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The modular sequence space of χ^2

N. Subramanian, P. Thirunavukkarasu and R. Babu

ABSTRACT: In this paper we introduce the modular sequence space of χ^2 and examine some topological properties of these space also establish some duals results among them. Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to define the sequence space ℓ_M which is called an Orlicz sequence space. Another generalization of Orlicz sequence spaces is due to Woo [9]. We define the sequence spaces $\chi^2_{f\lambda}$ and $\chi^{2\lambda}_g$, where $f = (f_{mn})$ and $g = (g_{mn})$ are sequences of modulus functions such that f_{mn} and g_{mn} be mutually complementary for each m, n.

Key Words: analytic sequence, modulus function, double sequences, χ^2 space, modular, duals.

Contents

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - |^{t_{mn}} = 1\, for\, some \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

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$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{allfinitesequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Orlicz [13] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (1 \leq p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], Bektas and Altin [3], Tripathy et al. [18], Rao and Subramanian [15], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An modulus function M is said to satisfy the Δ_2- condition for small u or at 0 if for each $k \in \mathbb{N}$, there exist $R_k > 0$ and $u_k > 0$ such that $M(ku) \leq R_k M(u)$ for all $u \in (0, u_k]$. Moreover, an modulus function M is said to satisfy the Δ_2- condition if and only if

$$lim_{u \to 0+} sup \frac{M(2u)}{M(u)} < \infty$$

Two Modulus functions M_1 and M_2 are said to be equivalent if there are positive constants α, β and b such that

$$M_1(\alpha u) \leq M_2(u) \leq M_1(\beta u) \text{ for all } u \in [0, b].$$

An modulus function M can always be represented in the following integral form

$$M(u) = \int_0^u \eta(t)\,dt,$$

where η, the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$ for $t > 0, \eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$ whenever $\frac{M(u)}{u} \uparrow \infty$ as $u \uparrow \infty$.

Consider the kernel η associated with the modulus function$M$ and let

$$\mu(s) = sup\{t : \eta(t) \leq s\}.$$

Then μ possesses the same properties as the function η. Suppose now

$$\Phi = \int_0^x \mu(s)\,ds.$$

Then, Φ is an modulus function. The functions M and Φ are called mutually complementary Orlicz functions.

Now, we give the following well-known results.

Let M and Φ are mutually complementary modulus functions. Then, we have:
(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality) \tag{1.2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \, for\, some\, \rho > 0\right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.
If X is a sequence space, we give the following definitions:

(i)$X^{'}=$ the continuous dual of X;

(ii)$X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \, for\, each\, x \in X\right\};$

(iii)$X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent, \, foreach\, x \in X\right\};$

(iv)$X^{\gamma} = \left\{a = (a_{mn}) : sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, foreachx \in X\right\}$;

(v)$let\, X\, beanFK - space \supset \phi;\, then\, X^{f} = \left\{f(\Im_{mn}) : f \in X^{'}\right\}$;

(vi)$X^{\delta} = \left\{a = (a_{mn}) : sup_{mn} \left|a_{mn}x_{mn}\right|^{1/m+n} < \infty,\, foreach\, x \in X\right\}$;

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha - (orK\ddot{o}the - Toeplitz)$dual of $X, \beta - (or\, generalized - K\ddot{o}the - Toeplitz)\, dual\, of\, X, \gamma -\, dual\, of\, X,\, \delta\, -\, dual\, of\, X\, respectively. X^{\alpha}$ is defined by Gupta and Kamptan [20]. It is clear that $x^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z\left(\Delta\right) = \left\{x = \left(x_{k}\right) \in w : \left(\Delta x_{k}\right) \in Z\right\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_{∞} denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Baar and Altay in [42] and in the case $0 < p < 1$ by Altay and Baar in [43]. The spaces $c\left(\Delta\right), c_0\left(\Delta\right), \ell_{\infty}\left(\Delta\right)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z\left(\Delta\right) = \left\{x = \left(x_{mn}\right) \in w^2 : \left(\Delta x_{mn}\right) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Definition and Preliminaries

Definition 2.1. *Let λ be a sequence space. Then λ is called*
(i) Solid (or normal) if $(\alpha_{mn}x_{mn}) \in \lambda$ whenever $(x_{mn}) \in \lambda$ for all sequences (α_{mn}) of scalars with $|\alpha_{mn}| \leq 1$.
(ii) Monotone if provided λ contains the canonical preimages of all its stepspaces.
(iii) Perfect if $\lambda = \lambda^{\alpha\alpha}$ [20].

