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Spectra of the Rhaly Operator on the Sequence Space $\overline{bv_0} \cap \ell_\infty$

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ABSTRACT: In this article we have determined the spectra of the Rhaly operator on the class of bounded statistically null bounded variation sequence space. We have also determined the dual of the bounded statistically null bounded variation sequence space.

Key Words: Spectra; Dual space; Rhaly operator; bounded variation.

Contents

1. Preliminaries and background

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

Throughout $B(X)$ denotes the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$, for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator $T_\lambda = T - \lambda I$,

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is
$T_\lambda^{-1} = (T - \lambda I)^{-1}$,

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and call it the resolvent operator of T.

Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A *regular value* λ of T is a complex number such that

$(R1)$ T_λ^{-1} exists,

$(R2)$ T_λ^{-1} is bounded

$(R3)$ T_λ^{-1} is defined on a set which is dense in X i.e. $\overline{R(T_\lambda)} = X$.

The *resolvent set* of T, denoted by $\rho(T, X)$, is the set of all regular values λ of T. Its complement $\sigma(T, X) = C - \rho(T, X)$ in the complex plane C is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point*(*discrete*) spectrum $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an eigenvalue of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and satisfies $(R3)$, but not $(R2)$, that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not), but does not satisfy $(R3)$, that is, the domain of T_λ^{-1} is not dense in X.

By w, we denote the space of all real or complex valued sequences. Throughout the paper c, c_0, bv, $\overline{c}$, $\overline{c_0}$, $\overline{bv}$, bs, ℓ_1, ℓ_∞ represent the spaces of all convergent, null, bounded variation, statistically convergent, statistically null, statistically bounded variation, bounded series, absolutely summable and bounded sequences respectively. Also $bv_0 = bv \cap c_0$ and $\overline{bv_0} = \overline{bv} \cap \overline{c_0}$.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in N_0 = \{0, 1, 2, ...\}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $A = \{(Ax)_n\}_{n \in N_0}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, n \in N_0. \tag{1}$$

For simplicity in notation, throughout the summation without limits runs from 0 to ∞. By (λ, μ), we denote the class of all matrices such that $A : \lambda \to \mu$. Thus, $A \in (\lambda, \mu)$ if and only if the series on the right hand side of (1) converges for each

$n \in N_0$ and every $x \in \lambda$ and we have $Ax = \{(Ax)_n\}_{n \in N_0} \in \mu$ for all $x \in \lambda$.

Our main focus in this paper is on the Rhaly matrix $A = R_a$, where

$$R_a = \begin{pmatrix} a_0 & 0 & 0 & 0 & \dots \\ a_1 & a_1 & 0 & 0 & \dots \\ a_2 & a_2 & a_2 & 0 & \dots \\ . & . & . & . & \dots \end{pmatrix},$$

where all the entries are real or complex.

On taking $L = \lim_{n\to\infty}(n+1)a_n$, Rhaly [9] determined the spectrum of R_a on the Hilbert space ℓ_2 of square summable sequences.

Mustafa Yildirim [18] determined the spectrum of R_a on the sequence spaces c_0 and c with the assumptions

(a) $L = \lim_{n\to\infty}(n+1)a_n$ exists, finite and nonzero,

(b) $a_n > 0$ for all n,

(c) $a_i \neq a_j$ for $i \neq j$.

Yildirim [20] under the same assumptions has determined the fine spectrum of R_a on the sequence space c_0.

Yildirim [16] determined the spectrum of R_a on the sequence space bv_0 with the assumptions

(a) $L = \lim_{n\to\infty}(n+1)a_n$ exists, finite and nonzero,

(b) $a_n > 0$ for all n,

(c) (a_n) is a monotone decreasing sequence.

The purpose of this paper is to determine the spectrum of R_a on the sequence space $\overline{bv_0} \cap \ell_\infty$ under the same conditions used by Yildirim in [16].

Recently the spectra of some matrix operators have been investigated by Tripathy and Paul ([12,13]), Tripathy and Saikia [14] and others.

2. The sequence space $\overline{bv_0} \cap \ell_\infty$

A sequence (x_n) is said to be bounded variation sequence if $(\Delta x_n) \in \ell_1$, where $\Delta x_n = x_n - x_{n+1}$, for all $n \in N_0$.

