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Biharmonic Constant $\mathbf{\Pi}_1-$Slope Curves according to Type-2 Bishop Frame in Heisenberg Group Heis^3

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ABSTRACT: In this paper, we introduce constant $\mathbf{\Pi}_1-$ slope curves according to type-2 Bishop frame in the Heisenberg group Heis^3. We characterize the biharmonic constant $\mathbf{\Pi}_1-$ slope curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the Heisenberg group Heis^3. Additionally, we illustrate our main theorem.

Key Words: Biharmonic curve, type-2 Bishop frame, Heisenberg group.

Contents

1. Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \mathrm{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi,$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps, [5,6,7].

This study is organised as follows: Firstly, we introduce constant $\mathbf{\Pi}_1-$ slope curves according to type-2 Bishop frame in the Heisenberg group Heis^3. We characterize the biharmonic constant $\mathbf{\Pi}_1-$ slope curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the Heisenberg group Heis^3. Additionally, we illustrate our main theorem.

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2. The Heisenberg Group Heis^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}) \tag{2.1}$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz-xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1=\frac{\partial}{\partial x},\quad \mathbf{e}_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\quad \mathbf{e}_3=\frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,\quad [\mathbf{e}_2,\mathbf{e}_3]=[\mathbf{e}_3,\mathbf{e}_1]=0$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1.$$

We obtain

$$\begin{aligned}
\nabla_{\mathbf{e}_1}\mathbf{e}_1 &= \nabla_{\mathbf{e}_2}\mathbf{e}_2=\nabla_{\mathbf{e}_3}\mathbf{e}_3=0,\\
\nabla_{\mathbf{e}_1}\mathbf{e}_2 &= -\nabla_{\mathbf{e}_2}\mathbf{e}_1=\frac{1}{2}\mathbf{e}_3,\\
\nabla_{\mathbf{e}_1}\mathbf{e}_3 &= \nabla_{\mathbf{e}_3}\mathbf{e}_1=-\frac{1}{2}\mathbf{e}_2,\\
\nabla_{\mathbf{e}_2}\mathbf{e}_3 &= \nabla_{\mathbf{e}_3}\mathbf{e}_2=\frac{1}{2}\mathbf{e}_1.
\end{aligned}$$

3. Biharmonic Constant $\mathbf{\Pi}_1-$Slope Curves according to New Type-2 Bishop Frame in Heisenberg Group Heis^3

Assume that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N},\\
\nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T}+\tau\mathbf{B},\\
\nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N},
\end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned}
g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{N},\mathbf{N})=1,\ g(\mathbf{B},\mathbf{B})=1,\\
g(\mathbf{T},\mathbf{N}) &= g(\mathbf{T},\mathbf{B})=g(\mathbf{N},\mathbf{B})=0.
\end{aligned}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{M}_1,\mathbf{M}_1) = 1,\ g(\mathbf{M}_2,\mathbf{M}_2) = 1, \\ g(\mathbf{T},\mathbf{M}_1) &= g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0. \end{aligned}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\mathfrak{U}(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \mathfrak{U}'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\cos\mathfrak{U}(s), \\ k_2 &= \kappa(s)\sin\mathfrak{U}(s). \end{aligned}$$

Theorem 3.1. *$\gamma : I \longrightarrow Heis^3$ is a unit speed biharmonic curve with Bishop frame if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= \textit{constant} = C \neq 0, \\ k_1'' - Ck_1 &= k_1\left[\frac{1}{4} - \left(M_2^3\right)^2\right] - k_2M_1^3M_2^3, \\ k_2'' - Ck_2 &= k_1M_1^3M_2^3 + k_2\left[\frac{1}{4} - \left(M_1^3\right)^2\right]. \end{aligned} \tag{3.3}$$

Let γ be a unit speed regular curve in Heis^3 and (3.1) be its Frenet–Serret frame. Let us express a relatively parallel adapted frame:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{\Pi}_1 &= -\epsilon_1\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{\Pi}_2 &= -\epsilon_2\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \epsilon_1\mathbf{\Pi}_1 + \epsilon_2\mathbf{\Pi}_2, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} g(\mathbf{B},\mathbf{B}) &= 1,\ g(\mathbf{\Pi}_1,\mathbf{\Pi}_1) = 1,\ g(\mathbf{\Pi}_2,\mathbf{\Pi}_2) = 1, \\ g(\mathbf{B},\mathbf{\Pi}_1) &= g(\mathbf{B},\mathbf{\Pi}_2) = g(\mathbf{\Pi}_1,\mathbf{\Pi}_2) = 0. \end{aligned}$$

