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Existence of solutions for a fourth order problem at resonance

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abstract: In this work, we are interested at the existence of nontrivial solutions of two fourth order problems governed by the weighted p-biharmonic operator. The first is the following

$$\Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda_1 m(x)|u|^{p-2}u + f(x,u) - h \text{ in } \Omega,\ u = \Delta u = 0 \text{ on } \partial\Omega,$$

where λ_1 is the first eigenvalue for the eigenvalue problem $\Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda m(x)|u|^{p-2}u$ in Ω, $u = \Delta u = 0$ on $\partial\Omega$. In the seconde problem, we replace λ_1 by λ such that $\lambda_1 < \lambda < \bar{\lambda}$, where $\bar{\lambda}$ is given bellow.

Key Words: p-biharmonic, weight, resonance, saddle point theorem.

Contents

1. Introduction and main results

In the present paper, we are concerned with the existence of weak solutions of the following problem

$$\begin{cases} \Delta(\rho|\Delta u|^{p-2}\Delta u) &= \lambda_1 m(x)|u|^{p-2}u + f(x,u) - h & \text{in} & \Omega \\ u &= \Delta u = 0 & \text{on} & \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$, Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $\rho \in C(\overline{\Omega})$, with $\inf_{\overline{\Omega}} \rho(x) > 0$, $f : \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a bounded Carathéodory function, $h \in L^{p'}(\Omega)$, $\left(p' = \frac{p}{p-1}\right)$, $m \in C(\overline{\Omega})$ is nonnegative weight function and λ_1 design the first eigenvalue for the eigenvalue problem

$$\begin{cases} \Delta(\rho|\Delta u|^{p-2}\Delta u) &= \lambda m(x)|u|^{p-2}u & \text{in} & \Omega \\ u &= \Delta u = 0 & \text{on} & \partial\Omega. \end{cases} \tag{1.2}$$

The investigation of existence of solutions for problems at resonance has drawn the attention of many authors, see for example [1,3,6,7,12]. In [7], Liu and Squassina study the following p-biharmic problem

$$\begin{cases} \Delta(\Delta u|^{p-2}\Delta u) &= g(x,u) & \text{in} & \Omega \\ u &= \Delta u = 0 & \text{on} & \partial\Omega. \end{cases}$$

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Under some conditions on $g(x,u)$ at resonance, the authors established the existence of at least one nontrivial solution.

According to the work of Talbi and Tsouli [10], the eigenvalue problem (1.2) has a nondecreasing and unbounded sequence of eigenvalues, and the first eigenvalue λ_1 is given by

$$\lambda_1 = \inf_{u\in X}\left\{\int_\Omega \rho|\Delta u|^p dx : \int_\Omega m(x)|u|^p dx = 1\right\},$$

where $X := W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ is the reflexive Banach space endowed with the norm

$$||u|| = \left(\int_\Omega \rho|\Delta u|^p dx\right)^{1/p}.$$

Since $m\in C(\overline{\Omega})$ and $m\geq 0$, λ_1 is positive, simple and isolated. Therefore

$$\int_\Omega \rho|\Delta u|^p dx \geq \lambda_1 \int_\Omega m(x)|u|^p dx \quad \text{for all}\, u\in X. \tag{1.3}$$

Moreover, there exists a unique positive eigenfunction φ_1 associated to λ_1, which can be chosen normalized. Let

$$\lambda_2 := \inf\left\{\lambda : \lambda \text{ is a eigenvalue of (1.2), with } \lambda > \lambda_1\right\}.$$

The fact that λ_1 is isolated implies that $\lambda_1 < \lambda_2$. It can also be shown (see Lemma 2.1) that there exists $\overline{\lambda}\in(\lambda_1,\lambda_2]$ such that

$$\int_\Omega \rho|\Delta u|^p dx \geq \overline{\lambda}\int_\Omega m(x)|u|^p dx, \tag{1.4}$$

for all $u\in X$ with $\int_\Omega m(x)\varphi_1^{p-1}u dx = 0$.