Proposition 2.2. *λ is perfect $\Rightarrow \lambda$ is normal $\Rightarrow \lambda$ is monotone [20].*

Proposition 2.3. *Let λ be a sequence space. If λ is monotone, then $\lambda^{\alpha} = \lambda^{\beta}$, and if λ is normal, then $\lambda^{\alpha} = \lambda^{\gamma}$.*

A Banach metric sequence space (λ, S) is called a BK-metric space if the topology S of λ is finer than the co-ordinatewise convergence topology, or equivalently, the projection maps $P_{mn} : \lambda \to K, P_{mn}(x) = x_{mn}, m, n \geq 1$ are continuous, where K is the scalar field $\mathbb{R}$ (the set of all reals) or (the complex plane). For

$$x = \begin{pmatrix} x_{11} & x_{12} & ...x_{1n} & 0 & ... & 0 \\ x_{21} & x_{22} & ...x_{2n} & 0 & ... & 0 \\ . \\ . \\ . \\ x_{m1} & x_{m2} & ...x_{mn} & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix}$$

and $m, n \in \mathbb{N}$, we write the $[mn]^{th}$ section of x as

$$x^{[mn]} = \begin{pmatrix} x_{11} & x_{12} & ...x_{1n} & 0 & ... & 0 \\ x_{21} & x_{22} & ...x_{2n} & 0 & ... & 0 \\ . \\ . \\ . \\ x_{m1} & x_{m2} & ...x_{mn} & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix}.$$

If $x^{[mn]} \to x$ in (λ, S) for each $x \in \lambda$, we say that (λ, S) is an AK-space. The metric $d(.,.)_\lambda$ generating the topology S of λ is said to be monotone metric if $d(x,0)_\lambda \leq d(y,0)_\lambda$ for $x = \{x_{mn}\}, y = (y_{mn}) \in \lambda$ with $|x_{mn}| \leq |y_{mn}|$, for all $m, n \geq 1$ [48].

Any Orlicz function M_{mn} always has the integral representation

$$M_k(x) = \int_0^x p_{mn}(t)\, dt,$$

where p_{mn}, known as the kernel of M_{mn} is non-decreasing, is right continuous for$t > 0, p_{mn}(0) = 0, p_{mn}(t) > 0$ for $t > 0$ and $p_{mn}(t) \to \infty$, as $t \to \infty$.

Given an Orlicz function M_{mn} with kernel $p_{mn}(t)$, define

$$q_{mn}(s) = sup\{t : p_{mn}(t) \leq s, s \geq 0\}$$

Then $q_{mn}(s)$ possesses the same properties as $p_{mn}(t)$ and the function N_{mn} defined as

$$N_{mn}(x) = \int_0^x q_{mn}(s)\, ds$$

is an Orlicz function. The functions M_{mn} and N_{mn} are called mutually complementary Orlicz functions.

For a sequence $M = (M_{mn})$ of Orlicz functions, the modular sequence class $\tilde{\ell}_M$ is defined by

$$\tilde{\ell}_M = \{x = (x_{mn}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}|) < \infty\}.$$

Using the sequence $N = (N_{mn})$ of Orlicz functions, similarly we define $\tilde{\ell}_N$. The class ℓ_M is defined by

$$\ell_M = \left\{ x = (x_{mn}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{mn} y_{mn}\, converges\, for\, all\, y \in \tilde{\ell}_N \right\}. \tag{2.1}$$

For a sequence $M = (M_{mn})$ of Orlicz functions, the modular sequence class ℓ_M is also defined as

$$\ell_M = \{x = (x_{mn}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}|) < \infty\}.$$

The space ℓ_M is a Banach space with respect to the norm $\|x\|_M$ defined as

$$\|x\|_M = inf\{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}|) \leq 1\}.$$

The single sequence spaces were introduced by Woo [49] around the year 1973, and generalized the Orlicz sequence ℓ_M and the modulared sequence space considered earlier by Nakano in [12].

