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Biharmonic Curves according to Parallel Transport Frame in $\mathbb{E}^4$

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ABSTRACT: In this paper, we study biharmonic curves according to parallel transport frame in $\mathbb{E}^4$. We give some characterizations for curvatures of a biharmonic curve in $\mathbb{E}^4$.

Key Words: Heisenberg group, fluid flow, biharmonic curve, parallel transport frame.

Contents

1. Introduction

In the last decade there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations, because biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE.

In this paper, we study biharmonic curves according to parallel transport frame in $\mathbb{E}^4$. We give some characterizations for curvatures of a biharmonic curve in $\mathbb{E}^4$.

2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $\mathbb{E}^4$ are briefly presented.

The Euclidean 4-space $\mathbb{E}^4$ provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}^4$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}^4$ is given by $\|a\| = \sqrt{\langle a, a \rangle}$. γ is called a unit speed curve if velocity vector v of γ satisfies $\|a\| = 1$.

Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ denotes the moving Frenet frame of the unit speed curve γ, where $\mathbf{T}, \mathbf{N}, \mathbf{B}_1$ and $\mathbf{B}_2$ are the tangent, the principal normal, the first binormal and the second binormal vector fields, respectively. Then the Serret-Frenet formulae

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are given by binormal and the second binormal vector fields, respectively. Then the Serret-Frenet formulae are given by

$$\begin{aligned} \mathbf{T}' &= \kappa\mathbf{N} \\ \mathbf{N}' &= -\kappa\mathbf{T} + \tau\mathbf{B}_1 \\ \mathbf{B}_1' &= -\tau\mathbf{N}+\sigma\mathbf{B}_2 \\ \mathbf{B}_2' &= -\sigma\mathbf{B}_1, \end{aligned}$$

where κ, τ and σ are called the first, the second and the third curvatures of the curve γ, respectively, [6].

$$\langle \mathbf{T},\mathbf{T}\rangle = \langle \mathbf{N},\mathbf{N}\rangle = \langle \mathbf{B}_1,\mathbf{B}_1\rangle = \langle \mathbf{B}_2,\mathbf{B}_2\rangle = 1.$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame, [4]. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. The Bishop frame is expressed as

$$\begin{aligned} \mathbf{T}' &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2 + k_3\mathbf{M}_3, \\ \mathbf{M}_1' &= -k_1\mathbf{T}, \\ \mathbf{M}_2' &= -k_2\mathbf{T}, \\ \mathbf{M}_3' &= -k_3\mathbf{T}, \end{aligned} \tag{2.1}$$

where k_1, k_2, k_3 are principal curvature functions acoording to parallel transport frame of the curve γ and their expression as follows:

$$\begin{aligned} k_1 &= \kappa\cos\theta\cos\psi, \\ k_2 &= \kappa\left(-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi\right), \\ k_3 &= \kappa\left(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi\right), \end{aligned}$$

and

$$\begin{gathered} \kappa = \sqrt{k_1^2 + k_2^2 + k_3^2}, \\ \tau = -\psi' + \phi'\sin\theta, \\ \sigma = \frac{\theta'}{\sin\psi}, \\ \phi'\cos\theta + \theta'\cot\psi = 0, \end{gathered}$$

where

$$\theta' = \frac{\sigma}{\sqrt{\kappa^2+\tau^2}}, \psi' = -(\tau + \sigma\frac{\sqrt{\sigma^2 - \left(\theta'\right)^2}}{\sqrt{\kappa^2+\tau^2}}),\ \phi' = \frac{\sqrt{\sigma^2 - \left(\theta'\right)^2}}{\cos\theta}$$

and κ, τ, σ are the principal curvature functions according to Frenet frame of the curve γ, [3].

We shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$ as parallel transport frame of γ.

3. Biharmonic curves according to parallel transport frame in $\mathbb{E}^4$

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R\left(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T}\right)\mathbf{T} = 0, \tag{3.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (3.1).