A subset E of N is said to have natural density $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k)$ exists, where χ_E is the characteristic function of E. Clearly, $\delta(E) = 0$ for all finite subset E of N and $\delta(E^c) = \delta(N - E) = 1 - \delta(E)$.

A sequence (x_n) is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \varepsilon\})$. We write $x_n \overset{stat}{\to} L$ or $stat - lim\ x_n = L$.

Alternatively, a sequence (x_n) is said to be statistically convergent to L if and only if there exists a subset $K = \{k_i : i \in N\}$ of N such that $\delta(K) = 1$ and $\lim_{i\to\infty} x_{k_i} = L$.

A sequence (x_n) is said to be a sequence of statistically bounded variation if $(\Delta x_{n_i}) \in \ell_1$ such that $\delta(\{n_i : i \in N\}) = 1$, where $\Delta x_{n_i} = x_{n_i} - x_{n_i+1}$ for all $i \in N$ and we denote $(x_n) \in \overline{bv}$.

Let us consider the sequence (x_n) defined by

$$x_n = \begin{cases} n, & \text{if } n = k^2, k \in N, \\ n^{-1}, & \text{otherwise.} \end{cases}$$

Clearly $(x_n) \in \overline{bv}$.

The above example shows that $\overline{bv}$ contains some unbounded sequence too.

Now, let us consider the sequence (x_n) given by

$$x_n = \begin{cases} 1, & \text{if } n = i^2, i \in N, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly (x_n) is bounded.

Now, let $K_1 = \{n \in N : n = i^2, i \in N\}$. Then $\delta(K_1) = 0$. If $K = N - K_1$, then $\delta(K) = 1$. If $K = \{k_i : i \in N\}$, then $\lim_{i\to\infty} x_{k_i} = 0$. That is, $x_n \overset{stat}{\to} 0$ and hence, $x_n \in \overline{c_0}$. Also $\sum_i |x_{k_i} - x_{k_i+1}| (= 0) < \infty$. So $(x_n) \in \overline{bv}$. Then $(x_n) \in \overline{bv} \cap \overline{c_0} \cap \ell_\infty$.

Let us denote $\overline{bv_0} = \overline{bv} \cap \overline{c_0}$.

In this paper we will mainly deal with this type of sequence spaces. Clearly, $\overline{bv_0} \cap \ell_\infty$ is a Banach space with respect to the norm $||x|| = ||(x_n)|| = \sum_n |x_n - x_{n+1}|$.

3. Some important results

We procure the following results those will be used in establishing the results of this article.

Lemma 3.1 (Tripathy [11], Theorem 5)**.** *$x = (x_n) \in \overline{bv}$ if and only if there exists sequences (u_n) and (v_n) such that $x_n = u_n + v_n$ for all $n \in N_0$ where $(u_n) \in bv$ and $\delta(\{k \in N : v_k \neq 0\}) = 0$.*

The following result is due to Connor [2] as well as Fast [4].

Lemma 3.2. *$x = (x_n) \in \overline{c} \cap \ell_\infty$ if and only if $x = y + z$, where $y = (y_n) \in c$ and $z = (z_n) \in \delta_0 \cap \ell_\infty$, where δ_0 is the space of all sequences $z = (z_n)$ such that $\delta(\{k \in N : z_k \neq 0\}) = 0$.*

We formulate the following result in view of Lemma 3.2.

Theorem 3.3. *$x = (x_n) \in \overline{c_0} \cap \ell_\infty$ if and only if $x = y + z$, where $y = (y_n) \in c$ and $z = (z_n) \in \delta_0 \cap \ell_\infty$, where δ_0 is the space of all sequences $z = (z_n)$ such that $\delta(\{k \in N : z_k \neq 0\}) = 0$.*

4. Matrix operators on $\overline{bv_0} \cap \ell_\infty$

Theorem 4.1. *If (a) $L = \lim_{n\to\infty} (n+1)a_n$ exists, finite and nonzero,*

(b) $a_n > 0$ for all n,

(c) (a_n) is a monotone decreasing sequence,

then $R_a \in B(\overline{bv_0} \cap \ell_\infty)$.

Proof: Let $A = R_a = (a_{nk})$, where

$$a_{nk} = \begin{cases} a_n, & \text{if } k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x = (x_n) \in \overline{bv_0} \cap \ell_\infty$. Then $x = (x_n) \in \overline{bv}$.