Proposition 2.4. *Let M_{mn} and N_{mn} be mutually complementary functions for each m, n. Then*
(i) For $x, y \geq 0, xy \leq M_{mn}(x) + N_{mn}(y)$.
(ii) For $x \geq 0, xp_{mn}(x) = M_{mn}(x) + N_{mn}(p_{mn}(x))$.

An important subspace of ℓ_M, which is an AK-space is the space h_M defined as

$$h_M = \{x \in \ell_M : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}| < \infty)\}$$

A sequence (M_{mn}) of Orlicz functions is said to satisfy uniform Δ_2- condition at $'0'$ if there exist $p > 1$ and $k_0 \in \mathbb{N}$ such that $x \in (0,1)$ and $k > k_0$, we have $\frac{xM'_{mn}(x)}{M_{mn}(x)} \leq p$, or equivalently, there exists a constant $K > 1$ and $k_0 \in \mathbb{N}$ such that $\frac{M_{mn}(2x)}{M_{mn}(x)} \leq K$ for all $k > k_0$ and $x \in \left(0, \frac{1}{2}\right]$. If the sequence (M_{mn}) satisfies uniform Δ_2- condition, then $h_M = \ell_M$ and vice versa [49].

Definition 2.5. *Let f_{mn} and g_{mn} be mutually complementary functions for each m, n and let $\lambda = (\lambda_{mn})$ be a sequence of strictly positve real numbers. Then we define the following sequence spaces:*

$$\chi^2_{f\lambda} = \left\{ x = (x_{mn}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}} \right) \to 0\, as\, m, n \to \infty \right\}.$$

The space $\chi^2_{f\lambda}$ is a metric space with the metric

$$d(x,y) = sup_{mn}\left\{ inf\left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn} - y_{mn}|)^{1/m+n}}{\lambda_{mn}} \right) \right\} \leq 1 \right\}$$

and

$$\chi_g^{2\lambda} = \left\{ x = (x_{mn}) : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}\left(\lambda_{mn}\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}\right) \to 0\, as\, m,n \to \infty \right\}.$$

The space $\chi_g^{2\lambda}$*is a metric space with the metric*

$$d(x,y) = sup_{mn}\left\{ inf \left\{ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}\left(\lambda_{mn}\left((m+n)!\left|x_{mn}-y_{mn}\right|\right)^{1/m+n}\right)\right) \le 1 \right\}$$

The spaces $\chi_{f\lambda}^2$ *and* $\chi_g^{2\lambda}$ *also can be written as*

$$\chi_{f\lambda}^2 = \left\{ x = (x_{mn}) : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}}\right) \in \chi_f^2 \right\}.$$

and

$$\chi_g^{2\lambda} = \left\{ x = (x_{mn}) : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}\left(\lambda_{mn}\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}\right) \in \chi_g^2 \right\}.$$

Throughout the paper we write $f_{mn}(1) = 1$ *and* $g_{mn}(1) = 1$ *for all* $m, n \in \mathbb{N}$.

3. Main Results

Theorem 3.1. *Let* $f = (f_{mn})$ *and* $g = (g_{mn})$ *be two sequences of Orlicz functions. Then* $\chi_{f\lambda}^2$ *and* $\chi_g^{2\lambda}$ *are linear spaces over the field of complex numbers.*

Proof: It is routine verfication. Therefore the proof is omitted. □

Theorem 3.2. $\chi_{f\lambda}^2$ *and* $\chi_g^{2\lambda}$ *are monotone metric*

Proof: $d(x,y) = sup_{mn}\left\{ inf \left\{ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}-y_{mn}|)^{1/m+n}}{\lambda_{mn}}\right)\right\} \le 1 \right\}$,
$d(x^r, y^r) = sup_{rr}\left\{ inf \left\{ \sum_{m=r}^{\infty}\sum_{n=r}^{\infty} f_{rr}\left(\frac{((2r)!|x_{rr}-y_{rr}|)^{1/2r}}{\lambda_{rr}}\right)\right\} \le 1 \right\}$, and
$d(x^s, y^s) = sup_{ss}\left\{ inf \left\{ \sum_{m=s}^{\infty}\sum_{n=s}^{\infty} f_{ss}\left(\frac{((2s)!|x_{ss}-y_{ss}|)^{1/2s}}{\lambda_{ss}}\right)\right\} \le 1 \right\}$,
Let $r > s$. Then
$sup_{mn}\left\{ inf \left\{ \sum_{m=r}^{\infty}\sum_{n=r}^{\infty} f_{rr}\left(\frac{((2r)!|x_{rr}-y_{rr}|)^{1/2r}}{\lambda_{rr}}\right)\right\} \le 1 \right\} \ge$
$sup_{ss}\left\{ inf \left\{ \sum_{m=s}^{\infty}\sum_{n=s}^{\infty} f_{ss}\left(\frac{((2s)!|x_{ss}-y_{ss}|)^{1/2s}}{\lambda_{ss}}\right)\right\} \le 1 \right\}$.