We shall call this frame as Type-2 Bishop Frame. In order to investigate this new frame's relation with Frenet–Serret frame, first we write

$$\tau = \sqrt{\epsilon_1^2 + \epsilon_2^2}. \tag{3.5}$$

The relation matrix between Frenet–Serret and type-2 Bishop frames can be expressed

$$\mathbf{T} = \sin\mathfrak{A}(s)\,\mathbf{\Pi}_1 - \cos\mathfrak{A}(s)\,\mathbf{\Pi}_2,$$
$$\mathbf{N} = \cos\mathfrak{A}(s)\,\mathbf{\Pi}_1 + \sin\mathfrak{A}(s)\,\mathbf{\Pi}_2,$$
$$\mathbf{B} = \mathbf{B}.$$

So by (3.5), we may express

$$\begin{aligned} \epsilon_1 &= -\tau\cos\mathfrak{A}(s), \\ \epsilon_2 &= -\tau\sin\mathfrak{A}(s). \end{aligned}$$

By this way, we conclude

$$\mathfrak{A}(s) = \arctan\frac{\epsilon_2}{\epsilon_1}.$$

The frame $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}\}$ is properly oriented, and τ and $\mathfrak{A}(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve γ. We shall call the set $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}, \epsilon_1, \epsilon_2\}$ as type-2 Bishop invariants of the curve γ, [19].

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{\Pi}_1 &= \pi_1^1\mathbf{e}_1 + \pi_1^2\mathbf{e}_2 + \pi_1^3\mathbf{e}_3, \\ \mathbf{\Pi}_2 &= \pi_2^1\mathbf{e}_1 + \pi_2^2\mathbf{e}_2 + \pi_2^3\mathbf{e}_3. \\ \mathbf{B} &= B^1\mathbf{e}_1 + B^2\mathbf{e}_2 + B^3\mathbf{e}_3, \end{aligned} \tag{3.6}$$

Theorem 3.2. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_1-$slope curves according to type-2 Bishop frame in the $Heis^3$. Then, the parametric equations of γ*

$$\begin{aligned} x(s) &= \frac{\sin\mathfrak{A}\cos[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1-\kappa)} - \frac{\sin\mathfrak{A}\cos[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} \\ &\quad -\frac{\cos\mathfrak{A}\sin[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{R}_1-\kappa)} - \frac{\cos\mathfrak{A}\sin[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} + \mathcal{L}_3, \end{aligned}$$

$$\begin{aligned} y(s) &= \frac{\cos\mathfrak{A}\cos[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1-\kappa)} + \frac{\cos\mathfrak{A}\cos[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} \\ &\quad +\frac{\sin\mathfrak{A}\sin[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{R}_1-\kappa)} - \frac{\sin\mathfrak{A}\sin[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} + \mathcal{L}_4, \end{aligned}$$

