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Bishop Equations of Smarandache $\mathbf{TM}_1$ Curves of Biharmonic $\mathfrak{B}$-Slant Helices in Heis^3

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ABSTRACT: In this paper, we study Bishop equations for Smarandache $\mathbf{TM}_1$ curves of biharmonic $\mathfrak{B}$-slant helices according to Bishop frame in the Heisenberg group Heis^3. Finally, we characterize the Smarandache $\mathbf{TM}_1$ curves of biharmonic $\mathfrak{B}$-slant helices in terms of Bishop frame in the Heisenberg group Heis^3.

Key Words: Biharmonic curve, Bishop frame, Heisenberg group.

Contents

1. Preliminaries

Definition 1.1. *Let $\mathbf{G}$ be a group. Define the sequence of groups $(\Gamma_n(\mathbf{G}))_{n\geq 1}$ by $\Gamma_0(\mathbf{G}) = \mathbf{G}$, $\Gamma_{n+1}(\mathbf{G}) = [\Gamma_n(\mathbf{G}), \mathbf{G}]$. $\mathbf{G}$ is called nilpotent if there is an $n \in \mathbb{N}$ such that $\Gamma_n(\mathbf{G}) = e$. The smallest integer n with the above property is called the class of nilpotence of $\mathbf{G}$, [3].*

The subset of $\mathbf{M}_3(\mathbb{R})$ given by

$$\mathbf{G} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

defines a noncommutative group with the usual matrix multiplication. Consider the matrices

$$A = \begin{pmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & b_1 & b_3 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Then

$$AB = \begin{pmatrix} 1 & a_1 + b_1 & a_3 + b_3 + a_1 b_2 \\ 0 & 1 & a_2 + b_2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} 1 & -a_1 & a_1 a_2 - a_3 \\ 0 & 1 & -a_2 \\ 0 & 0 & 1 \end{pmatrix}, \ B^{-1} = \begin{pmatrix} 1 & -b_1 & b_1 b_2 - b_3 \\ 0 & 1 & -b_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The commutator

$$[A, B] = ABA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 & a_1 b_2 - b_1 a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence the commutator subgroup is

$$\Gamma_1(\mathbf{G}) = [\mathbf{G}, \mathbf{G}] = \left\{ \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : k \in \mathbb{R} \right\}.$$

Let

$$C = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$AC = \begin{pmatrix} 1 & a & c+k \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = CA,$$

and therefore

$$[A, C] = AC\,(AC)^{-1} = I_3.$$

Hence

$$\Gamma_2(\mathbf{G}) = [\Gamma_1(\mathbf{G}), \mathbf{G}] = I_2 = e,$$

and the group $\mathbf{G}$ is nilpotent of class 2. $\mathbf{G}$ is called the Heisenberg group with 3 parameters, [3].

2. Biharmonic $\mathfrak{B}$-Slant Helices with Bishop Frame In The Heisenberg Group Heis^3

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{2.1}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{N},\mathbf{N}) = 1,\ g(\mathbf{B},\mathbf{B}) = 1, \\ g(\mathbf{T},\mathbf{N}) &= g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0. \end{aligned}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T}, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{M}_1,\mathbf{M}_1) = 1,\ g(\mathbf{M}_2,\mathbf{M}_2) = 1, \\ g(\mathbf{T},\mathbf{M}_1) &= g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0. \end{aligned}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures, $\theta(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_2^2 + k_1^2}$, [1]. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\cos\theta(s), \\ k_2 &= \kappa(s)\sin\theta(s). \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3. \end{aligned}$$

3. Smarandache $\mathbf{TM}_1$ Curves of Biharmonic $\mathfrak{B}$-Slant Helices with Bishop Frame In The Heisenberg Group $\mathbf{Heis}^3$

Definition 3.1. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}-$slant helix and $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_2\}$ be its moving Bishop frame. Smarandache $\mathbf{TM}_1$ curves are defined by*

$$\gamma^{\mathbf{TM}_1} = \frac{1}{\sqrt{2k_1^2 + k_2^2}}(\mathbf{T} + \mathbf{M}_1). \tag{3.1}$$

Lemma 3.2. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}-$slant helix. Then, the equation of Smarandache $\mathbf{TM}_1$ curves of γ is*

$$\begin{aligned} \gamma^{\mathbf{TM}_1} &= \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\cos\mathfrak{S}\cos[\mathfrak{C}s + \mathfrak{D}] + \sin\mathfrak{S}\cos[\mathfrak{C}s + \mathfrak{D}]]\mathbf{e}_1 \\ &+ \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\cos\mathfrak{S}\sin[\mathfrak{C}s + \mathfrak{D}] + \sin\mathfrak{S}\sin[\mathfrak{C}s + \mathfrak{D}]]\mathbf{e}_2 \\ &+ \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\cos\mathfrak{S} - \sin\mathfrak{S}]\mathbf{e}_3, \end{aligned} \tag{3.2}$$

where $\mathfrak{D}$ is constant of integration and

$$\mathfrak{C} = \sqrt{k_1^2 + k_2^2 - \cos^2 \mathfrak{S}} - \sin \mathfrak{S}.$$

Proof: Using Bishop formulas (3.3) and (2.1), we have (3.2), the lemma is proved. □

We need following theorem.