In addition, we study the existence of solutions for the following boundary value problem

$$\begin{cases} \Delta(\rho|\Delta u|^{p-2}\Delta u) &= \lambda m(x)|u|^{p-2}u + f(x,u) - h \quad \text{in} \quad \Omega \\ u &= \Delta u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

We assume that the function f satisfy the following hypothese:

(H) For almost every $x\in\Omega$, there exist

$$\lim_{s\to-\infty} f(x,s) = l(x), \quad \lim_{s\to+\infty} f(x,s) = k(x). \tag{1.6}$$

Let us recall the minimum principle and the saddle point theorem (see [9]).

Theorem 1.1. *Let X be a Banach space and $\Phi\in \mathcal{C}^1(X,\mathbb{R})$. Assume that*
(i) Φ satisfies the Palais-Smale condition,
(ii) Φ is bounded from below $c = \inf_X \Phi$.
Then there exists $u_0\in X$ such that $\Phi(u_0) = c$.

Theorem 1.2. *Let X be a Banach space. Let $\Phi : X \to \mathbb{R}$ be a C^1 functional that satisfies the Palais-Smale condition, and suppose that $X = V \oplus W$, with V a finite dimensional subspace of X. If there exists $R > 0$ such that*

$$\max_{v \in V, ||v|| = R} \Phi(v) < \inf_{w \in W} \Phi(w),$$

then Φ has a least a critical point on X.

Now, we are ready to state our main results.

Theorem 1.3. *Assume that (1.6) holds. Suppose that $h \in L^{p'}(\Omega)$ is such that either*

$$\int_\Omega k(x)\varphi_1 dx < \int_\Omega h(x)\varphi_1 dx < \int_\Omega l(x)\varphi_1 dx \tag{1.7}$$

or

$$\int_\Omega l(x)\varphi_1 dx < \int_\Omega h(x)\varphi_1 dx < \int_\Omega k(x)\varphi_1 dx. \tag{1.8}$$

Then problem (1.1) has at least a weak solution .

Theorem 1.4. *Assume that (1.6) holds. If $h \in L^{p'}(\Omega)$ satisfy (1.7) or (1.8), then problem (1.5) with $\lambda_1 < \lambda < \overline{\lambda}$, has at least one solution.*

2. Preliminaries and proofs of Theorems

We consider the following energy functional $\Phi : X \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p}\int_\Omega \rho|\Delta u|^p dx - \frac{\lambda_1}{p}\int_\Omega m(x)|u|^p dx - \int_\Omega F(x,u)dx + \int_\Omega h u dx,$$

where

$$F(x,t) = \int_0^t f(x,s)ds \text{ for almost every } x \in \Omega, \ \forall t \in \mathbb{R}.$$

It is well known that $\Phi \in \mathcal{C}^1(X, \mathbb{R})$, with derivative at point $u \in X$ is given by

$$\langle \Phi'(u), v\rangle = \int_\Omega \rho|\Delta u|^{p-2}\Delta u \Delta v dx - \lambda_1 \int_\Omega m(x)|u|^{p-2}uv dx - \int_\Omega f(x,u)v dx + \int_\Omega hv dx,$$

for every $v \in X$.

Let denote $V = \langle \varphi_1 \rangle$ the linear spans of φ_1 and

$$W = \left\{ u \in X : \int_\Omega m(x)\varphi_1^{p-1} u dx = 0 \right\}. \tag{2.1}$$

Then we can decompose X as a direct sum of V and W. In fact, let $u \in X$, writing

$$u = \alpha\varphi_1 + w,$$

where $w \in X$, and $\alpha = \lambda_1 \int_\Omega m(x)\varphi_1^{p-1} u dx$.

Since

$$\int_\Omega \rho|\Delta \varphi_1|^p dx = 1,$$

$$\int_\Omega m(x)\varphi_1^{p-1} w dx = 0.$$

Therefore $w \in W$, hence

$$X = V \oplus W.$$

We begin by establishing the existence of $\overline{\lambda}$ for which (1.4) holds.

Lemma 2.1. *There exists $\overline{\lambda} \in (\lambda_1, \lambda_2]$ such that*

$$\int_\Omega \rho |\Delta u|^p dx \geq \overline{\lambda} \int_\Omega m(x)|u|^p dx, \tag{2.2}$$

for all $u \in W$.

