



Heredity for triangular operators

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ABSTRACT: A proof is given that if the lower triangular infinite matrix T acts boundedly on ℓ^2 and U is the unilateral shift, the sequence $(U^*)^n T U^n$ inherits from T the following properties: posinormality, dominance, M -hyponormality, hyponormality, normality, compactness, and noncompactness. Also, it is demonstrated that the upper triangular matrix T^* is dominant if and only if T is a diagonal matrix.

Key Words: posinormal operator, dominant operator, compact operator, M -hyponormal operator, hyponormal operator, triangular matrix, terraced matrix

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1. Introduction

In this paper we extend the results of [2] to a much larger collection of operators, and we include more properties of those operators.

If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space H , then $A \in B(H)$ is said to be *posinormal* if $AA^* = A^*PA$ for some positive operator $P \in B(H)$, and A is *dominant* if

$$\text{Ran}(A - \lambda) \subset \text{Ran}(A - \lambda)^*$$

for all λ in the spectrum of A . In [1] it is shown that A is dominant if and only if $A - \lambda$ is posinormal for each complex number λ . The operator $A \in B(H)$ is *M -hyponormal* (see [5]) if there exists an $M > 0$ such that

$$\|(A - \lambda)^* f\|^2 \leq M \|(A - \lambda)f\|^2$$

for all complex numbers λ and all $f \in \ell^2$; if the inequality is satisfied for $M = 1$ and $\lambda = 0$, then A is *hyponormal*. Hyponormal and M -hyponormal operators are necessarily dominant.

We note that if $T = [t_{ij}]$ is a lower triangular infinite matrix and U is the unilateral shift, then U^*TU is the lower triangular infinite matrix that is obtained when the first row and first column are deleted from T . The lower triangular matrix T is *terraced* if its row segments are constant. In an earlier paper [2] it was shown that the hyponormality of a terraced matrix $T \in B(\ell^2)$ is inherited by U^*TU . Here, we will observe that because of a key technical lemma, a similar

result holds for lower triangular infinite matrices in general, and that an analogous result also holds for other properties: posinormality, dominance, M -hyponormality, normality, compactness, and noncompactness.

2. Main Results

The following lemma will play a key role in the proof of the main theorem.

Lemma 2.1. *Suppose that $T \in B(\ell^2)$ is a lower triangular infinite matrix and U is the unilateral shift. If Y is any infinite matrix that acts boundedly on ℓ^2 , then*

$$U^*(T^*Y)U = (U^*T^*U)(U^*YU).$$

Proof: First we calculate the entries of T^*Y , obtaining

$$[T^*Y]_{ij} = \sum_{k=i}^{\infty} \overline{t_{ki}} y_{kj}.$$

If we then delete the first row and first column from T^*Y , the entries of the resulting matrix $U^*(T^*Y)U$ agree with the corresponding entries of $(T^*)'Y'$ where $Y' = U^*YU$. $\square$

In contrast, since $t_{10}y_{01} + t_{11}y_{11} \neq t_{11}y_{11}$ when $t_{10}y_{01} \neq 0$, we have a different result for TY :

$$U^*(TY)U \neq (U^*TU)(U^*YU).$$

We are now ready for the main result.

Theorem 2.2. *Suppose that $T \in B(\ell^2)$ is a lower triangular infinite matrix and U is the unilateral shift. Then $T' := U^*TU$ inherits each of the following properties from T :*

- (a) *posinormality;*
- (b) *dominance;*
- (c) *M -hyponormality;*
- (d) *hyponormality;*
- (e) *normality;*
- (f) *compactness; and*
- (g) *noncompactness.*

Proof:

(a) Suppose T is posinormal. By [1, Theorem 2.1], $T = T^*B$ for some operator $B \in B(\ell^2)$. Now we apply Lemma 2.1 to obtain

$$T' = U^*TU = U^*(T^*B)U = (U^*T^*U)(U^*BU) = (T')^*(U^*BU);$$

by another application of [1, Theorem 2.1], T' is posinormal.

(b) Assume that T is dominant. This means that $T - \lambda$ is posinormal for all complex numbers λ , so $T - \lambda = (T - \lambda)^* B(\lambda)$ for some operator $B(\lambda) \in B(\ell^2)$. Since $T - \lambda$ is lower triangular, Lemma 2.1 once again applies:

$$\begin{aligned} T' - \lambda &= U^* T U - \lambda = U^* (T - \lambda) U = U^* ((T - \lambda)^* B(\lambda)) U \\ &= (U^* (T - \lambda)^* U) (U^* B(\lambda) U) = (T' - \lambda)^* (U^* B(\lambda) U), \end{aligned}$$

so again by [1, Theorem 2.1], $T' - \lambda$ is posinormal. This is true for all λ , so T' is dominant.

