

STRONG INSERTION OF A γ -CONTINUOUS FUNCTION¹

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Abstract

Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a γ -continuous function between two comparable real-valued functions. In this paper, is given a sufficient condition for the weak γ -insertion property. Also for a space with the weak γ -insertion property, we give necessary and sufficient conditions for the space to have the strong γ -insertion property. Several insertion theorems are obtained as corollaries of these results.

Keywords: Strong insertion, Strong binary relation, Preopen set, Semi-open set, γ -open set, Lower cut set.

1 Introduction

The concept of a preopen set in a topological space was introduced by H. H. Corson and E. Michael in 1964 [4]. A subset A of a topological space (X, τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq \text{Int}(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(\text{Int}(A)) \subseteq A$. The term ,preopen, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb [12], while the concept of a , locally dense, set was introduced by H. H. Corson and E. Michael [4].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [11]. A subset A of a topological space (X, τ) is called *semi-open* [11] if $A \subseteq Cl(\text{Int}(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $\text{Int}(Cl(A)) \subseteq A$.

Recall that a subset A of a topological space (X, τ) is called γ -open if $A \cap S$ is preopen, whenever S is preopen [1]. A set A is called γ -closed if its

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complement is γ -open or equivalently if $A \cup S$ is preclosed, whenever S is preclosed. we have that if a set is γ -open then it is semi-open and preopen.

Recall that a real-valued function f defined on a topological space X is called A -continuous [14] if the preimage of every open subset of \mathbb{R} belongs to A , where A is a collection of subset of X . Most of the definitions of function used throughout this paper are consequences of the definition of A -continuity. However, for unknown concepts the reader may refer to [5, 6].

Hence, a real-valued function f defined on a topological space X is called precontinuous (resp. semi-continuous or γ -continuous) if the preimage of every open subset of \mathbb{R} is preopen (resp. semi-open or γ -open) subset of X .

Precontinuity was called by V. Ptak nearly continuity [15]. Nearly continuity or precontinuity is known also as almost continuity by T. Husain [7]. Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922 [2].

Results of Katětov [8, 9] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give necessary and sufficient conditions for the strong insertion of a γ -continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X , we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X .

The following definitions are modifications of conditions considered in [10].

A property P defined relative to a real-valued function on a topological space is a γ -property provided that any constant function has property P and provided that the sum of a function with property P and any γ -continuous function also has property P . If P_1 and P_2 are γ -property, the following terminology is used: (i) A space X has the *weak γ -insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2 , then there exists a γ -continuous function h such that $g \leq h \leq f$. (ii) A space X has the *strong γ -insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2 , then there exists a γ -continuous function h such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X , then $g(x) < h(x) < f(x)$.

In this paper, is given a sufficient condition for the weak γ -insertion property. Also for a space with the weak γ -insertion property, we give necessary and sufficient conditions for the space to have the strong γ -insertion

property. Several insertion theorems are obtained as corollaries of these results.

2 The Main Result

Before giving a sufficient condition for insertability of a γ -continuous function, the necessary definitions and terminology are stated.

The abbreviations pc , sc and γc are used for precontinuous, semicontinuous and γ -continuous, respectively.

Let (X, τ) be a topological space, the family of all γ -open, γ -closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\gamma O(X, \tau)$, $\gamma C(X, \tau)$, $sO(X, \tau)$, $sC(X, \tau)$, $pO(X, \tau)$ and $pC(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ) . Respectively, we define the γ -closure, γ -interior, s -closure, s -interior, p -closure and p -interior of a set A , denoted by $\gamma Cl(A)$, $\gamma Int(A)$, $sCl(A)$, $sInt(A)$, $pCl(A)$ and $pInt(A)$ as follows:

$$\begin{aligned}\gamma Cl(A) &= \cap \{F : F \supseteq A, F \in \gamma C(X, \tau)\}, \\ \gamma Int(A) &= \cup \{O : O \subseteq A, O \in \gamma O(X, \tau)\}, \\ sCl(A) &= \cap \{F : F \supseteq A, F \in sC(X, \tau)\}, \\ sInt(A) &= \cup \{O : O \subseteq A, O \in sO(X, \tau)\}, \\ pCl(A) &= \cap \{F : F \supseteq A, F \in pC(X, \tau)\} \text{ and} \\ pInt(A) &= \cup \{O : O \subseteq A, O \in pO(X, \tau)\}.\end{aligned}$$

Respectively, we have $\gamma Cl(A)$, $sCl(A)$, $pCl(A)$ are γ -closed, semi-closed, preclosed and $\gamma Int(A)$, $sInt(A)$, $pInt(A)$ are γ -open, semi-open, preopen.

