Bol. Soc. Paran. Mat. (3s.) **v. 32** 1 (2014): 51–59.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v32i1.19242

Smarandache Curves on S^2

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ABSTRACT: In this paper, we introduce special Smarandache curves according to Sabban frame on S^2 and we give some characterization of Smarandache curves. Besides, we illustrate examples of our results.

Key Words: Smarandache Curves, Sabban Frame, Geodesic Curvature.

Contents

1. Introduction

The differential geometry of curves is usual starting point of students in field of differential geometry which is the field concerned with studying curves, surfaces, etc. with the use the concept of derivatives in calculus. Thus, implicit in the discussion, we assume that the defining functions are sufficiently differentiable, i.e., they have no concerns of cusps, etc. Curves are usually studied as subsets of an ambient space with a notion of equivalence. For example, one may study curves in the plane, the usual three dimensional space, curves on a sphere, etc. There are many important consequences and properties of curves in differential geometry. In the light of the existing studies, authors introduced new curves. Special Smarandache curves are one of them. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [8]. Special Smarandache curves have been studied by some authors ([1], [2], [3], [4], [8]).

In this paper, we study special Smarandache curves such as γt, td, γtd - Smarandache curves according to Sabban frame in Euclidean unit sphere S^2. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

2000 *Mathematics Subject Classification:* 53A04, 53C40

2. Preliminaries

The Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $X \in E^3$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$. The curve α is called a unit speed curve if velocity vector α' of α satisfies $\|\alpha'\| = 1$. For vectors $v, w \in E^3$, it is said to be orthogonal if and only if $\langle v, w \rangle = 0$. The sphere of radius $r = 1$ and with center in the origin in the space E^3 is defined by

$$S^2 = \{P = (P_1, P_2, P_3) | \langle P, P \rangle = 1\}.$$

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve α in E^3. For an arbitrary curve $\alpha \in E^3$, with first and second curvature, κ and τ respectively, the Frenet formulae is given by [5]

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N. \end{aligned}$$

Now, we give a new frame different from Frenet frame. Let γ be a unit speed spherical curve. We denote s as the arc-length parameter of γ. Let us denote $t(s) = \gamma'(s)$, and we call $t(s)$ a unit tangent vector of γ. We now set a vector $d(s) = \gamma(s) \wedge t(s)$ along γ. This frame is called the Sabban frame of γ on S^2 (Sphere of unit radius). Then we have the following spherical Frenet formulae of γ:

$$\begin{aligned} \gamma' &= t \\ t' &= -\gamma + \kappa_g d \\ d' &= -\kappa_g t \end{aligned}$$

where is called the geodesic curvature of κ_g on S^2 and $\kappa_g = \langle t', d \rangle$ [6].

3. Smarandache Curves According to Sabban Frame on S^2

In this section, we investigate Smarandache curves according to the Sabban frame on S^2 .Let $\gamma = \gamma(s)$ and $\beta = \beta(s^*)$ be a unit speed regular spherical curves on S^2, and $\{\gamma, t, d\}$ and $\{\gamma_\beta, t_\beta, d_\beta\}$ be the Sabban frame of these curves, respectively.

3.1. γt-Smarandache Curves

Definition 3.1. *Let S^2 be a unit sphere in E^3 and suppose that the unit speed regular curve $\gamma = \gamma(s)$ lying fully on S^2. In this case, γt - Smarandache curve can be defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\gamma + t). \tag{3.1}$$

Now we can compute Sabban invariants of γt - Smarandache curves. Differentiating the equation 3.1 with respect to s, we have

$$\beta'(s^*) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\gamma' + t')$$

and

$$t_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(t + \kappa_g d - \gamma)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2 + {\kappa_g}^2}{2}}. \tag{3.2}$$

Thus, the tangent vector of curve β is to be

$$t_\beta = \frac{1}{\sqrt{2 + {\kappa_g}^2}}(-\gamma + t + \kappa_g d). \tag{3.3}$$