Proof: Let

$$\lambda = \inf \left\{ \int_\Omega \rho |\Delta u|^p dx : u \in W, \int_\Omega m(x)|u|^p dx = 1 \right\}.$$

This value is attained in W. To see why this is so, let (u_n) be a sequence in W, satisfying $\int_\Omega m(x)|u_n|^p dx = 1$ for all n, and $\int_\Omega \rho|\Delta u_n|^p dx \to \lambda$. It follows that (u_n) is bounded in X and therefore, up to subsequence, we may assume that

$$u_n \rightharpoonup u \text{ weakly in } X \quad \text{and} \quad u_n \to u \text{ strongly in } L^p(\Omega).$$

From the strong convergence of the sequence in $L^p(\Omega)$ we obtain

$$\int_\Omega m(x)|u|^p dx = \lim_{n\to\infty} \int_\Omega m(x)|u_n|^p dx = 1$$

and

$$\int_\Omega m(x)\varphi_1^{p-1} u dx = \lim_{n\to\infty} \int_\Omega m(x)\varphi_1^{p-1} u_n dx = 0,$$

so that $u \in W$. By the weakly lower semicontinuity of the norm $||.||$, we get

$$\lambda \leq \int_\Omega \rho|\Delta u|^p dx \leq \liminf_{n\to\infty} \int_\Omega \rho|\Delta u_n|^p dx = \lambda,$$

and hence λ is attained at u.

Now we claim that $\lambda > \lambda_1$. It follows from (1.3) that $\lambda \geq \lambda_1$. If $\lambda = \lambda_1$, by simplicity of λ_1 there is $\alpha \in \mathbb{R}$ such that $u = \alpha\varphi_1$. Since $u \in W$,

$$\alpha \int_\Omega m(x)\varphi_1^p dx = 0,$$

which implies $\alpha = 0$. This contradicts the fact that $\int_\Omega m(x)|u|^p dx = 1$. So, choose $\overline{\lambda} = \min\{\lambda, \lambda_2\}$. It is clear that $\overline{\lambda}$ satisfies (2.2) and the proof of lemma is complete. □

Lemma 2.2. *Assume that (1.6) and (1.7) or (1.8) are verified. Then the functional Φ satisfies the Palais-Smale condition on X.*

Proof: Let (u_n) be a sequence in X, and c a real number such that:

$$|\Phi(u_n)| \leq c \ \ for\ all\ n, \tag{2.3}$$

$$\Phi'(u_n) \to 0. \tag{2.4}$$

We claim that (u_n) is bounded in X. Indeed, suppose by contradiction that

$$||u_n|| \to +\infty, \quad as\ n \to +\infty.$$

Put $v_n = u_n/||u_n||$, thus (v_n) is bounded, for a subsequence still denoted (v_n), we can assume that $v_n \rightharpoonup v$ weakly in X, by Sobelev injection theorem we have $v_n \to v$ strongly in $L^p(\Omega)$, and $v_n \to v$ a.e. in Ω. Dividing (2.3) by $||u_n||^p$, we get

$$\lim_{n\to+\infty}\left(\frac{1}{p}\int_\Omega \rho|\Delta v_n|^p dx - \frac{\lambda_1}{p}\int_\Omega m(x)|v_n|^p dx - \int_\Omega \frac{F(x,u_n)}{||u_n||^p}dx + \int_\Omega h\frac{u_n}{||u_n||^p}dx\right) = 0. \tag{2.5}$$

By the hypotheses on f, h and (u_n), we obtain

$$\lim_{n\to+\infty}\int_\Omega \frac{F(x,u_n)}{||u_n||^p}dx = \lim_{n\to+\infty}\int_\Omega h\frac{u_n}{||u_n||^p}dx = 0,$$

while

$$\lim_{n\to+\infty}\int_\Omega m(x)|v_n|^p dx = \int_\Omega m(x)|v|^p dx$$

then, from (2.5) we deduce that

$$1 = \lim_{n\to+\infty}\int_\Omega \rho|\Delta v_n|^p dx = \lambda_1\int_\Omega m(x)|v|^p dx.$$