$$
\begin{aligned}
z(s) \quad = \quad & \frac{\sin^2 \mathfrak{A}}{2\left(\mathcal{L}_1-\kappa\right) 8\mathcal{L}_1\left(\mathcal{L}_1-\kappa\right)\kappa}[-\mathcal{L}_1(\mathcal{L}_1-\kappa)\sin[2\kappa s]+\kappa((-\mathcal{L}_1+\kappa) \\
& \sin[2(\mathcal{L}_2+\mathcal{L}_1 s)]+\mathcal{L}_1(2\mathcal{L}_1 s-2\kappa s+\sin[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)]))] \\
& -\frac{1}{2\left(\mathcal{L}_1-\kappa\right) 8\mathcal{L}_1\left(-\mathcal{L}_1+\kappa\right)\kappa}\sin\mathfrak{A}\cos\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\cos[2\kappa s] \\
& +\kappa((\mathcal{L}_1-\kappa)\cos[2(\mathcal{L}_2+\mathcal{L}_1 s)]+\mathcal{L}_1\cos[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)])] \\
& -\frac{1}{2\left(\mathcal{L}_1+\kappa\right) 8\mathcal{L}_1\left(\mathcal{L}_1-\kappa\right)\kappa}\sin^2\mathfrak{A}[-\mathcal{L}_1(\mathcal{L}_1-\kappa)\sin[2\kappa s]+\kappa((-\mathcal{L}_1 \\
& +\kappa)\sin[2(\mathcal{L}_2+\mathcal{L}_1 s)]+\mathcal{L}_1(2\mathcal{L}_1 s-2\kappa s+\sin[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)]))] \\
& +\frac{1}{2\left(\mathcal{L}_1+\kappa\right) 8\mathcal{L}_1\left(-\mathcal{L}_1+\kappa\right)\kappa}\sin\mathfrak{A}\cos\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\cos[2\kappa s] \\
& +\kappa((\mathcal{L}_1-\kappa)\cos[2(\mathcal{L}_2+\mathcal{L}_1 s)]+\mathcal{L}_1\cos[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)])] \\
& -\frac{1}{2\left(\mathcal{L}_1-\kappa\right) 8\mathcal{L}_1\left(-\mathcal{L}_1+\kappa\right)\kappa}\cos\mathfrak{A}\sin\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\cos[2\kappa s] \\
& +\kappa((-\mathcal{L}_1+\kappa)\cos[2(\mathcal{L}_2+\mathcal{L}_1 s)]+\mathcal{L}_1\cos[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)])] \\
& +\frac{1}{2\left(\mathcal{L}_1-\kappa\right) 8\mathcal{L}_1\left(\mathcal{L}_1-\kappa\right)\kappa}\cos^2\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\sin[2\kappa s]-\kappa((\mathcal{L}_1 \\
& -\kappa)\sin[2(\mathcal{L}_2+\mathcal{L}_1 s)]+\mathcal{L}_1(-2\mathcal{L}_1 s+2\kappa s+\sin[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)]))] \\
& -\frac{1}{2\left(\mathcal{L}_1+\kappa\right) 8\mathcal{L}_1\left(-\mathcal{L}_1+\kappa\right)\kappa}\cos\mathfrak{A}\sin\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\cos[2\kappa s] \\
& +\kappa((-\mathcal{L}_1+\kappa)\cos[2(\mathcal{L}_2+\mathcal{L}_1 s)]+\mathcal{L}_1\cos[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)])] \\
& +\frac{1}{2\left(\mathcal{L}_1+\kappa\right) 8\mathcal{L}_1\left(\mathcal{L}_1-\kappa\right)\kappa}\cos^2\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\sin[2\kappa s]-\kappa((\mathcal{L}_1 \\
& -\kappa)\sin[2(\mathcal{L}_2+\mathcal{L}_1 s)]+\mathcal{L}_1(-2\mathcal{L}_1 s+2\kappa s+\sin[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)]))] \\
& +\mathcal{L}_3\sin\mathfrak{A}\frac{\sin\left[\left(\mathcal{L}_1-\kappa\right)s+\mathcal{L}_2\right]}{2\left(\mathcal{L}_1-\kappa\right)}-\mathcal{L}_3\sin\mathfrak{A}\frac{\sin\left[\left(\mathcal{L}_1+\kappa\right)s+\mathcal{L}_2\right]}{2\left(\mathcal{L}_1+\kappa\right)} \\
& +\mathcal{L}_3\cos\mathfrak{A}\frac{\cos\left[\left(\mathcal{L}_1-\kappa\right)s+\mathcal{L}_2\right]}{2\left(\mathcal{L}_1-\kappa\right)}+\mathcal{L}_3\cos\mathfrak{A}\frac{\cos\left[\left(\mathcal{L}_1-\kappa\right)s+\mathcal{L}_2\right]}{2\left(\mathcal{L}_1-\kappa\right)} \\
& -\frac{1}{\kappa}\cos\left[\kappa s\right]\cos\mathfrak{A}+\frac{1}{\kappa}\sin\left[\kappa s\right]\sin\mathfrak{A}+\mathcal{L}_4,
\end{aligned}
$$

where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are constants of integration.