The following first two definitions are modifications of conditions considered in [8, 9].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S .

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

- 1) If $A_i \rho B_j$ for any $i \in \{1, \dots, m\}$ and for any $j \in \{1, \dots, n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \dots, m\}$ and any $j \in \{1, \dots, n\}$.
- 2) If $A \subseteq B$, then $A \bar{\rho} B$.
- 3) If $A \rho B$, then $\gamma Cl(A) \subseteq B$ and $A \subseteq \gamma Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ , then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ .

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on a topological space X with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$, then there exists a γ -continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$.

Define functions F and G mapping the rational numbers \mathbb{Q} into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of \mathbb{Q} with $t_1 < t_2$, then $F(t_1) \bar{\rho} F(t_2)$, $G(t_1) \bar{\rho} G(t_2)$, and $F(t_1) \rho G(t_2)$. By Lemmas 1 and 2 of [9] it follows that there exists a function H mapping \mathbb{Q} into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1) \rho H(t_2)$, $H(t_1) \rho H(t_2)$ and $H(t_1) \rho G(t_2)$.

For any x in X , let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = \gamma Int(H(t_2)) \setminus \gamma Cl(H(t_1))$. Hence $h^{-1}(t_1, t_2)$ is a γ -open subset of X , i. e., h is a γ -continuous function on X . ■

The above proof used the technique of proof of Theorem 1 of [8].

If a space has the strong γ -insertion property for (P_1, P_2) , then it has the weak γ -insertion property for (P_1, P_2) . The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak γ -insertion property to satisfy the strong γ -insertion property.

Theorem 2.2. Let P_1 and P_2 be γ -property and X be a space that satisfies the weak γ -insertion property for (P_1, P_2) . Also assume that g and f are functions on X such that $g \leq f$, g has property P_1 and f has property P_2 . The space X has the strong γ -insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f - g, 2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of X such that (i) for each n , F_n and $A(f - g, 2^{-n})$ are completely separated by γ -continuous functions, and (ii) $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.

Proof. Theorem 3.1, of [13]. ■

Theorem 2.3. Let P_1 and P_2 be γ -properties and assume that the space X satisfied the weak γ -insertion property for (P_1, P_2) . The space X satisfies the strong γ -insertion property for (P_1, P_2) if and only if X satisfies the strong γ -insertion property for $(P_1, \gamma c)$ and for $(\gamma c, P_2)$.

Proof. Theorem 3.2, of [13]. ■

3 Applications

Corollary 3.1. If for each pair of disjoint preclosed (resp. semi-closed) sets F_1, F_2 of X , there exist γ -open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then X has the weak γ -insertion property for (pc, pc) (resp. (sc, sc)).

Proof. Let g and f be real-valued functions defined on the X , such that f and g are pc (resp. sc), and $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $pCl(A) \subseteq pInt(B)$ (resp. $sCl(A) \subseteq sInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X . If t_1 and t_2 are any elements of \mathbb{Q} with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a preclosed (resp. semi-closed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (resp. semi-open) set, it follows that $pCl(A(f, t_1)) \subseteq pInt(A(g, t_2))$ (resp. $sCl(A(f, t_1)) \subseteq sInt(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1. ■

Corollary 3.2. If for each pair of disjoint preclosed (resp. semi-closed) sets F_1, F_2 , there exist γ -open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then every precontinuous (resp. semi-continuous) function is γ -continuous.