Differentiating the equation 3.3 with respect to s, we get

$${t_\beta}' \frac{ds^*}{ds} = \frac{1}{(2 + {\kappa_g}^2)^{\frac{3}{2}}}(\lambda_1 \gamma + \lambda_2 t + \lambda_3 d) \tag{3.4}$$

where

$$\begin{aligned} \lambda_1 &= \kappa_g {\kappa_g}' - {\kappa_g}^2 - 2 \\ \lambda_2 &= -\kappa_g {\kappa_g}' - 2 - 2{\kappa_g}^2 - {\kappa_g}^4 \\ \lambda_3 &= 2\kappa_g + 2{\kappa_g}' + {\kappa_g}^3. \end{aligned}$$

Substituting the equation 3.2 into equation 3.4, we reach

$${t_\beta}' = \frac{\sqrt{2}}{(2 + {\kappa_g}^2)^2}(\lambda_1 \gamma + \lambda_2 t + \lambda_3 d). \tag{3.5}$$

Considering the equations 3.1 and 3.3, it easily seen that

$$d_\beta = \beta \wedge t_\beta = \frac{1}{\sqrt{4 + 2{\kappa_g}^2}}(\kappa_g \gamma + (-1 - \kappa_g) t + 2d). \tag{3.6}$$

From the equation 3.5 and 3.6, the geodesic curvature of $\beta(s^*)$ is

$$\begin{aligned} {\kappa_g}^\beta &= \langle {t_\beta}', d_\beta \rangle \\ &= \frac{1}{(2+{\kappa_g}^2)^{\frac{3}{2}}}(\lambda_1 \kappa_g + \lambda_2 (-1 - \kappa_g) + 2\lambda_3). \end{aligned}$$

3.2. *td*-Smarandache Curves

Definition 3.2. *Let S^2 be a unit sphere in E^3 and suppose that the unit speed regular curve $\gamma = \gamma(s)$ lying fully on S^2. In this case, td - Smarandache curve can be defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}}(t + d). \tag{3.7}$$

Now we can compute Sabban invariants of td - Smarandache curves. Differentiating the equation 3.7 with respect to s, we have

$$\beta'(s^*) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(t' + d')$$

and

$$t_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-\gamma + \kappa_g d - \kappa_g t)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{1 + 2{\kappa_g}^2}{2}}. \tag{3.8}$$

In that case, the tangent vector of curve β is as follows

$$t_\beta = \frac{1}{\sqrt{1 + 2{\kappa_g}^2}}(-\gamma - \kappa_g t + \kappa_g d). \tag{3.9}$$

Differentiating the equation 3.9 with respect to s, it is obtained that

$$t_\beta{}' \frac{ds^*}{ds} = \frac{1}{(1 + 2{\kappa_g}^2)^{\frac{3}{2}}}(\lambda_1\gamma + \lambda_2 t + \lambda_3 d) \tag{3.10}$$

where

$$\begin{aligned} \lambda_1 &= 2\kappa_g{\kappa_g}' + \kappa_g + 2{\kappa_g}^3 \\ \lambda_2 &= -1 - {\kappa_g}' - 3{\kappa_g}^2 - 2{\kappa_g}^4 \\ \lambda_3 &= -{\kappa_g}^2 + {\kappa_g}' - 2{\kappa_g}^4. \end{aligned}$$

Substituting the equation 3.8 into equation 3.10, we get

$$t_\beta{}' = \frac{\sqrt{2}}{(1 + 2{\kappa_g}^2)^2}(\lambda_1\gamma + \lambda_2 t + \lambda_3 d). \tag{3.11}$$

Using the equations 3.7 and 3.9, we easily find

$$d_\beta = \beta \wedge t_\beta = \frac{1}{\sqrt{2 + 4{\kappa_g}^2}}(\kappa_g\gamma - t + (1 + \kappa_g)d). \tag{3.12}$$