Theorem 3.3. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}-$slant helix. Then, the parametric equations of Smarandache $\mathbf{TM}_1$ curves of γ are*

$$\begin{aligned}
x_{\gamma \mathbf{TM}_1} &= \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\cos \mathfrak{S} \cos[\mathfrak{C}s + \mathfrak{D}] + \sin \mathfrak{S} \cos[\mathfrak{C}s + \mathfrak{D}]], \\
y_{\gamma \mathbf{TM}_1} &= \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\cos \mathfrak{S} \sin[\mathfrak{C}s + \mathfrak{D}] + \sin \mathfrak{S} \sin[\mathfrak{C}s + \mathfrak{D}]], \\
z_{\gamma \mathbf{TM}_1} &= \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\cos \mathfrak{S} \cos[\mathfrak{C}s + \mathfrak{D}] + \sin \mathfrak{S} \cos[\mathfrak{C}s + \mathfrak{D}]] \\
&\quad \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\cos \mathfrak{S} \sin[\mathfrak{C}s + \mathfrak{D}] + \sin \mathfrak{S} \sin[\mathfrak{C}s + \mathfrak{D}]] \\
&\quad + \frac{1}{\sqrt{2k_1^2 + k_2^2}}[\cos \mathfrak{S} - \sin \mathfrak{S}],
\end{aligned}$$

where $\mathfrak{D}$ is constant of integration and

$$\mathfrak{C} = \sqrt{k_1^2 + k_2^2 - \cos^2 \mathfrak{S}} - \sin \mathfrak{S}.$$

Proof: Using orthonormal basis we easily have above system. Hence, the proof is completed. □

The equations of a unit speed biharmonic $\mathfrak{B}$-slant helix and its the equation of Smarandache $\mathbf{TM}_1$ curve are illustrated colour Blue, Red, respectively.

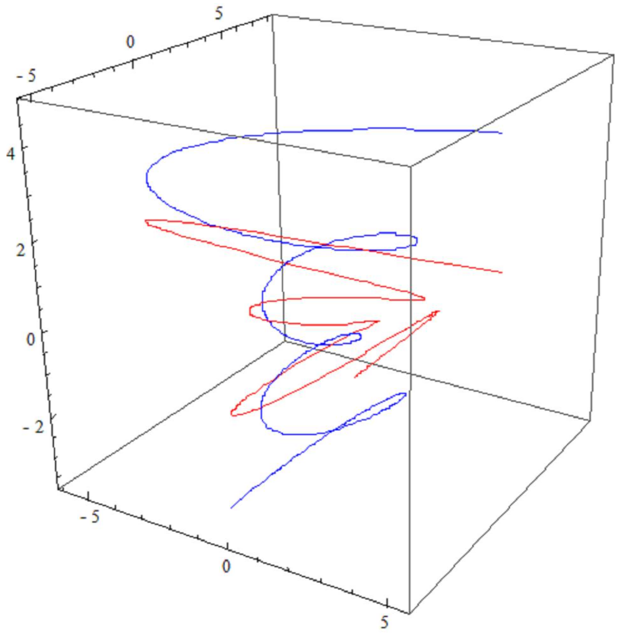

Fig. 1

In this section, we shall call the set $\{\tilde{\mathbf{T}}, \tilde{\mathbf{M}}_1, \tilde{\mathbf{M}}_2\}$ as Bishop trihedra, $\tilde{k}_1$ and $\tilde{k}_2$ as Bishop curvatures of Smarandache $\mathbf{TM}_1$ curve.

We can now state the main result of the paper.

Theorem 3.4. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}$-slant helix with constant Bishop curvatures. Then, the Bishop equations of Smarandache $\mathbf{TM}_1$ curves of γ are*

$$\begin{aligned}
\nabla_{\tilde{\mathbf{T}}}\tilde{\mathbf{T}} &= \mathfrak{W}[\left(-k_1^2 - k_2^2\right)\cos\mathfrak{S}\cos[\mathfrak{C}s + \mathfrak{D}] - k_1^2 \sin\mathfrak{S}\cos[\mathfrak{C}s + \mathfrak{D}] \\
&\quad - k_1 k_2 \sin[\mathfrak{C}s + \mathfrak{D}]]\mathbf{e}_1 + \mathfrak{W}[\left(-k_1^2 - k_2^2\right)\cos\mathfrak{S}\sin[\mathfrak{C}s + \mathfrak{D}] \\
&\quad - k_1^2 \sin\mathfrak{S}\sin[\mathfrak{C}s + \mathfrak{D}] + k_1 k_2 \cos[\mathfrak{C}s + \mathfrak{D}]]\mathbf{e}_2 \\
&\quad + \mathfrak{W}[\left(k_1^2 + k_2^2\right)\sin\mathfrak{S} - k_1 k_2 \cos\mathfrak{S}]\mathbf{e}_3,
\end{aligned}$$