Proof. Let f be a real-valued precontinuous (resp. semi-continuous) function defined on the X . Set $g = f$, then by Corollary 3.1, there exists a γ -continuous function h such that $g = h = f$. ■

Corollary 3.3. If for each pair of disjoint preclosed (resp. semi-closed) sets F_1, F_2 of X , there exist γ -open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then X has the strong γ -insertion property for (pc, pc) (resp. (sc, sc)).

Proof. Let g and f be real-valued functions defined on the X , such that f and g are pc (resp. sc), and $g \leq f$. Set $h = (f + g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X , then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are γ -continuous functions hence h is γ -continuous function. ■

Corollary 3.4. If for each pair of disjoint subsets F_1, F_2 of X , such that F_1 is preclosed and F_2 is semi-closed, there exist γ -open subsets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then X have the weak γ -insertion property for (pc, sc) and (sc, pc) .

Proof. Let g and f be real-valued functions defined on the X , such that g is pc (resp. sc) and f is sc (resp. pc), with $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $sCl(A) \subseteq pInt(B)$ (resp. $pCl(A) \subseteq sInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X . If t_1 and t_2 are any elements of \mathbb{Q} with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a semi-closed (resp. preclosed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (resp. semi-open) set, it follows that $sCl(A(f, t_1)) \subseteq pInt(A(g, t_2))$ (resp. $pCl(A(f, t_1)) \subseteq sInt(A(g, t_2))$). Hence

$t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1. ■

Before stating the consequences of Theorems 2.2, and 2.3, we state and prove the necessary lemmas.

Lemma 3.1. The following conditions on the space X are equivalent:

(i) For each pair of disjoint subsets F_1, F_2 of X , such that F_1 is preclosed and F_2 is semi-closed, there exist γ -open subsets G_1, G_2 of X such that $F_1 \subseteq G_1, F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$.

(ii) If F is a semi-closed (resp. preclosed) subset of X which is contained in a preopen (resp. semi-open) subset G of X , then there exists an γ -open subset H of X such that $F \subseteq H \subseteq \gamma Cl(H) \subseteq G$.

Proof. (i) \Rightarrow (ii) Suppose that $F \subseteq G$, where F and G are semi-closed (resp. preclosed) and preopen (resp. semi-open) subsets of X , respectively. Hence, G^c is a preclosed (resp. semi-closed) and $F \cap G^c = \emptyset$.

By (i) there exists two disjoint γ -open subsets G_1, G_2 of X s.t., $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since G_2^c is a γ -closed set containing G_1 we conclude that $\gamma Cl(G_1) \subseteq G_2^c$, i. e.,

$$F \subseteq G_1 \subseteq \gamma Cl(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) \Rightarrow (i) Suppose that F_1, F_2 are two disjoint subsets of X , such that F_1 is preclosed and F_2 is semi-closed.

This implies that $F_2 \subseteq F_1^c$ and F_1^c is a preopen subset of X . Hence by (ii) there exists a γ -open set H s. t., $F_2 \subseteq H \subseteq \gamma Cl(H) \subseteq F_1^c$.

But

$$H \subseteq \gamma Cl(H) \Rightarrow H \cap (\gamma Cl(H))^c = \emptyset$$

and

$$\gamma Cl(H) \subseteq F_1^c \Rightarrow F_1 \subseteq (\gamma Cl(H))^c.$$

Furthermore, $(\gamma Cl(H))^c$ is a γ -open set of X . Hence $F_2 \subseteq H, F_1 \subseteq (\gamma Cl(H))^c$ and $H \cap (\gamma Cl(H))^c = \emptyset$. This means that condition (i) holds. ■

Lemma 3.2. Suppose that X is a topological space. If each pair of disjoint subsets F_1, F_2 of X , where F_1 is preclosed and F_2 is semi-closed, can separate by γ -open subsets of X then there exists a γ -continuous function $h : X \rightarrow [0, 1]$ s. t., $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.