Then $v \not\equiv 0$. According to the definition of λ_1 and the weak lower semi continuity of norm, one has

$$\lambda_1\int_\Omega m(x)|v|^p dx \leq \int_\Omega \rho|\Delta v|^p dx \leq \liminf_{n\to+\infty}\int_\Omega \rho|\Delta v_n|^p dx = \lambda_1\int_\Omega m(x)|v|^p dx.$$

This implies that

$$v_n \to v \text{ strongly in } X \quad \text{and} \quad \int_\Omega \rho|\Delta v|^p dx = \lambda_1\int_\Omega m|v|^p dx.$$

By the definition of φ_1, we deduce that $v = \pm\varphi_1$.
On the other hand, from (2.3) we have

$$-cp \leq \int_\Omega \rho|\Delta u_n|^p dx - \lambda_1\int_\Omega m(x)|u_n|^p dx - p\int_\Omega F(x,u_n)dx + p\int_\Omega hu_n dx \leq cp \tag{2.6}$$

In view of (2.4), for all $\varepsilon > 0$ and n large enough, we have

$$-\varepsilon\|u_n\| \leq -\int_\Omega \rho|\Delta u_n|^p dx + \lambda_1 \int_\Omega m|u_n|^p dx + \int_\Omega f(x,u_n)u_n dx - \int_\Omega hu_n dx \leq \varepsilon\|u_n\| \tag{2.7}$$

Let

$$g(x,s) = \begin{cases} \frac{F(x,s)}{s} & \text{if} \quad s \neq 0 \\ f(x,0) & \text{if} \quad s = 0. \end{cases} \tag{2.8}$$

Suppose that $v_n \to -\varphi_1$ (for example), then $u_n(x) \to -\infty$ for a.e. $x \in \Omega$, it follows from (1.6) that

$$\begin{cases} f(x,u_n) \to l(x) & \text{a.e } x \in \Omega \\ g(x,u_n) \to l(x) & \text{a.e } x \in \Omega, \end{cases}$$

Moreover, the Lebesgue's theorem imply

$$\lim_{n\to+\infty} \int_\Omega (f(x,u_n)v_n - pg(x,u_n)v_n)dx = (p-1)\int_\Omega l(x)\varphi_1 dx. \tag{2.9}$$

Combining (2.6) and (2.7), we get

$$-cp - \varepsilon\|u_n\| \leq \int_\Omega f(x,u_n)u_n dx - p\int_\Omega F(x,u_n)dx + (p-1)\int_\Omega hu_n dx \leq cp + \varepsilon\|u_n\|.$$

Dividing by $||u_n||$ the last inequalities, we obtain

$$\frac{-cp}{||u_n||} - \varepsilon \leq \int_\Omega f(x,u_n)v_n dx - p\int_\Omega g(x,u_n)v_n dx + (p-1)\int_\Omega hv_n dx \leq \frac{cp}{||u_n||} + \varepsilon,$$

and passing to the limits, we deduce from (2.9) that

$$\int_\Omega l(x)\varphi_1 dx = \int_\Omega h(x)\varphi_1 dx,$$

which contradicts both (1.7) and (1.8). Thus (u_n) is bounded in X, for a subsequence denoted also (u_n), there exists $u \in X$ such that $u_n \rightharpoonup u$ weakly in X, and strongly in $L^p(\Omega)$. From

$$\lim_{n\to+\infty} \langle \Phi'(u_n), (u_n - u)\rangle = 0,$$

that is

$$\begin{aligned} \langle \Phi'(u_n), (u_n - u)\rangle \quad &= \int_\Omega \rho|\Delta u_n|^{p-2}\Delta u_n \Delta(u_n - u)dx \\ &\quad -\lambda_1 \int_\Omega m(x)|u_n|^{p-2}u_n(u_n - u)dx \\ &\quad -\int_\Omega f(x,u_n)(u_n - u)dx + \int_\Omega h(u_n - u)dx \\ &= o_n(1). \end{aligned}$$

