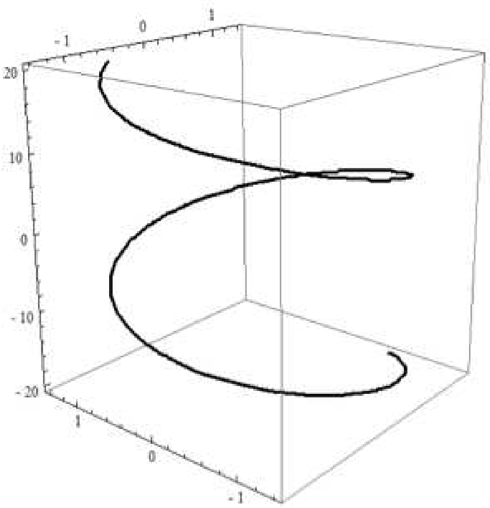

Fig. 1

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