Proof. Suppose F_1 and F_2 are two disjoint subsets of X , where F_1 is preclosed and F_2 is semi-closed. Since $F_1 \cap F_2 = \emptyset$, hence $F_2 \subseteq F_1^c$. In particular, since F_1^c is a preopen subset of X containing semi-closed subset F_2 of X , by Lemma 3.1, there exists a γ -open subset $H_{1/2}$ of X s. t.,

$$F_2 \subseteq H_{1/2} \subseteq \gamma Cl(H_{1/2}) \subseteq F_1^c.$$

Note that $H_{1/2}$ is also a preopen subset of X and contains F_2 , and F_1^c is a preopen subset of X and contains a semi-closed subset $\gamma Cl(H_{1/2})$ of X . Hence, by Lemma 3.1, there exists γ -open subsets $H_{1/4}$ and $H_{3/4}$ s. t.,

$$F_2 \subseteq H_{1/4} \subseteq \gamma Cl(H_{1/4}) \subseteq H_{1/2} \subseteq \gamma Cl(H_{1/2}) \subseteq H_{3/4} \subseteq \gamma Cl(H_{3/4}) \subseteq F_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain γ -open subsets H_t of X with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_1$ and $h(x) = 1$ for $x \in F_1$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i. e., h maps X into $[0, 1]$. Also, we note that for any $t \in D, F_2 \subseteq H_t$; hence $h(F_2) = \{0\}$. Furthermore, by definition, $h(F_1) = \{1\}$. It remains only to prove that h is a γ -continuous function on X . For every $\beta \in \mathbb{R}$, we have if $\beta \leq 0$ then $\{x \in X : h(x) < \beta\} = \emptyset$ and if $0 < \beta$ then $\{x \in X : h(x) < \beta\} = \cup\{H_t : t < \beta\}$, hence, they are γ -open subsets of X . Similarly, if $\beta < 0$ then $\{x \in X : h(x) > \beta\} = X$ and if $0 \leq \beta$ then $\{x \in X : h(x) > \beta\} = \cup\{(\gamma Cl(H_t))^c : t > \beta\}$ hence, every of them is a γ -open subset of X . Consequently h is a γ -continuous function. ■

Lemma 3.3. Suppose that X is a topological space. If each pair of disjoint subsets F_1, F_2 of X , where F_1 is preclosed and F_2 is semi-closed, can separate by γ -open subsets of X , and F_1 (resp. F_2) is a countable intersection of γ -open subsets of X , then there exists a γ -continuous function $h : X \rightarrow [0, 1]$ s. t., $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$) and $h(F_2) = \{1\}$ (resp. $h(F_1) =$

$\{1\}$).

Proof. Suppose that $F_1 = \bigcap_{n=1}^{\infty} G_n$ (resp. $F_2 = \bigcap_{n=1}^{\infty} G_n$), where G_n is a γ -open subset of X . We can suppose that $G_n \cap F_2 = \emptyset$ (resp. $G_n \cap F_1 = \emptyset$), otherwise we can substitute G_n by $G_n \setminus F_2$ (resp. $G_n \setminus F_1$). By Lemma 3.2, for every $n \in \mathbb{N}$, there exists a γ -continuous function $h_n : X \rightarrow [0, 1]$ s. t., $h_n(F_1) = \{0\}$ (resp. $h_n(F_2) = \{0\}$) and $h_n(X \setminus G_n) = \{1\}$. We set $h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$.

Since the above series is uniformly convergent, it follows that h is a γ -continuous function from X to $[0, 1]$. Since for every $n \in \mathbb{N}$, $F_2 \subseteq X \setminus G_n$ (resp. $F_1 \subseteq X \setminus G_n$), therefore $h_n(F_2) = \{1\}$ (resp. $h_n(F_1) = \{1\}$) and consequently $h(F_2) = \{1\}$ (resp. $h(F_1) = \{1\}$). Since $h_n(F_1) = \{0\}$ (resp. $h_n(F_2) = \{0\}$), hence $h(F_1) = \{0\}$ (resp. $h(F_2) = \{0\}$). It suffices to show that if $x \notin F_1$ (resp. $x \notin F_2$), then $h(x) \neq 0$.