By Lemma 3.1, $x = y + z$ where $y = (y_n) \in bv$ and $z = (z_n)$ be such that $\delta(\{k \in N : z_k \neq 0\}) = 0$. Then clearly we have $A_n(x) = A_n(y) + A_n(z)$.

Since, $y = (y_n) \in bv$ and $A = R_a \in B(bv)$ (Yildirim [17]), so it follows that $Ay = \{A_n(y)\} \in bv$.

Next, let $k \in N$ be such that $A_k(z) \neq 0$.

Now, $A_k(z) \neq 0 \Rightarrow \sum_j a_{kj} z_j \neq 0$

$\Rightarrow a_{k0}z_0 + a_{k1}z_1 + a_{k2}z_2 + ... + a_{kn}z_n + a_{k,n+1}z_{n=1} + ... \neq 0$

$\Rightarrow a_k z_0 + a_k z_1 + a_k z_2 + ... + a_k z_k \neq 0$

$\Rightarrow z_0 + z_1 + z_2 + ... + z_k \neq 0$

$\Rightarrow$ at least one $z_i \neq 0$, for $0 \leq i \leq k$.

$\Rightarrow$ either $z_k \neq 0$ or $z_k = 0$.

If $z_k = 0$, then $A_0(z) \neq 0 \Rightarrow z_0 = 0$.

But, $A_0(z) = \sum_k a_{nk} z_k = a_0 z_0$. So, $A_0(z) \neq 0 \Rightarrow a_0 z_0 \neq 0 \Rightarrow z_0 \neq 0$.

Hence, $A_k(z) \neq 0 \Rightarrow z_k \neq 0$.

This implies, $\{k : A_k(z) \neq 0\} \subseteq \{k : z_k \neq 0\}$.

Hence $\delta(\{k : A_k(z) \neq 0\}) \leq \delta(\{k : z_k \neq 0\}) = 0$ i.e. $\delta(\{k : A_k(z) \neq 0\}) = 0$. i.e. $Az = \{A_n(z)\} \in \delta_0$.

Therefore, by Lemma 3.1, $Ax = \{A_n(x)\} \in \overline{bv}$.

Again, $x = (x_n) \in \overline{bv_0} \cap \ell_\infty \Rightarrow x \in \overline{c_0} \cap \ell_\infty$.

Thus by Theorem 3.3, $x = y + z$ where $y = (y_n) \in c_0$ and $z = (z_n) \in \delta_0 \cap \ell_\infty$, where δ_0 is the space of all sequences $z = (z_n)$ such that $\delta(\{k \in N_0 : z_k \neq 0\}) = 0$.

Now, $A_n(x) = \sum_k a_{nk} x_k$

$= A_n(y) + A_n(z)$.

Since, $y = (y_n) \in c_0$ and $A = R_a \in B(c_0)$ (Yildirim [18]), so it follows that $Ay = \{A_n(y)\} \in c_0$.

Exactly as above, we can show that $\delta(\{k : A_k(z) \neq 0\}) = 0$ i.e. $Az = \{A_n(z)\} \in \delta_0$. Again, $|A_n(z)| = |\sum_k a_{nk} z_k|$

$= |a_{n0}z_0 + a_{n1}z_1 + a_{n2}z_2 + ... + a_{nn}z_n + ...|$

$= |a_n z_0 + a_n z_1 + a_n z_2 + ... + a_n z_n$

$\leq \sum_{k=0}^{n} |a_n||z_k|$

$$\leq (n+1)a_n \sup_n |z_n| \tag{2}$$

Since, $L = \lim_{n\to\infty} (n+1)a_n$, so the sequence $\{(n+1)a_n\}$ is bounded.

Hence from (2), we have $|A_n(z)| \leq \sup_n (n+1)a_n \sup_n |z_n|$ and therefore, $Az = \{A_n(z)\} \in \ell_\infty$.

Thus, $Az = \{A_n(z)\} \in \delta_0 \cap \ell_\infty$.

Therefore, by Theorem 3.3, $Ax = \{A_n(x)\} \in \overline{c_0} \cap \ell_\infty$.

Hence, $Ax = \{A_n(x)\} \in \overline{bv_0} \cap \ell_\infty$, for all $x = (x_n) \in \overline{bv_0} \cap \ell_\infty$.