Using the hypotheses on m, h and f, we see that

$$\lim_{n\to+\infty}\int_\Omega m(x)|u_n|^{p-2}u_n(u_n-u)dx=0,\quad \lim_{n\to+\infty}\int_\Omega f(x,u_n)(u_n-u)dx=0$$

$$\lim_{n\to+\infty}\int_\Omega h(u_n-u)dx=0.$$

Consequently,

$$\lim_{n\to+\infty}\int_\Omega \rho|\Delta u_n|^{p-2}\Delta u_n\Delta(u_n-u)dx=0.$$

In the same way, we obtain

$$\lim_{n\to\infty}\int_\Omega \rho|\Delta u|^{p-2}\Delta u\Delta(u_n-u)dx=0.$$

Therefore

$$\begin{aligned}0&=\lim_{n\to\infty}\int_\Omega(\rho|\Delta u_n|^{p-2}\Delta u_n-\rho|\Delta u|^{p-2}\Delta u)\Delta(u_n-u)dx\\&\geq\lim_{n\to\infty}(||u_n||^{p-1}-||u||^{p-1})(||u_n||-||u||)\geq 0,\end{aligned}$$

hence $||u_n||\to||u||$. By the uniform convexity of X, it follows that $u_n\to u$ strongly in X and Φ satisfies the (PS) condition. □

Lemma 2.3. *Assume that (1.6) and (1.7) are satisfied. Then the functional Φ is coercive on X.*

Proof: Suppose by contadiction that Φ is not coercive, then there exists a sequence (u_n) such that $||u_n||\to+\infty$, and $|\Phi(u_n)|\leq C$.
In the proof of lemma 2.2, we have showed that $v_n=u_n/||u_n||\to\pm\varphi_1$.
Since

$$0\leq\int_\Omega\rho|\Delta u_n|^pdx-\lambda_1\int_\Omega m|u_n|^pdx,$$

$$-\int_\Omega F(x,u_n)dx+\int_\Omega hu_ndx\leq\Phi(u_n)\leq C. \tag{2.10}$$

Assume $v_n\to-\varphi_1$ (for example). Dividing (2.10) by $||u_n||$, we get

$$-\int_\Omega\frac{F(x,u_n)}{||u_n||}dx+\int_\Omega h\frac{u_n}{||u_n||}dx\leq\frac{C}{||u_n||}.$$

Passing to the limits, we have

$$\int_\Omega l(x)\varphi_1dx\leq\int_\Omega h(x)\varphi_1dx$$

which contradicts (1.7).

□

Proof: [**Proof of Theorem 1.3**]. If (1.7) holds, the coerciveness of the functional Φ and the Palais-Smale condition entrain, from theorem 1.1, that Φ attains its minimum, so problem (1.1) admits at least a weak solution in X.

If (1.8) holds, then Φ has the geometry of the saddle point theorem 1.2. Indeed, splitting $X = V \oplus W$. Let $u \in W$, using Höder inequality and the properties of F, since $\overline{\lambda} > \lambda_1$

$$\begin{aligned}\Phi(u) &\geq \frac{1}{p}\int_\Omega \rho|\Delta u|^p dx - \frac{\lambda_1}{p}\int_\Omega m(x)|u|^p dx - \int_\Omega F(x,u)dx + \int_\Omega h(x)u dx \\ &\geq \frac{1}{p}\left(1 - \frac{\lambda_1}{\overline{\lambda}}\right)||u||^p - C(\|b\|_\infty |\Omega|^{\frac{1}{p'}} + \|h\|_{p'})||u||, \end{aligned} \tag{2.11}$$

where C is the embedding constants of Sobolev, $\|.\|_{p'}$ and $\|.\|_\infty$ denote the norms in $L^{p'}(\Omega)$ and $L^\infty(\Omega)$ respectively. Then Φ is bounded from below on W, is a consequence of the assumption that $p > 1$, so that

$$\inf_{w \in W} \Phi(w) > -\infty. \tag{2.12}$$

On the other hand, for every $t \in \mathbb{R}$, one has

$$\begin{aligned}\Phi(t\varphi_1) &= -\int_\Omega F(x, t\varphi_1)dx + t\int_\Omega h(x)\varphi_1 dx \\ &= t\left(\int_\Omega h(x)\varphi_1 dx - \int_\Omega g(x, t\varphi_1)\varphi_1 dx\right)\end{aligned}$$