Proof: The vector $\mathbf{\Pi}_1$ is a unit vector, we have the following equation

$$\mathbf{\Pi}_1 = \sin\mathfrak{A}\cos\left[\mathcal{L}_1 s+\mathcal{L}_2\right]\mathbf{e}_1+\sin\mathfrak{A}\sin\left[\mathcal{L}_1 s+\mathcal{L}_2\right]\mathbf{e}_2+\cos\mathfrak{A}\mathbf{e}_3, \tag{3.7}$$

where $\mathcal{L}_1, \mathcal{L}_2 \in \mathbb{R}$.

Then by type-2 Bishop formulas (3.4) and (2.1), we have

$$\mathbf{\Pi}_2 = \cos\mathfrak{A}\cos\left[\mathcal{L}_1 s+\mathcal{L}_2\right]\mathbf{e}_1+\cos\mathfrak{A}\sin\left[\mathcal{L}_1 s+\mathcal{L}_2\right]\mathbf{e}_2-\sin\mathfrak{A}\mathbf{e}_3.$$

Applying above equation and (3.9), we get

$$\mathbf{B} = -\sin\left[\mathcal{L}_1 s + \mathcal{L}_2\right]\mathbf{e}_1 + \cos\left[\mathcal{L}_1 s + \mathcal{L}_2\right]\mathbf{e}_2.$$

Then, a combination of these equations with the second equation of (5.1) would give us

$$\begin{aligned}\mathbf{T} &= [\sin[\kappa s]\sin\mathfrak{A}\cos[\mathcal{L}_1 s + \mathcal{L}_2] - \cos[\kappa s]\cos\mathfrak{A}\cos[\mathcal{L}_1 s + \mathcal{L}_2]]\mathbf{e}_1 \\ &\quad +[\sin[\kappa s]\sin\mathfrak{A}\sin[\mathcal{L}_1 s + \mathcal{L}_2] - \cos[\kappa s]\cos\mathfrak{A}\sin[\mathcal{L}_1 s + \mathcal{L}_2]]\mathbf{e}_2 \\ &\quad +[\sin[\kappa s]\cos\mathfrak{A} + \cos[\kappa s]\sin\mathfrak{A}]\mathbf{e}_3. \end{aligned} \tag{3.8}$$

From (2.2) and (3.8), we have

$$\begin{aligned} x(s) &= \frac{\sin\mathfrak{A}\cos[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1-\kappa)} - \frac{\sin\mathfrak{A}\cos[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} \\ &\quad -\frac{\cos\mathfrak{A}\sin[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathfrak{R}_1-\kappa)} - \frac{\cos\mathfrak{A}\sin[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} + \mathcal{L}_3, \end{aligned}$$

where $\mathcal{L}_3$ is constant of integration.

Also,

$$\begin{aligned} y(s) &= \frac{\cos\mathfrak{A}\cos[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1-\kappa)} + \frac{\cos\mathfrak{A}\cos[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} \\ &\quad +\frac{\sin\mathfrak{A}\sin[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1-\kappa)} - \frac{\sin\mathfrak{A}\sin[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} + \mathcal{L}_4, \end{aligned}$$

where $\mathcal{L}_4$ is constant of integration.