Also, $||Ax|| = \sup_n \left| \sum_k a_{nk} x_k \right|$

$= \sup_n \left| \sum_{k=0}^{n} a_{nk} x_k \right|$

$= \sup_n \left| \sum_{k=0}^{n} a_n x_k \right|$

$= \sup_n (n+1)a_n \sup_n |x_n| = M$, say.

Since, $L = \lim_{n\to\infty} (n+1)a_n$, so the sequence $\{(n+1)a_n\}$ is bounded and hence, $\sup_n (n+1)a_n$ exists.

Thus, there exists a constant $M > 0$ such that $||Ax|| \leq M$, for all $x = (x_n) \in \overline{bv_0} \cap \ell_\infty$.

Hence, $A = R_a \in \overline{bv_0} \cap \ell_\infty$. □

In view of Theorem 2.1 of [16], we state the following result.

Theorem 4.2. *If the sequence* $\{(n+1)a_n\}$ *is monotone and* $\lim_{n\to\infty} (n+1)a_n = L < \infty$, *then*

$$||R_a|| = \sup_i \sum_n \left| \sum_{k=0}^{i} (a_{nk} - a_{n,k+1}) \right|.$$

5. Dual space of $\overline{bv_0} \cap \ell_\infty$

The generalized duals of some sequence spaces has recently been investigated by Chandra and Tripathy [1].

Theorem 5.1. $\left(\overline{bv_0} \cap \ell_\infty\right)^*$ *is isometrically isomorphic to bs.*

Proof: Let us define $T : \left(\overline{bv_0} \cap \ell_\infty\right)^* \to bs$ by $Tf = \left(f(e^{(0)}, f(e^{(1)}, f(e^{(2)}, ...\right)$ where $e^{(k)} = (0, 0, ..., 1, 0, ...)$, the only one appears at the k-th place.

Trivially, T is linear.

<u>T is one-to-one</u>

$Tf = 0 \Rightarrow \left(f(e^{(0)}, f(e^{(1)}, f(e^{(2)}, ...\right) = (0, 0, 0, ...)$.

$\Rightarrow f(e^{(j)}) = 0$, for all $j = 0, 1, 2, ...$

Let $x = (x_n) \in \overline{bv_0} \cap \ell_\infty$.

Then, $x = \sum\limits_{j=0}^{\infty} x_j e^{(j)}$

$\Rightarrow f(x) = \sum\limits_{j=0}^{\infty} x_j f(e^{(j)}) = 0.$

$\Rightarrow f = \overline{0}$, where $\overline{0}$ is the null operator.

Thus, $\ker(T) = \{\overline{0}\}$ and hence, T is one-to-one.

<u>T is onto</u>

Let $g = (g_n) = (g_0, g_1, g_2, ...) \in bs$.

Let us define $h : \overline{bv_0} \cap \ell_\infty \to C$ by $h(x) = \sum\limits_{k=0}^{\infty} (\Delta x)_k \sum\limits_{j=0}^{k} g_j$, where $(\Delta x)_k = x_k - x_{k+1}$.

Clearly, h is linear.

Now, $x = (x_n) = \sum\limits_{j=0}^{\infty} x_j e^{(j)}$

$\Rightarrow x = (x_0 - x_1)(1, 0, 0, ...) + (x_1 - x_2)(1, 1, 0, ...) + (x_2 - x_3)(1, 1, 1, ...) + ...$

$\Rightarrow x = \sum\limits_{k=0}^{\infty} (\Delta x)_k \sum\limits_{j=0}^{k} e^{(j)}.$

$\Rightarrow h(x) = \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=0}^{k} h(e^{(j)})$.

Then, $\sum_{j=0}^{k} h(e^{(j)}) = \sum_{j=0}^{k} g_j, k = 0, 1, 2, ...$

That is,

$h(e^{(0)}) = g_0$, $h(e^{(0)}) + h(e^{(1)}) = g_0 + g_1$, $h(e^{(0)}) + h(e^{(1)}) + h(e^{(2)}) = g_0 + g_1 + g_2$, ...

Solving, we get $h(e^{(j)}) = g_j$, for all $j = 0, 1, 2, ...$

Again, $|h(x)| = \left| \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=0}^{k} g_j \right|$

$\leq \sum_{k=0}^{\infty} |(\Delta x)_k| \left| \sum_{j=0}^{k} g_j \right|$

$\leq \sum_{k=0}^{\infty} |(\Delta x)_k| \sup_n \left| \sum_{j=0}^{n} g_j \right|$

$\leq ||x||_{\overline{bv_0} \cap \ell_\infty} ||g||_{bs}$

Hence, $h \in \left(\overline{bv_0} \cap \ell_\infty\right)^*$.