$$d(x^r, y^r) \ge d(x^s, y^s), r > s \tag{3.1}$$

Also $\{d(x^s, y^s) : r = 1, 2, 3, \cdots\}$ is a monotonically increasing sequence bounded by $d(x,y)$. For such a sequence

$$sup_{ss}\left\{ inf \left\{ \sum_{m=s}^{\infty}\sum_{n=s}^{\infty} f_{ss}\left(\frac{((2s)!\left|x_{ss}-y_{ss}\right|)^{1/2s}}{\lambda_{ss}}\right)\right\} \le 1 \right\} = lim_{s\to\infty} d(x^s, y^s) = d(x,y) \tag{3.2}$$

From (3.1) and (3.2) it follows that
$d(x,y) = sup_{mn}\left\{ inf \left\{ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}-y_{mn}|)^{1/m+n}}{\lambda_{mn}}\right)\right\} \le 1 \right\}$ is monotone metric for $\chi_{f\lambda}^2$.
The proof similar for $\chi_g^{2\lambda}$. □

Theorem 3.3. *$\chi^2_{f\lambda}$ has AK*

Proof: Let $x = (x_{mn}) \in \chi^2_{f\lambda}$ and take the $[m,n]^{th}$ sectional sequence of x. We have $d\left(x, x^{[r,s]}\right) = sup_{mn}\left\{inf\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}}\right)\right\} \le 1 : m \ge r, n \ge s\right\} \to 0\, as\ [r,s] \to \infty$. Therefore $x^{[r,s]} \to x$ in $\chi^2_{f\lambda}$ as $r, s \to \infty$. Thus $\chi^2_{f\lambda}$ has AK. □

Theorem 3.4. *$\chi^2_{f\lambda}$ is solid*

Proof: Let $|x_{mn}| \le |y_{mn}|$ and let $y_{mn} \in \chi^2_{f\lambda}$. We have $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}}\right) \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|y_{mn}|)^{1/m+n}}{\lambda_{mn}}\right)$.
But $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|y_{mn}|)^{1/m+n}}{\lambda_{mn}}\right) \in \chi^2_{f\lambda}$, because $y \in \chi^2_{f\lambda}$. That is $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|y_{mn}|)^{1/m+n}}{\lambda_{mn}}\right) \to 0\, as\, m, n \to \infty \Rightarrow$
$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}}\right) \to 0\, as\, m, n \to \infty$. Therefore $x = (x_{mn}) \in \chi^2_{f\lambda}$. Hence $\chi^2_{f\lambda}$ is solid. □

Theorem 3.5. *The spaces $\left(\chi^2_{f\lambda}, d\,(.,.)^f_\lambda\right)$ and $\left(\chi^{2\lambda}_g, d\,(.,.)^\lambda_g\right)$ are Banach metric spaces.*

Theorem 3.6. *The sequence spaces $\chi^2_{f\lambda}$ and $\chi^{2\lambda}_g$ are BK-spaces*

Proof: The space $\left(\chi^2_{f\lambda}, d\,(.,.)^f_\lambda\right)$ is a Banach space by Theorem 3.5. Now let

$$d\left(x^{[r,s]}, x\right)^f_\lambda \to 0\, as\, r, s \to \infty.$$

Then

$$\left((m+n)!\left|x^{[rs]}_{mn} - x_{mn}\right|\right)^{1/m+n} \to 0\, as\, r, s \to \infty$$

$$sup_{mn}\left\{inf\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{\left((m+n)!\left|x^{[rs]}_{mn} - x_{mn}\right|\right)^{1/m+n}}{\lambda_{mn}}\right)\right\} \le 1\right\} \to 0\, as\, r, s \to \infty$$