Again, by combining (2.2) and (3.8) we have

$$\begin{aligned} \frac{dz}{ds} &= \frac{1}{2(\mathcal{L}_1-\kappa)}\sin^2\mathfrak{A}\cos[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]\sin[\kappa s]\sin[\mathcal{L}_1 s+\mathcal{L}_2] \\ &\quad -\frac{1}{2(\mathcal{L}_1-\kappa)}\sin\mathfrak{A}\cos\mathfrak{A}\cos[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]\cos[\kappa s]\sin[\mathcal{L}_1 s+\mathcal{L}_2] \\ &\quad -\frac{1}{2(\mathcal{L}_1+\kappa)}\sin^2\mathfrak{A}\cos[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]\sin[\kappa s]\sin[\mathcal{L}_1 s+\mathcal{L}_2] \\ &\quad +\frac{1}{2(\mathcal{L}_1+\kappa)}\sin\mathfrak{A}\cos\mathfrak{A}\cos[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]\cos[\kappa s]\sin[\mathfrak{R}_1 s+\mathcal{L}_2] \\ &\quad -\frac{1}{2(\mathcal{L}_1-\kappa)}\cos\mathfrak{A}\sin\mathfrak{A}\sin[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]\sin[\kappa s]\sin[\mathcal{L}_1 s+\mathcal{L}_2] \\ &\quad +\frac{1}{2(\mathcal{L}_1-\kappa)}\cos^2\mathfrak{A}\sin[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]\cos[\kappa s]\sin[\mathfrak{R}_1 s+\mathcal{L}_2] \\ &\quad -\frac{1}{2(\mathcal{L}_1+\kappa)}\cos\mathfrak{A}\sin\mathfrak{A}\sin[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]\sin[\kappa s]\sin[\mathcal{L}_1 s+\mathcal{L}_2] \\ &\quad +\frac{1}{2(\mathcal{L}_1+\kappa)}\cos^2\mathfrak{A}\sin[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]\cos[\kappa s]\sin[\mathcal{L}_1 s+\mathcal{L}_2] \\ &\quad +\mathcal{L}_3[\sin[\kappa s]\sin\mathfrak{A}\sin[\mathcal{L}_1 s+\mathcal{L}_2] - \cos[\kappa s]\cos\mathfrak{A}\sin[\mathcal{L}_1 s+\mathcal{L}_2]] \\ &\quad +[\sin[\kappa s]\cos\mathfrak{A} + \cos[\kappa s]\sin\mathfrak{A}]. \end{aligned}$$

Integrating both sides, we have theorem. Thus, the proof of theorem is completed. □

Theorem 3.3. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_1-$slope curves according to type-2 Bishop frame in the $Heis^3$. Then, the position vector of γ is*