Thus, for $g = (g_n) \in bs$, there exists $h \in \left(\overline{bv_0} \cap \ell_\infty\right)^*$ such that

$Th = (h(e^{(0)}), h(e^{(1)}), h(e^{(2)}), ...) = (g_0, g_1, g_2, ...) = g.$

Therefore, T is onto.

Again, $|f(x)| = \left| f\left(\sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=0}^{k} e^{(j)} \right) \right|$

$= \left| \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=0}^{k} f\left(e^{(j)}\right) \right|$

$\leq \sum_{k=0}^{\infty} |(\Delta x)_k| \left| \sum_{j=0}^{k} f\left(e^{(j)}\right) \right|$

$\leq \sum_{k=0}^{\infty} |(\Delta x)_k| \sup_n \left| \sum_{j=0}^{n} f\left(e^{(j)}\right) \right|$

$\leq ||x||_{\overline{bv_0} \cap \ell_\infty} ||Tf||_{bs}$.

So, $||f|| \leq ||Tf||_{bs}$.

Also,

$\left|\sum_{j=0}^{n} f\left(e^{(j)}\right)\right| = \left|f\left(\sum_{j=0}^{n} e^{(j)}\right)\right| = |f(1,1,...,1,0,0,...)|$, where the first n terms are 1 each

$\leq ||f||||(1,1,...,1,0,0,...)||$

$\leq ||f||$.

$\Rightarrow \sup_n \left|\sum_{j=0}^{n} f\left(e^{(j)}\right)\right| \leq ||f||$.

$\Rightarrow ||Tf||_{bs} \leq ||f||$.

Therefore, $||Tf||_{bs} = ||f||$.

Hence, $\left(\overline{bv_0} \cap \ell_\infty\right)^*$ is isometrically isomorphic to bs. □

6. Spectra of the Rhaly operator on the sequence space $\overline{bv_0} \cap \ell_\infty$

Theorem 6.1. *If the sequence $\{(n+1)a_n\}$ is monotone and $\lim_{n\to\infty}\{(n+1)a_n\} = L < \infty$, then $S \cap (2L, \infty) \subseteq \sigma_p(R_a, \overline{bv_0} \cap \ell_\infty$, where $S = \{a_n : n = 0, 1, 2, ...\}$.*

Proof: Since $bv \subseteq \overline{bv}$, $c_0 \subseteq \overline{c_0}$, therefore

$bv \cap c_0 \subseteq \overline{bv} \cap \overline{c_0}$.

$\Rightarrow bv_0 \subseteq \overline{bv_0}$.

$\Rightarrow bv_0 \cap \ell_\infty \subseteq \overline{bv_0} \cap \ell_\infty$.

$\Rightarrow bv_0 \subseteq \overline{bv_0} \cap \ell_\infty$ (since $bv_0 \cap \ell_\infty = bv_0$).

Therefore, $\sigma_p(R_a, \overline{bv_0}) \subseteq \sigma_p(R_a, \overline{bv_0} \cap \ell_\infty)$.

But, by Theorem 2.2 of Yildirim [16],we have $S \cap (2L, \infty) \subseteq \sigma_p(R_a, bv_0)$.

Hence, $S \cap (2L, \infty) \subseteq \sigma_p(R_a, \overline{bv_0} \cap \ell_\infty)$. □

Lemma 6.2. *If the sequence $\{(n+1)a_n\}$ is monotone and $\lim_{n\to\infty}(n+1)a_n = L < \infty$, then R_a^*, the adjoint operator of R_a, is the transpose of the matrix R_a on $\overline{bv_0} \cap \ell_\infty$ and $R_a^* \in ((\overline{bv_0} \cap \ell_\infty)^* \cong bs)$.*

Proof: We establish the proof as proved by Yildirim ([16], Lemma 2.3). Since $\overline{bv_0} \cap \ell_\infty$ is an AK-space and $(\overline{bv_0} \cap \ell_\infty)^* \cong bs$, therefore from Wilansky ([15], p.266), we have

$$R_a^* = R_a^t = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_1 & a_2 & a_3 & \dots \\ 0 & 0 & a_2 & a_3 & \dots \\ 0 & 0 & 0 & a_3 & \dots \\ . & . & . & . & \dots \end{pmatrix},$$