(c) Suppose that T is M -hyponormal, so for some $M > 0$ and for all $f \in \ell^2$,

$$M \langle (T - \lambda)^* (T - \lambda) f, f \rangle - \langle (T - \lambda) (T - \lambda)^* f, f \rangle \geq 0$$

for all complex numbers λ . Besides Lemma 2.1, we will use the fact that the unilateral shift U satisfies $I - U U^* \geq 0$. Then

$$\begin{aligned} &M \langle (T' - \lambda)^* (T' - \lambda) f, f \rangle - \langle (T' - \lambda) (T' - \lambda)^* f, f \rangle = \\ &M \langle (U^* (T - \lambda)^* U) (U^* (T - \lambda) U) f, f \rangle - \langle (U^* (T - \lambda) U) (U^* (T - \lambda) U)^* f, f \rangle = \\ &M \langle (U^* (T - \lambda)^* (T - \lambda) U) f, f \rangle - \langle (U^* (T - \lambda) U) (U^* (T - \lambda) U)^* f, f \rangle = \\ &M \langle (T - \lambda)^* (T - \lambda) U f, U f \rangle - \langle (T - \lambda) (T - \lambda)^* U f, U f \rangle \\ &\quad + \langle (T - \lambda) (T - \lambda)^* U f, U f \rangle - \langle (T - \lambda) U U^* (T - \lambda)^* U f, U f \rangle = \\ &M \langle (T - \lambda)^* (T - \lambda) U f, U f \rangle - \langle (T - \lambda) (T - \lambda)^* U f, U f \rangle \\ &\quad + \langle (I - U U^*) (T - \lambda)^* U f, (T - \lambda)^* U f \rangle \geq 0 \end{aligned}$$

for all f and all λ , and therefore T' is M -hyponormal.

(d) This proof involves only a minor modification of the preceding argument with $M = 1$ and $\lambda = 0$.

(e) If T is normal, then $T T^* = T^* T$, so it is not hard to show that T must be a diagonal matrix. Consequently, T' is also a diagonal matrix and thus also normal.

(f) Trivial.

(g) We prove the contrapositive. Assume that T' is compact, so $U T' U^*$ is also compact. We note that $T - U T' U^*$ has nonzero entries only in the first column. Since T is bounded, we must have $\sum_{i=0}^{\infty} |t_{i0}|^2 = \|T e_0\|^2 < \infty$, where e_0 belongs to the standard orthonormal basis for ℓ^2 ; consequently, $T - U T' U^*$ is a Hilbert-Schmidt operator on ℓ^2 and is therefore compact. Thus

$$T = U T' U^* + (T - U T' U^*)$$

is compact, since it is the sum of two compact operators. $\square$

If n is a positive integer, then $(U^*)^n T U^n$ is obtained by deleting the first n rows and the first n columns from T .

Corollary 2.3. *For each positive integer n , $(U^*)^n T U^n$ inherits from T any of the properties (a) through (g) held by T .*

It is not hard to construct an example showing that non-normality is not inherited from a triangular operator T , and that is left to the interested reader. Next we consider an example that helps settle other similar questions.

Example 2.4. Let T denote the terraced matrix with row segments given by the sequence $\{r_n\}$ with $r_0 = 0$ and $r_n = 1/(n+1)$ for $n \geq 1$. If $f = e_0 - e_1$, then $f \in \text{Ker}T$ but $f \notin \text{Ker}T^*$, so by [1, Corollary 2.3], T is not posinormal; consequently, T is also not dominant, not M -hyponormal, and not hyponormal. But the terraced matrix T' is hyponormal (see [1, Theorem 5.2] or [4]) and hence also M -hyponormal, dominant, and posinormal. So this example has shown that non-posinormality, non-dominance, non- M -hyponormality, and non-hyponormality are not inherited in general by triangular operators.

Finally, we turn now to the adjoint of a lower triangular operator T . The inheritance of compactness or noncompactness by $(T^*)'$ from the upper triangular matrix T^* is clear. On the other hand, the question of whether posinormality is inherited in general by $(T^*)'$ from T^* is unresolved, but the next result gives a nontrivial case in which posinormality is inherited.

Proposition 2.5. If T is the terraced matrix associated with a positive decreasing sequence $\{r_n\}$ that converges to 0 and such that $\{(n+1)r_n\}$ is an increasing sequence that converges to $L < \infty$, then T^* and $(T^*)'$ are both posinormal.

Proof: This result follows from [3, Corollary 3.1]. □

The next theorem shows that if $t_{ij} \neq 0$ for some $i > j$, then T^* cannot be dominant; consequently, T^* also cannot be hyponormal or M -hyponormal.

Theorem 2.6. If $T \in B(\ell^2)$ is a lower triangular infinite matrix and $t_{ij} \neq 0$ for some $i > j$, then T^* is not dominant.

Proof: Let j_0 designate the first column in which $t_{i_0j_0} \neq 0$ for some $i_0 > j_0$. We find that $e_{j_0} \in \text{Ker}(T - t_{j_0j_0})^*$ but $e_{j_0} \notin \text{Ker}(T - t_{j_0j_0})$. Thus $(T - t_{j_0j_0})^*$ is not posinormal, so T^* is not dominant. □

Thus we see that the adjoint T^* of a lower triangular infinite matrix is dominant if and only if T is a diagonal matrix; in that case, $(T^*)'$ trivially inherits posinormality, dominance, M -hyponormality, hyponormality, and normality from T^* .

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