for all $m, n \in \mathbb{N}$.
If $f_{mn}\left(\frac{\left((m+n)!\left|x^{[rs]}_{mn} - x_{mn}\right|\right)^{1/m+n}}{\lambda_{mn} d(.,.)^f_\lambda}\right) \le 1$ then $\left(\frac{\left((m+n)!\left|x^{[rs]}_{mn} - x_{mn}\right|\right)^{1/m+n}}{\lambda_{mn} d(.,.)^f_\lambda}\right) \le 1$ for all m, n. Therefore we also obtain

$$\left((m+n)!\left|x^{[rs]}_{mn} - x_{mn}\right|\right)^{1/m+n} \le \lambda_{mn} d\left(x^{[r,s]}, x\right)^f_\lambda.$$

Since $d\left(x^{[r,s]}, x\right)^f_\lambda \to 0\, as\, r, s \to \infty$, then $\left((m+n)!\left|x^{[rs]}_{mn} - x_{mn}\right|\right)^{1/m+n} \to 0$ as $r, s \to \infty$ for all $m, n \in \mathbb{N}$. Hence $\left(\chi^2_{f\lambda}, d\,(.,.)^f_\lambda\right)$ is a BK-space.

The proof is similar for $\left(\chi^{2\lambda}_g, d\,(.,.)^\lambda_g\right)$. □

Corollary 3.7. $\chi^2_{f\lambda}$ *has AD*

$\chi^2_{f\lambda}$ is a BK-AK by Theorem 3.3 and 3.6. Consequently $\chi^2_{f\lambda}$ has AD. Also AK implies AD by [50].

Corollary 3.8. $\chi^2_{f\lambda}$ *has FAK*

Every space with monotone metric has AK and also AB implies FAK by [Wilansky].
$\chi^2_{f\lambda}$ has AB, consequently $\chi^2_{f\lambda}$ has FAK.

4. χ^{2f}_{λ} and $\chi^{2\lambda}_{f}$

If $\lambda_{mn} = 1$ for all $m, n \in \mathbb{N}$, then the sequence space $\chi^{2\lambda}_f$ reduces to the sequence space

$$\chi^2_f = \left\{x = (x_{mn}) : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(((m+n)!\,|x_{mn}|)^{1/m+n}\right) \to 0\, as\, m, n \to \infty\right\}.$$

Theorem 4.1. *If* $\lambda = (\lambda_{mn})$ *is a double gai sequence such that* $inf \lambda_{mn} > 0$*, then* $\chi^{2f}_{\lambda} = \chi^{2\lambda}_f = \chi^2_f$

Proof: Let $x \in \chi^2_f$. Then$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(((m+n)!\,|x_{mn}|)^{1/m+n}\right) \to 0$as $m, n \to \infty$. Since $\lambda = (\lambda_{mn})$ is double gai, we can write $a \leq \lambda_{mn} \leq b$ for some $b > a \geq 0$. Since f_{mn} is non-decreasing , it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left((\lambda_{mn}\,(m+n)!\,|x_{mn}|)^{1/m+n}\right) \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(((m+n)!\,|x_{mn}|)^{1/m+n}\right) \to 0$$

as $m, n \to \infty$. Hence $\chi^2_f \subset \chi^{2\lambda}_f$. The other inclusion $\chi^{2\lambda}_f \subset \chi^2_f$ follows from the inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(((m+n)!\,|x_{mn}|)^{1/m+n}\right) \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left((\lambda_{mn}\,(m+n)!\,|x_{mn}|)^{1/m+n}\right) \to 0$$

as $m, n \to \infty$. Therefore $\chi^{2\lambda}_f = \chi^2_f$. Similarly, we can prove $\chi^{2f}_{\lambda} = \chi^2_f$. $\square$

Theorem 4.2. *If* $\{\lambda_{mn}\} \in \Lambda^2$ *with* $a = sup_{m,n}\lambda_{mn} \geq 1$ *and* $\left\{\lambda^{-1}_{mn}\right\}$ *is unbounded, then* $\chi^{2f}_{\lambda} \subset \chi^{2\lambda}_f$ *and the inclusion map* $\zeta : \chi^{2f}_{\lambda} \to \chi^{2\lambda}_f$ *is continuous with* $d(\zeta, 0) \leq \eta^2$

Proof: We have $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\lambda_{mn}\,((m+n)!\,|x_{mn}|)^{1/m+n}\right) \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(((m+n)!\,|x_{mn}|)^{1/m+n}\right) \to 0$ as $m, n \to \infty$. Hence $\chi^{2f}_{\lambda} \subset \chi^{2\lambda}_f$.