$$\begin{aligned}
\gamma(s) = & [\frac{\sin\mathfrak{A}\cos[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1-\kappa)} - \frac{\sin\mathfrak{A}\cos[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} \\
& - \frac{\cos\mathfrak{A}\sin[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathfrak{R}_1-\kappa)} - \frac{\cos\mathfrak{A}\sin[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} + \mathcal{L}_3]\mathbf{e}_1 \\
& +[\frac{\cos\mathfrak{A}\cos[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1-\kappa)} + \frac{\cos\mathfrak{A}\cos[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} \\
& + \frac{\sin\mathfrak{A}\sin[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathfrak{R}_1-\kappa)} - \frac{\sin\mathfrak{A}\sin[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} + \mathcal{L}_4]\mathbf{e}_2 \\
& +[-[\frac{\sin\mathfrak{A}\cos[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1-\kappa)} - \frac{\sin\mathfrak{A}\cos[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} \\
& - \frac{\cos\mathfrak{A}\sin[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathfrak{R}_1-\kappa)} - \frac{\cos\mathfrak{A}\sin[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} + \mathcal{L}_3] \\
& [\frac{\cos\mathfrak{A}\cos[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1-\kappa)} + \frac{\cos\mathfrak{A}\cos[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} \\
& + \frac{\sin\mathfrak{A}\sin[(\mathcal{L}_1-\kappa)s+\mathcal{L}_2]}{2(\mathfrak{R}_1-\kappa)} - \frac{\sin\mathfrak{A}\sin[(\mathcal{L}_1+\kappa)s+\mathcal{L}_2]}{2(\mathcal{L}_1+\kappa)} + \mathcal{L}_4] \\
& + \frac{\sin^2\mathfrak{A}}{2(\mathcal{L}_1-\kappa)8\mathcal{L}_1(\mathcal{L}_1-\kappa)\kappa}[-\mathcal{L}_1(\mathcal{L}_1-\kappa)\sin[2\kappa s] + \kappa((-\mathcal{L}_1+\kappa) \\
& \sin[2(\mathcal{L}_2+\mathcal{L}_1 s)] + \mathcal{L}_1(2\mathcal{L}_1 s - 2\kappa s + \sin[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)]))] \\
& - \frac{1}{2(\mathcal{L}_1-\kappa)8\mathcal{L}_1(-\mathcal{L}_1+\kappa)\kappa}\sin\mathfrak{A}\cos\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\cos[2\kappa s] \\
& +\kappa((\mathcal{L}_1-\kappa)\cos[2(\mathcal{L}_2+\mathcal{L}_1 s)] + \mathcal{L}_1\cos[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)])] \\
& - \frac{1}{2(\mathcal{L}_1+\kappa)8\mathcal{L}_1(\mathcal{L}_1-\kappa)\kappa}\sin^2\mathfrak{A}[-\mathcal{L}_1(\mathcal{L}_1-\kappa)\sin[2\kappa s] + \kappa((-\mathcal{L}_1 \\
& +\kappa)\sin[2(\mathcal{L}_2+\mathcal{L}_1 s)] + \mathcal{L}_1(2\mathcal{L}_1 s - 2\kappa s + \sin[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)]))] \\
& + \frac{1}{2(\mathcal{L}_1+\kappa)8\mathcal{L}_1(-\mathcal{L}_1+\kappa)\kappa}\sin\mathfrak{A}\cos\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\cos[2\kappa s] \\
& +\kappa((\mathcal{L}_1-\kappa)\cos[2(\mathcal{L}_2+\mathcal{L}_1 s)] + \mathcal{L}_1\cos[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)])] \\
& - \frac{1}{2(\mathcal{L}_1-\kappa)8\mathcal{L}_1(-\mathcal{L}_1+\kappa)\kappa}\cos\mathfrak{A}\sin\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\cos[2\kappa s] \\
& +\kappa((-\mathcal{L}_1+\kappa)\cos[2(\mathcal{L}_2+\mathcal{L}_1 s)] + \mathcal{L}_1\cos[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)])] \\
& + \frac{1}{2(\mathcal{L}_1-\kappa)8\mathcal{L}_1(\mathcal{L}_1-\kappa)\kappa}\cos^2\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\sin[2\kappa s] - \kappa((\mathcal{L}_1 \\
& -\kappa)\sin[2(\mathcal{L}_2+\mathcal{L}_1 s)] + \mathcal{L}_1(-2\mathcal{L}_1 s + 2\kappa s + \sin[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)]))]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2\left(\mathcal{L}_1+\kappa\right)8\mathcal{L}_1\left(-\mathcal{L}_1+\kappa\right)\kappa}\cos\mathfrak{A}\sin\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\cos[2\kappa s]\\
&+\kappa((-\mathcal{L}_1+\kappa)\cos[2(\mathcal{L}_2+\mathcal{L}_1 s)]+\mathcal{L}_1\cos[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)])]\\
&+\frac{1}{2\left(\mathcal{L}_1+\kappa\right)8\mathcal{L}_1\left(\mathcal{L}_1-\kappa\right)\kappa}\cos^2\mathfrak{A}[\mathcal{L}_1(\mathcal{L}_1-\kappa)\sin[2\kappa s]-\kappa((\mathcal{L}_1\\
&-\kappa)\sin[2(\mathcal{L}_2+\mathcal{L}_1 s)]+\mathcal{L}_1(-2\mathcal{L}_1 s+2\kappa s+\sin[2(\mathcal{L}_2+\mathcal{L}_1 s-\kappa s)]))]\\
&+\mathcal{L}_3\sin\mathfrak{A}\frac{\sin\left[\left(\mathcal{L}_1-\kappa\right)s+\mathcal{L}_2\right]}{2\left(\mathcal{L}_1-\kappa\right)}-\mathcal{L}_3\sin\mathfrak{A}\frac{\sin\left[\left(\mathcal{L}_1+\kappa\right)s+\mathcal{L}_2\right]}{2\left(\mathcal{L}_1+\kappa\right)}\\
&+\mathcal{L}_3\cos\mathfrak{E}\frac{\cos\left[\left(\mathcal{L}_1-\kappa\right)s+\mathcal{L}_2\right]}{2\left(\mathcal{L}_1-\kappa\right)}+\mathcal{L}_3\cos\mathfrak{A}\frac{\cos\left[\left(\mathcal{L}_1-\kappa\right)s+\mathcal{L}_2\right]}{2\left(\mathcal{L}_1-\kappa\right)}\\
&-\frac{1}{\kappa}\cos\left[\kappa s\right]\cos\mathfrak{A}+\frac{1}{\kappa}\sin\left[\kappa s\right]\sin\mathfrak{A}+\mathcal{L}_4]\mathbf{e}_3,
\end{aligned}
$$

where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ *are constants of integration.*

Proof: Substituting (2.2) in Theorem 3.2, we have above equation. This completes the proof. □