So, the geodesic curvature of $\beta(s^*)$ is as follows

$$\kappa_g{}^\beta = \langle t_\beta{}', d_\beta \rangle$$
$$= \frac{1}{(1+2\kappa_g{}^2)^{\frac{3}{2}}} \left(\lambda_1 \kappa_g - \lambda_2 + \lambda_3 \left(1+\kappa_g\right)\right).$$

3.3. γtd-Smarandache Curves

Definition 3.3. *Let S^2 be a unit sphere in E^3 and suppose that the unit speed regular curve $\gamma = \gamma(s)$ lying fully on S^2. Denote the Sabban frame of $\gamma(s)$, $\{\gamma, t, d\}$. In this case, γtd - Smarandache curve can be defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\gamma + t + d). \tag{3.13}$$

Lastly, let us calculate Sabban invariants of γtd - Smarandache curves. Differentiating the equation 3.13 with respect to s, we have

$$\beta'(s^*) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left(\gamma' + t' + d'\right)$$

and

$$t_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left(t - \gamma + \kappa_g d - \kappa_g t\right)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2\left(1-\kappa_g+\kappa_g{}^2\right)}{3}}. \tag{3.14}$$

Thus, the tangent vector of curve β is

$$t_\beta = \frac{1}{\sqrt{2\left(1-\kappa_g+\kappa_g{}^2\right)}}\left(-\gamma + \left(1-\kappa_g\right)t + \kappa_g d\right). \tag{3.15}$$

Differentiating the equation 3.15 with respect to s, it is obtained that

$$t_\beta{}' \frac{ds^*}{ds} = \frac{1}{2\sqrt{2}(1-\kappa_g+\kappa_g{}^2)^{\frac{3}{2}}}\left(\lambda_1\gamma + \lambda_2 t + \lambda_3 d\right) \tag{3.16}$$

where

$$\lambda_1 = -\kappa_g{}' + 2\kappa_g\kappa_g{}' - 2 + 4\kappa_g - 4\kappa_g{}^2 + 2\kappa_g{}^3$$
$$\lambda_2 = -\kappa_g{}' - \kappa_g\kappa_g{}' - 2 - 4\kappa_g{}^2 + 2\kappa_g + 2\kappa_g{}^3 - 2\kappa_g{}^4$$
$$\lambda_3 = -\kappa_g\kappa_g{}' + 2\kappa_g - 4\kappa_g{}^2 + 2\kappa_g{}' + 4\kappa_g{}^3 - 2\kappa_g{}^4.$$

Substituting the equation 3.14 into equation 3.16, we reach

$$t_\beta{}' = \frac{\sqrt{3}}{4(1-\kappa_g+\kappa_g{}^2)^2}\left(\lambda_1\gamma + \lambda_2 t + \lambda_3 d\right). \tag{3.17}$$

Using the equations 3.13 and 3.15, we have

$$d_\beta = \beta \wedge t_\beta = \frac{1}{\sqrt{6}\sqrt{1-\kappa_g+\kappa_g{}^2}}\left(\left(2\kappa_g - 1\right)\gamma + \left(-1-\kappa_g\right)t + \left(2-\kappa_g\right)d\right). \tag{3.18}$$

From the equation 3.17 and 3.18, the geodesic curvature of $\beta(s^*)$ is

$$\kappa_g{}^{\beta} = \langle t_\beta{}', d_\beta \rangle$$
$$= \frac{1}{4\sqrt{2}(1-\kappa_g+\kappa_g{}^2)^{\frac{3}{2}}} \left(\lambda_1 \left(2\kappa_g - 1\right) + \lambda_2 \left(-1-\kappa_g\right) + \lambda_3 \left(2-\kappa_g\right)\right).$$

3.4. Example

Let us consider the unit speed spherical curve:

$$\gamma\left(s\right) = \left\{cos\left(s\right)tanh\left(s\right), sin\left(s\right)tanh\left(s\right), sech\left(s\right)\right\}.$$

It is rendered in Figure 1.