Now if $x \notin F_1$ (resp. $x \notin F_2$), since $F_1 = \bigcap_{n=1}^{\infty} G_n$ (resp. $F_2 = \bigcap_{n=1}^{\infty} G_n$), therefore there exists $n_0 \in \mathbb{N}$ s. t., $x \notin G_{n_0}$, hence $h_{n_0}(x) = 1$, i. e., $h(x) > 0$. Therefore $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$). ■

Lemma 3.4. Suppose that X is a topological space such that every two disjoint semi-closed and preclosed subsets of X can be separated by γ -open subsets of X . The following conditions are equivalent:

(i) For every two disjoint subsets F_1 and F_2 of X , where F_1 is preclosed and F_2 is semi-closed, there exists a γ -continuous function $h : X \rightarrow [0, 1]$ s. t., $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$) and $h^{-1}(1) = F_2$ (resp. $h^{-1}(1) = F_1$).

(ii) Every preclosed (resp. semi-closed) subset of X is a countable intersection of γ -open subsets of X .

(iii) Every preopen (resp. semi-open) subset of X is a countable union of γ -closed subsets of X .

Proof. (i) \Rightarrow (ii) Suppose that F is a preclosed (resp. semi-closed) subset of X . Since \emptyset is a semi-closed (resp. preclosed) subset of X , by (i) there exists a γ -continuous function $h : X \rightarrow [0, 1]$ s. t., $h^{-1}(0) = F$. Set $G_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, G_n is a γ -open subset of X and $\bigcap_{n=1}^{\infty} G_n = \{x \in X : h(x) = 0\} = F$.

(ii) \Rightarrow (i) Suppose that F_1 and F_2 are two disjoint subsets of X , where F_1 is preclosed and F_2 is semi-closed. By Lemma 3.3, there exists a γ -continuous function $f : X \rightarrow [0, 1]$ s. t., $f^{-1}(0) = F_1$ and $f(F_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two γ -closed subsets of X and $(G \cup F) \cap F_2 = \emptyset$. By Lemma 3.3, there exists a γ -continuous function $g : X \rightarrow [\frac{1}{2}, 1]$ s. t.,

$g^{-1}(1) = F_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define h by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then h is well-defined and a γ -continuous function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence h defined on X and maps to $[0, 1]$. Also, we have $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

(ii) \Leftrightarrow (iii) By De Morgan law and noting that the complement of every γ -open subset of X is a γ -closed subset of X and complement of every γ -closed subset of X is a γ -open subset of X , the equivalence is hold. ■

Corollary 3.5. If for every two disjoint subsets F_1 and F_2 of X , where F_1 is preclosed (resp. semi-closed) and F_2 is semi-closed (resp. preclosed), there exists a γ -continuous function $h : X \rightarrow [0, 1]$ s.t., $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$ then X has the strong γ -insertion property for (pc, sc) (resp. (sc, pc)).

Proof. Since for every two disjoint subsets F_1 and F_2 of X , where F_1 is preclosed (resp. semi-closed) and F_2 is semi-closed (resp. preclosed), there exists a γ -continuous function $h : X \rightarrow [0, 1]$ s. t., $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$, define $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then G_1 and G_2 are two disjoint γ -open subsets of X that contain F_1 and F_2 , respectively. Hence by Corollary 3.4, X has the weak γ -insertion property for (pc, sc) and (sc, pc) . Now, assume that g and f are functions on X such that $g \leq f$, g is pc (resp. sc) and f is γc . Since $f - g$ is pc (resp. sc), therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$ is a preclosed (resp. semi-closed) subset of X . By Lemma 3.4, we can choose a sequence $\{F_n\}$ of γ -closed subsets of X s. t., $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$ and for every $n \in \mathbb{N}$, F_n and $A(f - g, 2^{-n})$ are disjoint subsets of X . By Lemma 3.2, F_n and $A(f - g, 2^{-n})$ can be completely separated by γ -continuous functions. Hence by Theorem 2.2, X has the strong γ -insertion property for $(pc, \gamma c)$ (resp. $(sc, \gamma c)$).

By an analogous argument, we can prove that X has the strong γ -insertion property for $(\gamma c, sc)$ (resp. $(\gamma c, pc)$). Hence, by Theorem 2.3, X has the strong γ -insertion property for (pc, sc) (resp. (sc, pc)). ■

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