We now show that the containment $\chi^{2f}_{\lambda} \subset \chi^{2\lambda}_{f}$ is proper. From the unboundedness of the sequence $\left\{\lambda^{-1}_{mn}\right\}$, choose a sequence $m_p n_q$ of positive integers such that $\lambda^{-1}_{m_p n_q} \geq pq$. Define $x = \{x_{mn}\}$ as follows:

$$x_{mn} = \begin{cases} \frac{1}{(pq)^{m+n}(m+n)!}, mn = m_p n_q, p, q = 1, 2, \cdots \\ 0, otherwise. \end{cases}$$

Then $x \in \chi^{2\lambda}_{f}$; but $x \notin \chi^{2f}_{\lambda}$.

To prove the continuity of the inclusion map ζ, for $x \in \chi^{2f}_{\lambda}$, we write

$$_{A}\chi^2_f = sup_{mn} \left\{inf \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}}\right)\right) \leq 1\right\} \to 0\, as\, m, n \to \infty.$$

and

$$_{B}\chi^{2\lambda}_{f} = \left\{inf \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\lambda_{mn}\left((m+n)!\,|x_{mn}|\right)^{1/m+n}\right)\right) \leq 1\right\} \to 0\, as\, m, n \to \infty.$$

Since f_{mn} is nondecreasing, we get $_{A}\chi^2_f \subset_B \chi^{2\lambda}_{f}$. Hence

$$d\left(x, 0\right)^{\lambda}_{f} = sup\left(inf\left(_{B}\chi^{2\lambda}_{f}\right)\right) \leq sup\left(inf\left(_{A}\chi^{2f}_{\lambda}\right)\right) = d\left(x, 0\right)^{f}_{\lambda} \qquad (4.1)$$

(i.e)$d\left(\zeta, 0\right)^{\lambda}_{f} \leq d\left(x, 0\right)^{f}_{\lambda}$. Thus ζ is continuous with $d\left(\zeta, 0\right) \leq 1 = \eta^2$.

Define $\beta_{mn} = \lambda_{mn}, m, n \in \mathbb{N}$. Then $\beta_{mn} \leq 1$ and from (4.1), it follows that

$$d\left(x, 0\right)^{\beta}_{f} \leq d\left(x, 0\right)^{f}_{\beta}\; for\, x \in \chi^{2f}_{\lambda} \qquad (4.2)$$

Hence from (4.2)

$$d\left(\zeta, 0\right)^{2\lambda}_{f} = d\left(x, 0\right)^{\lambda}_{f} \leq \eta^2 d\left(\zeta, 0\right)^{2f}_{\lambda}.$$

(i.e) ζ is continuous with $d\left(\zeta, 0\right) \leq \eta^2$. □

Theorem 4.3. $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}a_{mn}$ *for all* $x = \{x_{mn}\} \in \chi^2_{f\lambda} \Leftrightarrow \{a_{mn}\} \in \chi^{2\lambda}_{g}$

Proof: $|x_{mn}a_{mn}| \leq f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}}\right) + g_{mn}\left(\lambda_{mn}\left((m+n)!\,|a_{mn}|\right)^{1/m+n}\right)$
$\Leftrightarrow \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}a_{mn}| \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}}\right) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}\left(\lambda_{mn}\left((m+n)!\,|a_{mn}|\right)^{1/m+n}\right)$.
Since $a = \{a_{mn}\} \in \chi^{2\lambda}_{g}$ we have $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}\left(\lambda_{mn}\left((m+n)!\,|x_{mn}|\right)^{1/m+n}\right) \to 0\, as\, m, n \to \infty$ and
$x = \{x_{mn}\} \in \chi^2_{f\lambda}$ we have $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}}\right) \to 0\, as\, m, n \to \infty$.
Hence $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}a_{mn}\, converges \Leftrightarrow \{a_{mn}\} \in \chi^{2\lambda}_{g}$. □