Since $\overline{bv_0} \cap \ell_\infty$ is a Banach space, so $||R_a||_{(\overline{bv_0} \cap \ell_\infty)} = ||R_a^*||_{(\overline{bv_0} \cap \ell_\infty)^*} = ||R_a^t||_{bs}$. Hence, from Theorem 4.1 and Theorem 4.2, $R_a^t \in B((\overline{bv_0} \cap \ell_\infty)^*)$. □

Lemma 6.3. *Yildirim ([16], Lemma 2.4). Let* $0 < L = \lim_{n\to\infty}(n+1)a_n < \infty$ *and* $Z_n = \prod_{\nu=0}^{n}\left(1 - \frac{a_\nu}{\lambda}\right), \lambda \neq 0, \lambda \in C$. *Then the partial sums of* $\sum_{\nu=1}^{\infty} Z_\nu$ *are bounded if and only if* $LRe\frac{1}{\lambda} \geq 1, \lambda \neq L$.

Theorem 6.4. *If the sequence* $\{(n+1)a_n\}$ *is monotone and* $0 < L = \lim_{n\to\infty}(n+1)a_n < \infty$, *then* $S \cup \left(\{\lambda : \left|\lambda - \frac{L}{2}\right| \leq \frac{L}{2}\} - \{0\}\right) \subset \sigma_p(R_a^*, (\overline{bv_0} \cap \ell_\infty)^* \cong bs)$.

Proof: From Yildirim ([16], Lemma 2.5), we have

$$S \cup \left(\{\lambda : \left|\lambda - \tfrac{L}{2}\right| \leq \tfrac{L}{2}\} - \{0\}\right) \subset \sigma_p(R_a^*, bv_0^* \cong bs).$$

By Theorem 5.1, we have $(\overline{bv_0} \cap \ell_\infty)^* \cong bs)$, therefore

$$S \cup \left(\{\lambda : \left|\lambda - \tfrac{L}{2}\right| \leq \tfrac{L}{2}\} - \{0\}\right) \subset \sigma_p(R_a^*, (\overline{bv_0} \cap \ell_\infty)^* \cong bs).$$ □

Theorem 6.5. *If the sequence* $\{(n+1)a_n\}$ *is monotone and* $0 < L = \lim_{n\to\infty}(n+1)a_n < \infty$, *then* $\sigma(R_a, \overline{bv_0} \cap \ell_\infty) = S \cup \{\lambda : \left|\lambda - \frac{L}{2}\right| \leq \frac{L}{2}\}$.

Proof: From Yildirim ([16], Lemma 2.8), we have

$$S \cup \left(\{\lambda : \left|\lambda - \tfrac{L}{2}\right| \leq \tfrac{L}{2}\}\right) \subseteq \sigma_p(R_a^*, {bv_0}^* \cong bs).$$

By Theorem 5.1, we have $(\overline{bv_0} \cap \ell_\infty)^* \cong bs$, therefore

$$S \cup \{\lambda : \left|\lambda - \tfrac{L}{2}\right| \leq \tfrac{L}{2}\} \subseteq \sigma_p(R_a^*, (\overline{bv_0} \cap \ell_\infty)^* \cong bs).$$

But, $\sigma_p(R_a^*, (\overline{bv_0} \cap \ell_\infty)^*) \subseteq \sigma(R_a^*, (\overline{bv_0} \cap \ell_\infty)^*) = \sigma(R_a, \overline{bv_0} \cap \ell_\infty)$.

Therefore, $S \cup \left(\{\lambda : \left|\lambda - \frac{L}{2}\right| \leq \frac{L}{2}\}\right) \subseteq \sigma(R_a, \overline{bv_0} \cap \ell_\infty)$.

Proceeding exactly as in the theorem 2.8 of Yildirim [16], we can show that

$\sigma(R_a, \overline{bv_0} \cap \ell_\infty) \subseteq S \cup \{\lambda : |\lambda - \frac{L}{2}| \leq \frac{L}{2}\}$.

Hence the result. □

Conclusion. In this article we have determined the spectra of the Rally operator on the class of bounded statistically null bounded variation sequences. This is a new direction and the work can be applied for investigating spectra of other matrix operators too.

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