We can use Mathematica in Theorem 3.2, yields

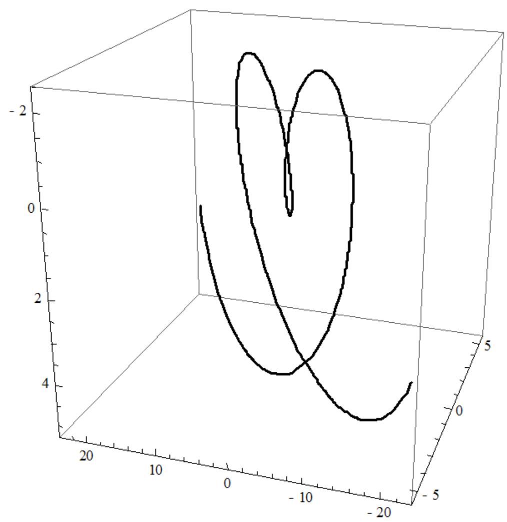

Fig. 1

References

1. L. R. Bishop: *There is More Than One Way to Frame a Curve,* Amer. Math. Monthly 82 (3) (1975) 246-251.

2. B. Bükcü, M.K. Karacan: *Special Bishop motion and Bishop Darboux rotation axis of the space curve,* J. Dyn. Syst. Geom. Theor. 6 (1) (2008) 27–34.

3. B. Bükcü, M.K. Karacan, *The slant helices according to Bishop frame,* Int. J. Math. Comput. Sci. 3 (2) (2009) 67–70.

4. R. Caddeo, C. Oniciuc, P. Piu: *Explicit formulas for non-geodesic biharmonic curves of the Heisenberg group,* Rend. Sem,. Mat. Univ. Politec. Torino 62 (2004), 265-278.

5. J. Eells and L. Lemaire: *A report on harmonic maps,* Bull. London Math. Soc. 10 (1978), 1–68.

6. J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109–160.

7. G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130–144.

8. T. Körpınar, E. Turhan, V. Asil: *Biharmonic B-General Helices with Bishop Frame In The Heisenberg Group $Heis^3$,* World Applied Sciences Journal 14 (10) (2010), 1565-1568.

9. T. Körpınar, E. Turhan, V. Asil: *Tangent Bishop spherical images of a biharmonic B-slant helix in the Heisenberg group $Heis^3$*, Iranian Journal of Science & Technology, A (4) (2011), 265-271.

10. T. Körpınar and E. Turhan: *Bertrand mate of biharmonic curves in the special three-dimensional Kenmotsu manifold K with η-parallel Ricci tensor,* TWMS Jour. Pure Appl. Math., 2 (2) (2011), 247-255

11. T. Körpınar and E. Turhan: *Position Vectors of Smarandache D-Tg Curves of Timelike Biharmonic D-Helices According to Darboux Frame on Non Degenerate Timelike Surfaces in the Lorentzian Heisenberg Group H*, Advanced Modeling and Optimization, 14(2) (2012), 417-424.

12. T. Körpınar and E. Turhan: *Integral Equations of Biharmonic Constant Π_1-Slope Curves according to New Type-2 Bishop Frame in Sol Space $\mathfrak{SOL}^3$,* Bol. Soc. Paran. Mat., 31 2 (2013), 205–212.

13. Y. Ou and Z. Wang: *Linear Biharmonic Maps into Sol, Nil and Heisenberg Spaces,* Mediterr. j. math. 5 (2008), 379–394

14. S. Rahmani: *Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois*, Journal of Geometry and Physics 9 (1992), 295-302.

15. D. J. Struik: *Lectures on Classical Differential Geometry,* Dover, New-York, 1988.

16. E. Turhan and T. Körpınar: *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group $Heis^3$*, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

17. E. Turhan and T. Körpınar: *Parametric equations of general helices in the sol space,* Bol. Soc. Paran. Mat., 31 (1) (2013), 99–104.

18. E. Turhan and T. Körpınar: *Position Vector Of Biharmonic Curves in the 3-Dimensional Locally ϕ-Quasiconformally Symmetric Sasakian Manifold,* Electronic Journal of Theoretical Physics, 8 (25) (2011), 169-176.

19. S. Yılmaz and M. Turgut: *A new version of Bishop frame and an application to spherical images,* J. Math. Anal. Appl., 371 (2010), 764-776.

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