Theorem 3.1. $\gamma : I \longrightarrow \mathbb{E}^4$ *is a unit speed biharmonic curve if and only if*

$$\begin{aligned} k_1^2 + k_2^2 + k_3^2 &= \mathfrak{A}, \\ k_1'' - k_1\left(k_1^2 + k_2^2 + k_3^2\right) &= 0, \\ k_2'' - k_2\left(k_1^2 + k_2^2 + k_3^2\right) &= 0, \\ k_3'' - k_3\left(k_1^2 + k_2^2 + k_3^2\right) &= 0, \end{aligned} \tag{3.2}$$

where $\mathfrak{A}$ *is non-zero constant of integration.*

Proof: Using the Bishop equations (2.1) and biharmonic equation (3.1), we obtain

$$\begin{aligned} &(-3k_1'k_1 - 3k_2'k_2 - 3k_3'k_3)\mathbf{T} + (k_1'' - k_1\left(k_1^2 + k_2^2 + k_3^2\right))\mathbf{M}_1 \\ &+(k_2'' - k_2\left(k_1^2 + k_2^2 + k_3^2\right))\mathbf{M}_2 - R\left(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T}\right)\mathbf{T} = 0. \end{aligned} \tag{3.3}$$

In $\mathbb{E}^4$, the Riemannian curvature is zero, we have

$$\begin{aligned} &(-3k_1'k_1 - 3k_2'k_2 - 3k_3'k_3)\mathbf{T} + (k_1'' - k_1\left(k_1^2 + k_2^2 + k_3^2\right))\mathbf{M}_1 \\ &+(k_2'' - k_2\left(k_1^2 + k_2^2 + k_3^2\right))\mathbf{M}_2 = 0. \end{aligned} \tag{3.4}$$

By (3.4), we see that γ is a unit speed biharmonic curve if and only if

$$\begin{aligned} -3k_1'k_1 - 3k_2'k_2 - 3k_3'k_3 &= 0, \\ k_1'' - k_1\left(k_1^2 + k_2^2 + k_3^2\right) &= 0, \\ k_2'' - k_2\left(k_1^2 + k_2^2 + k_3^2\right) &= 0, \\ k_3'' - k_3\left(k_1^2 + k_2^2 + k_3^2\right) &= 0. \end{aligned} \tag{3.5}$$

These, together with (3.5), complete the proof of the theorem. □

Corollary 3.2. $\gamma : I \longrightarrow \mathbb{E}^4$ *is a unit speed biharmonic curve if and only if*

$$\begin{aligned} k_1^2 + k_2^2 + k_3^2 &= \mathfrak{A}, \\ k_1'' - k_1\mathfrak{A} &= 0, \\ k_2'' - k_2\mathfrak{A} &= 0, \\ k_3'' - k_3\mathfrak{A} &= 0, \end{aligned} \tag{3.6}$$

where $\mathfrak{A}$ *is constant of integration.*

Theorem 3.3. *Let $\gamma : I \longrightarrow \mathbb{E}^4$is a unit speed biharmonic curve, then*

$$\begin{aligned} k_1^2(s) + k_2^2(s) + k_3^2(s) &= \mathfrak{A}, \\ k_1(s) &= c_1 e^{\sqrt{\mathfrak{A}}s} + c_2 e^{-\sqrt{\mathfrak{A}}s}, \\ k_2(s) &= c_3 e^{\sqrt{\mathfrak{A}}s} + c_4 e^{-\sqrt{\mathfrak{A}}s}, \\ k_3(s) &= c_5 e^{\sqrt{\mathfrak{A}}s} + c_6 e^{-\sqrt{\mathfrak{A}}s}, \end{aligned} \tag{3.7}$$

where $\mathfrak{A}$, c_1, c_2, c_3, c_4, c_5, c_6 are constants of integration.

Proof: Using (3.6), we have (3.7). □

Corollary 3.4. *If $c_1 = c_3 = c_5$ and $c_2 = c_4 = c_6$, then*

$$k_1(s) = k_2(s) = k_3(s).$$

Corollary 3.5. *Let $\gamma : I \longrightarrow \mathbb{E}^4$ is a unit speed biharmonic curve, then*

$$\begin{aligned} c_1 e^{\sqrt{\mathfrak{A}}s} + c_2 e^{-\sqrt{\mathfrak{A}}s} &= \sqrt{\mathfrak{A}} \sin \Lambda \cos \Pi, \\ c_3 e^{\sqrt{\mathfrak{A}}s} + c_4 e^{-\sqrt{\mathfrak{A}}s} &= \sqrt{\mathfrak{A}} \sin \Lambda \sin \Pi, \\ c_5 e^{\sqrt{\mathfrak{A}}s} + c_6 e^{-\sqrt{\mathfrak{A}}s} &= \sqrt{\mathfrak{A}} \cos \Lambda, \end{aligned}$$

where $\Lambda \in [0, \pi]$, $\Pi \in [0, 2\pi)$ and $\mathfrak{A}$, c_1, c_2, c_3, c_4, c_5, c_6 are constants of integration.

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