$$\begin{aligned}
\nabla_{\tilde{\mathbf{T}}}\tilde{\mathbf{M}}_1 &= -\tilde{k}_1\mathfrak{W}[-k_1\cos\mathfrak{S}\cos[\mathfrak{C}s + \mathfrak{D}] + k_1 \sin\mathfrak{S}\cos[\mathfrak{C}s + \mathfrak{D}] \\
&\quad + k_2 \sin[\mathfrak{C}s + \mathfrak{D}]]\mathbf{e}_1 - \tilde{k}_1\mathfrak{W}[-k_1\cos\mathfrak{S}\sin[\mathfrak{C}s + \mathfrak{D}] \\
&\quad + k_1 \sin\mathfrak{S}\sin[\mathfrak{C}s + \mathfrak{D}] - k_2\cos[\mathfrak{C}s + \mathfrak{D}]]\mathbf{e}_2 \\
&\quad - \tilde{k}_1\mathfrak{W}[k_1\sin\mathfrak{S} + k_2\cos\mathfrak{S}]\mathbf{e}_3],
\end{aligned}$$

$$\begin{aligned}\nabla_{\tilde{\mathbf{T}}}\tilde{\mathbf{M}}_2 &= -\tilde{k}_2\mathfrak{W}[-k_1\cos\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]+k_1\sin\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]\\ &\quad +k_2\sin[\mathfrak{C}s+\mathfrak{D}]]\mathbf{e}_1-\tilde{k}_2\mathfrak{W}[-k_1\cos\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]\\ &\quad +k_1\sin\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]-k_2\cos[\mathfrak{C}s+\mathfrak{D}]]\mathbf{e}_2\\ &\quad -\tilde{k}_2\mathfrak{W}[k_1\sin\mathfrak{S}+k_2\cos\mathfrak{S}]\mathbf{e}_3],\end{aligned}$$

where $\tilde{k}_1, \tilde{k}_2$ are Bishop curvatures of $\tilde{\gamma}$ and $\mathfrak{D}$ is constant of integration and

$$\mathfrak{C}=\sqrt{k_1^2+k_2^2-\cos^2\mathfrak{S}}-\sin\mathfrak{S} \ \ and\ \mathfrak{W}=\frac{1}{\sqrt{2k_2^2+k_1^2}}.$$

Proof: Differentiating (3.1) and using (3.2), we easily have

$$\begin{aligned}\tilde{\mathbf{T}} &= \mathfrak{W}[-k_1\cos\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]+k_1\sin\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]+k_2\sin[\mathfrak{C}s+\mathfrak{D}]]\mathbf{e}_1\\ &\quad +\mathfrak{W}[-k_1\cos\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]+k_1\sin\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]-k_2\cos[\mathfrak{C}s+\mathfrak{D}]]\mathbf{e}_2\\ &\quad +\mathfrak{W}[k_1\sin\mathfrak{S}+k_2\cos\mathfrak{S}]\mathbf{e}_3], \qquad (3.3)\end{aligned}$$

where $\mathfrak{W}=\frac{1}{\sqrt{2k_2^2+k_1^2}}$.

From the above system of equations, we have the following equation

$$\begin{aligned}\nabla_{\tilde{\mathbf{T}}}\tilde{\mathbf{T}} &= \mathfrak{W}[\left(-k_1^2-k_2^2\right)\cos\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]-k_1^2\sin\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]\\ &\quad -k_1k_2\sin[\mathfrak{C}s+\mathfrak{D}]]\mathbf{e}_1+\mathfrak{W}[\left(-k_1^2-k_2^2\right)\cos\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]\\ &\quad -k_1^2\sin\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]+k_1k_2\cos[\mathfrak{C}s+\mathfrak{D}]]\mathbf{e}_2\\ &\quad +\mathfrak{W}[\left(k_1^2+k_2^2\right)\sin\mathfrak{S}-k_1k_2\cos\mathfrak{S}]\mathbf{e}_3. \qquad (3.4)\end{aligned}$$

Combining (3.3) and (3.4), we have theorem. This concludes the proof of theorem. □