Proposition 4.4. *The $\beta-$ dual space of $\chi^2_{f\lambda}$ is $\Lambda^2_{f\lambda}$*

Proof: First, we observe that $\chi^2_{f\lambda} \subset \Gamma^2_{f\lambda}$, Theorefore $\left(\Gamma^2_{f\lambda}\right)^\beta \subset \left(\chi^2_{f\lambda}\right)^\beta$. But $\left(\Gamma^2_{f\lambda}\right)^\beta \underset{\neq}{\subset} \Lambda^2_{f\lambda}$ Hence

$$\Lambda^2_{f\lambda} \subset \left(\chi^2_{f\lambda}\right)^\beta \tag{4.3}$$

Next we show that $\left(\chi^2_{f\lambda}\right)^\beta \subset \Lambda^2_{f\lambda}$. Let $y = (y_{mn}) \in \left(\chi^2_{f\lambda}\right)^\beta$. Consider $f(x) = \sum_{m=1}^\infty \sum_{n=1}^\infty x_{mn} y_{mn}$ with $x = (x_{mn}) \in \chi^2_{f\lambda}$
$x = [(\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1})]$

$$= \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...\frac{\lambda_{mn}}{(m+n)!} & \frac{-\lambda_{mn}}{(m+n)!} & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...\frac{\lambda_{mn}}{(m+n)!} & \frac{-\lambda_{mn}}{(m+n)!} & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix}$$

$$f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}}\right) = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0, & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...f_{mn}\left(\frac{\lambda_{mn}}{(m+n)!}\right) & f_{mn}\left(\frac{-\lambda_{mn}}{(m+n)!}\right) & ... & 0 \\ \\ 0 & 0 & ...f_{mn}\left(\frac{-\lambda_{mn}}{(m+n)!}\right) & f_{mn}\left(\frac{\lambda_{mn}}{(m+n)!}\right) & ... & 0 \\ 0 & 0 & ...0 & 0, & ... & 0 \end{pmatrix}.$$

Hence converges to zero.

Therefore $[(\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1})] \in \chi^2_{f\lambda}$.

Hence $d\left((\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1}), 0\right) = 1$. But

$|y_{mn}| \leq \|f\| \, d\left((\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1}), 0\right) \leq \|f\| \cdot 1 < \infty$ for each m, n. Thus (y_{mn}) is a double modular bounded sequence and hence an modular analytic sequence. In other words $y \in \Lambda^2_{f\lambda}$. But $y = (y_{mn})$ is arbitrary in $\left(\chi^2_{f\lambda}\right)^\beta$. Therefore

$$\left(\chi^2_{f\lambda}\right)^\beta \subset \Lambda^2_{f\lambda} \tag{4.4}$$

From (4.3) and (4.4) we get $\left(\chi^2_{f\lambda}\right)^\beta = \Lambda^2_{f\lambda}$. □

Proposition 4.5. *The dual space of* $\chi^2_{f\lambda}$ *is* $\Lambda^2_{f\lambda}$.*In other words* $(\chi^2_{f\lambda})^* = \Lambda^2_{f\lambda}$.

Proof: We recall that $\Im_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...\frac{\lambda_{mn}}{(m+n)!} & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$
with $\frac{\lambda_{mn}}{(m+n)!}$ in the $(m,n)th$ position and zero's else where, with $x = \Im_{mn}$,

$$\left\{ f\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}} \right) \right\} = \begin{pmatrix} 0. & . & . & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & f\left(\frac{\lambda_{mn}}{(m+n)!}\right)^{1/m+n} & . & 0 \\ & (m,n)^{th} & & \\ 0 & . & . & 0 \end{pmatrix}$$

which is a modular double gai sequence. Hence, $\Im_{mn} \in \chi^2_{f\lambda}.f(x) = \sum_{m,n=1}^{\infty} x_{mn} y_{mn}$ with $x \in \chi^2_{f\lambda}$ and $f \in (\chi^2_{f\lambda})^*$, where $(\chi^2_{f\lambda})^*$ is the dual space of $\chi^2_{f\lambda}$. Take $x = (x_{mn}) = \Im_{mn} \in \chi^2_{f\lambda}$.Then,

$$|y_{mn}| \leq \|f\| \, d(\Im_{mn}, 0) < \infty \qquad \forall m, n \tag{4.5}$$

Thus, (y_{mn}) is a modular bounded sequence and hence an modular double analytic sequence. In other words, $y \in \Lambda^2_{f\lambda}$. Therefore $(\chi^2_{f\lambda})^* = \Lambda^2_{f\lambda}$.This completes the proof. □