where g has been defined by (2.8). From the Lebesgue theorem, it follows that

$$\lim_{t\to+\infty}\left(\int_\Omega h(x)\varphi_1 dx - \int_\Omega g(x,t\varphi_1)\varphi_1 dx\right) = \int_\Omega (h(x) - k(x))\varphi_1 dx, \tag{2.13}$$

and the limit is negative by (1.8). Analogously, if t tends to $-\infty$, we have the same result with $k(x)$ exchanged with $l(x)$, so that the limit is positive by (1.8). In both cases we get

$$\lim_{t\to\pm\infty} \Phi(t\varphi_1) = -\infty \tag{2.14}$$

By (2.12) and (2.14), there exists $R > 0$ such that

$$\max_{v\in V, ||v||=R} \Phi(v) < \inf_{w\in W} \Phi(w).$$

Hence, Φ satisfies the hypotheses of Theorem 1.2, and there exists a critical point of Φ, that is a solution of (1.1). □

Proof: [**Proof of Theorem 1.4**]. The result of lemma 2.2 holds true for the Euler functional associated to problem (1.5), that is

$$\Phi_\lambda(u) = \frac{1}{p}\int_\Omega \rho|\Delta u|^p dx - \frac{\lambda}{p}\int_\Omega m(x)|u|^p dx - \int_\Omega F(x,u)dx + \int_\Omega hudx \tag{2.15}$$

for every $u \in X$. Indeed, Let (u_n) be a sequence satisfying (2.3) and (2.4), suppose that (u_n) is unbounded, and define $v_n = u_n/||u_n||$, so that, up to subsequence, (v_n) converges weakly to a function v in X. Dividing (2.4) by $||u_n||^{p-1}$, and then taking $\langle \Phi'_\lambda(u_n), v_n - v\rangle = o_n(1)$, we get

$$\lim_{n\to+\infty} \int_\Omega \rho|\Delta v_n|^{p-2}\Delta v_n \Delta(v_n - v)dx = 0$$

this fact implies (as in proof of lemma 2.2) that $v_n \to v$ strongly in X. since $\langle \Phi'_\lambda(u_n), \psi/||u_n||^{p-1}\rangle = o_n(1)$, with $\psi \in X$,

$$\int_\Omega \rho|\Delta v|^{p-2}\Delta v \Delta\psi dx = \lambda \int_\Omega m|v|^{p-2}v\psi dx,$$

so that v solve the problem $\Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda m(x)|u|^{p-2}u$ with Navier boundary condition on $\partial\Omega$. But this equation, being $\lambda \in (\lambda_1, \overline{\lambda}) \subset (\lambda_1, \lambda_2)$, has zero as the only solution by definition of $\overline{\lambda}$. Thus $v = 0$, a contradiction with the strong convergence of v_n to v. Hence (u_n) is bounded. This implies, by same argument in proof of lemma 2.2, that (u_n) is strongly convergent.
On the other hand, as in the second part of the proof of Theorem 1.3, rewrite everything with λ instead of λ_1 and use the fact that $\lambda < \overline{\lambda}$, we get the coerciveness of Φ_λ on W.
Now, recalling that

$$\int_\Omega \rho|\Delta t\varphi_1|^p dx = \lambda_1 \int_\Omega m(x)|t\varphi_1|^p dx, \quad \text{for every } t \in \mathbb{R}$$

thus

$$\Phi_\lambda(t\varphi_1) = \frac{\lambda_1 - \lambda}{p}|t|^p \int_\Omega m|\varphi_1|^p dx + t\left(\int_\Omega h(x)\varphi_1 dx - \int_\Omega g(x, t\varphi_1)\varphi_1 dx\right),$$

since $\lambda > \lambda_1$ and $p > 1$, we have, as before

$$\lim_{t\to\pm\infty} \Phi_\lambda(t\varphi_1) = -\infty.$$

Using again the saddle point theorem, the desired result follows. □

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