From the above theorem, one concludes

Corollary 3.5. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}$-slant helix with constant Bishop curvatures. Then, the Bishop vectors of Smarandache* $\mathbf{TM}_1$ *curves of γ are*

$$\begin{aligned}\tilde{\mathbf{T}} &= \mathfrak{W}[-k_1\cos\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]+k_1\sin\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]+k_2\sin[\mathfrak{C}s+\mathfrak{D}]]\mathbf{e}_1\\ &\quad +\mathfrak{W}[-k_1\cos\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]+k_1\sin\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]-k_2\cos[\mathfrak{C}s+\mathfrak{D}]]\mathbf{e}_2\\ &\quad +\mathfrak{W}[k_1\sin\mathfrak{S}+k_2\cos\mathfrak{S}]\mathbf{e}_3],\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{M}}_1 \quad = \quad & [\frac{\mathfrak{W}}{\tilde{\kappa}}\cos[\tau s+\varpi]\left[\left(-k_1^2-k_2^2\right)\cos\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]-k_1k_2\sin\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]\right.\\
& -k_2^2\sin[\mathfrak{C}s+\mathfrak{D}]]-\frac{\mathfrak{W}^2}{\tilde{\kappa}}\sin[\tau s+\varpi][[-k_1^2k_2-\left(k_1^2+k_2^2\right)k_2]\sin\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]\\
& +[k_1^3+k_1\left(k_1^2+k_2^2\right)]\sin[\mathfrak{C}s+\mathfrak{D}]]]\mathbf{e}_1+[\frac{\mathfrak{W}}{\tilde{\kappa}}\cos\theta[\tau s+\varpi]\\
& [\left(-k_1^2-k_2^2\right)\cos\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]-k_1k_2\sin\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]+k_2^2\cos[\mathfrak{C}s+\mathfrak{D}]]\\
& -\frac{\mathfrak{W}^2}{\tilde{\kappa}}\sin[\tau s+\varpi][[-k_1^2k_2-\left(k_1^2+k_2^2\right)k_2]\sin\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]-[k_1^3\\
& +k_1\left(k_1^2+k_2^2\right)]\cos[\mathfrak{C}s+\mathfrak{D}]]]\mathbf{e}_2+[\frac{\mathfrak{W}}{\tilde{\kappa}}\cos[\tau s+\varpi][\left(k_1^2+k_2^2\right)\sin\mathfrak{S}\\
& -k_2^2\cos\mathfrak{S}]-\frac{\mathfrak{W}^2}{\tilde{\kappa}}\sin\theta(s)[-k_1^2k_2-\left(k_1^2+k_2^2\right)k_2]\cos\mathfrak{S}]\mathbf{e}_3,
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{M}}_2 \quad = \quad & [\frac{\mathfrak{W}}{\tilde{\kappa}}\sin[\tau s+\varpi][\left(-k_1^2-k_2^2\right)\cos\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]-k_1k_2\sin\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]\\
& -k_2^2\sin[\mathfrak{C}s+\mathfrak{D}]]+\frac{\mathfrak{W}^2}{\tilde{\kappa}}\cos[\tau s+\varpi][[-k_1^2k_2-\left(k_1^2+k_2^2\right)k_2]\sin\mathfrak{S}\cos[\mathfrak{C}s+\mathfrak{D}]\\
& +[k_1^3+k_1\left(k_1^2+k_2^2\right)]\sin[\mathfrak{C}s+\mathfrak{D}]]]\mathbf{e}_1+[\frac{\mathfrak{W}}{\tilde{\kappa}}\sin[\tau s+\varpi][\left(-k_1^2-k_2^2\right)\cos\mathfrak{S}\\
& \sin[\mathfrak{C}s+\mathfrak{D}]-k_1k_2\sin\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]+k_2^2\cos[\mathfrak{C}s+\mathfrak{D}]]\\
& +\frac{\mathfrak{W}^2}{\tilde{\kappa}}\cos[\tau s+\varpi][[-k_1^2k_2-\left(k_1^2+k_2^2\right)k_2]\sin\mathfrak{S}\sin[\mathfrak{C}s+\mathfrak{D}]\\
& -[k_1^3+k_1\left(k_1^2+k_2^2\right)]\cos[\mathfrak{C}s+\mathfrak{D}]]]\mathbf{e}_2+[\frac{\mathfrak{W}}{\tilde{\kappa}}\sin[\tau s+\varpi][\left(k_1^2+k_2^2\right)\sin\mathfrak{S}\\
& -k_2^2\cos\mathfrak{S}]+\frac{\mathfrak{W}^2}{\tilde{\kappa}}\cos\theta(s)[-k_1^2k_2-\left(k_1^2+k_2^2\right)k_2]\cos\mathfrak{S}]\mathbf{e}_3,
\end{aligned}$$

where $\mathfrak{D},\varpi$ *are constants of integration and*

$$\mathfrak{C}=\sqrt{k_1^2+k_2^2-\cos^2\mathfrak{S}}-\sin\mathfrak{S}.$$

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