Theorem 4.6. *Let* f_{mn} *and* g_{mn} *for each* mn *be mutually complementary functions. Then*

$$\left[\chi^2_{f\lambda}\right]^{\beta} = \left[\chi^2_{f\lambda}\right]^{f} = \left[\chi^2_{f\lambda}\right]^{\alpha} = \left[\chi^2_{f\lambda}\right]^{\gamma} = \chi^{2\lambda}_{g}.$$

Proof: From Proposition 2.2, Proposition 2.3 and Theorem 3.4. □

Theorem 4.7. *(i) If the sequence* f_{mn} *satisfies uniform* Δ_2- *condition, then* $\left[\chi^2_{f\lambda}\right]^{\alpha} = \chi^{2\lambda}_{g}$. *(ii) If the sequence* g_{mn} *satisfies uniform* Δ_2- *condition, then* $\left[\chi^{2\lambda}_{g}\right]^{\alpha} = \chi^2_{f\lambda}$.

Proof: Let the sequence f_{mn} satisfies uniform Δ_2- condition. Then for any $x \in \chi^2_{f\lambda}$ and $a \in \chi^{2\lambda}_{g}$, we have

$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}a_{mn}| \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\lambda_{mn}}\right) +$
$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}\left(\lambda_{mn}\left((m+n)!\,|a_{mn}|\right)^{1/m+n}\right) \to 0\, as\, m,n \to \infty.$
Thus $a \in \left[\chi^2_{f\lambda}\right]^{\alpha}$. Hence $\chi^{2\lambda}_g \subset \left[\chi^2_{f\lambda}\right]^{\alpha}$.

To prove the inclusion $\left[\chi^2_{f\lambda}\right]^{\alpha} \subset \chi^{2\lambda}_g$, let $a \in \left[\chi^2_{f\lambda}\right]^{\alpha}$. Then for all $\{x_{mn}\}$ with $\left(\frac{((m+n)!x_{mn})^{1/m+n}}{\lambda_{mn}}\right) \in \chi^2_f$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}a_{mn}|\; converges \Leftrightarrow \{a_{mn}\} \in \chi^{2\lambda}_g.$$

Since the sequence (f_{mn}) satisfies uniform Δ_2- condition, we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\lambda_{mn}y_{mn}a_{mn}|\; converges \Leftrightarrow \{a_{mn}\} \in \chi^{2\lambda}_g.$$

Thus $(\lambda_{mn}a_{mn}) \in \left[\chi^2_f\right]^{\alpha} = \chi^2_g$ and hence $\{a_{mn}\} \in \chi^{2\lambda}_g$. This gives that $\left[\chi^2_{f\lambda}\right]^{\alpha} = \chi^{2\lambda}_g$.

(ii) Similarly, one can prove that $\left[\chi^{2\lambda}_g\right]^{\alpha} = \chi^2_{f\lambda}$ if the sequence (g_{mn}) satisfies uniform Δ_2- condition. □

Theorem 4.8. *Let Y be an FAK-space$\supset \Phi$. Then $y \supset \chi^2_f \Leftrightarrow (\Im_{mn})$ is a member of the mutually complementary Orlicz sequence space*

Proof: $Y \supset \chi^2_f$
$\Leftrightarrow Y^f \subset \left(\chi^2_f\right)^f$
$\Leftrightarrow Y^f \subset \chi^{2\lambda}_g$ by Theorem 4.6
$\Leftrightarrow (\Im_{mn})$ is a member of the mutually complementary Orlicz sequence space.
□

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N. Subramanian
Department of Mathematics,
SASTRA University,
Thanjavur-613 401, India
E-mail address: nsmaths@yahoo.com

and

P. Thirunavukkarasu
P.G. and Research Department of Mathematics,
Periyar E.V.R. College (Autonomous),
Tiruchirappalli–620 023, India.
E-mail address: ptavinash1967@gmail.com

and

R. Babu
Department of Mathematics,
Shanmugha Polytechnic College,
Thanjavur-613 401, India
E-mail address: babunagar1968@gmail.com