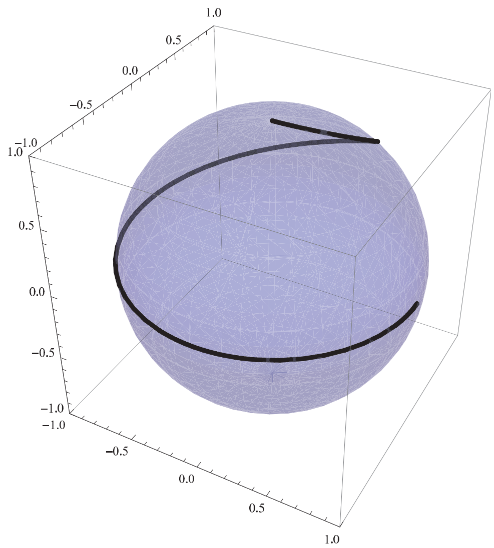

Figure 1: $\gamma = \gamma(s)$

Considering the Sabban frame and the Sabban formulae, we have geodesic curvature of γ:

$$\kappa_g = 2sech\left(t\right).$$

In terms of the definitions, we obtain Smarandache curves according to Sabban frame on S^2, see Figures 2-4.

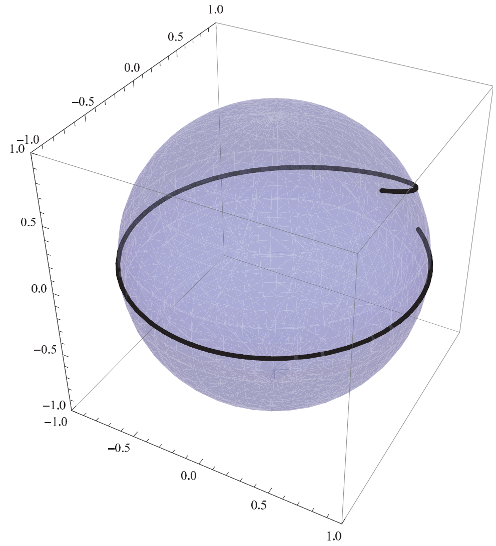

Figure 2: γt - Smarandache Curve

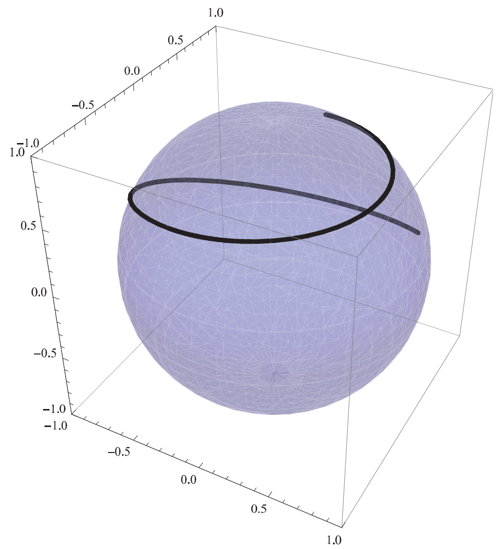

Figure 3: td - Smarandache Curve

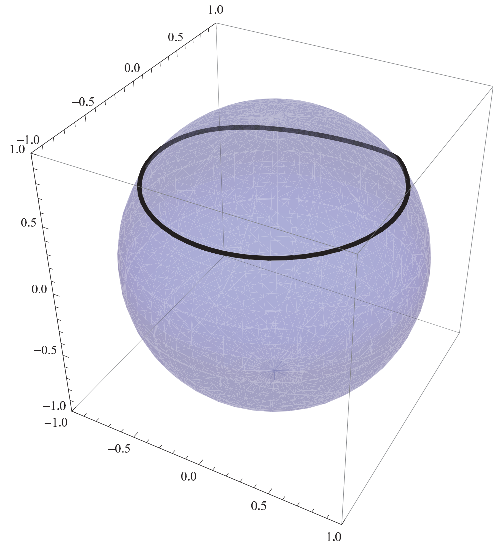

Figure 4: γtd - Smarandache Curve

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