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Approximation for Inextensible Flows of Curves in $\mathbb{E}^3$

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abstract: In this paper, we construct a new method for inextensible flows of curves in $\mathbb{E}^3$. Using the Frenet frame of the given curve, we present partial differential equations. We give some characterizations for curvatures of a curve in $\mathbb{E}^3$.

Key Words: Fluid flow, $\mathbb{E}^3$, partial differential equation.

Contents

1. Introduction

Flows has received considerable recent attention because of its relatively simple geometry and its relevance to a large variety of engineering applications [20]. It enables direct insight into fundamental turbulence physics, as well as direct verification by local (such as profiles of mean velocity, Reynolds stress, fluctuation intensities, etc.) and integral measurements (such as skin friction and heat transfer). The fundamental challenge is to predict the mean flow properties, including the mean velocity profile (MVP), the Reynolds stress, the kinetic energy, etc; however, a deductive theory of this kind is still missing, [17].

In the past two decades, for the need to explain certain physical phenomena and to solve practical problems, geometers and geometric analysis have begun to deal with curves and surfaces which are subject to various forces and which flow or evolve with time in response to those forces so that the metrics are changing. Now, various geometric flows have become one of the central topics in geometric analysis. Many authors have studied geometric flow problems, [12].

This study is organised as follows: Firstly, we study inextensible flows of curves in Euclidean 3-space. Secondly, using the Frenet frame of the given curve, we present partial differential equations. Finally, we give some characterizations for curvatures of a curve in Euclidean 3-space.

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2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $\mathbb{E}^3$ are briefly presented; a more complete elementary treatment can be found in [8].

The Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}^3$ is given by

$$\|a\| = \sqrt{\langle a, a \rangle}.$$

α is called a unit speed curve if velocity vector v of α satisfies $\|v\| = 1$. Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$. If the tangent vector of this curve forms a constant angle with a fixed constant vector U, then this curve is called a general helix or an inclined curve. The sphere of radius $r > 0$ and with center in the origin in the space $\mathbb{E}^3$ is defined by

$$\mathbb{S}^2 = \left\{ p = (p_1, p_2, p_3) \in \mathbb{E}^3 : \langle p, p \rangle = r^2 \right\}.$$

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet-Serret frame along the curve α in the space $\mathbb{E}^3$. For an arbitrary curve α with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet-Serret formulae are given in [8] written under matrix form

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 1, \\ \langle \mathbf{T}, \mathbf{N} \rangle &= \langle \mathbf{N}, \mathbf{B} \rangle = \langle \mathbf{T}, \mathbf{B} \rangle = 0. \end{aligned}$$

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{E}^3$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be positive oriented natural basis of $\mathbb{E}^3$. Cross product of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Mixed product of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is defined by the determinant

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Torsion of the curve α is given by the aid of the mixed product

$$\tau = \frac{[\alpha', \alpha'', \alpha''']}{\kappa^2}.$$

3. Inextensible Flows of Curves in $\mathbb{E}^3$

Physically, inextensible curve and surface flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications, [11,12,13].

Let $\alpha(u,t)$ is a one parameter family of smooth curves in $\mathbb{E}^3$.

Any flow of α can be represented as

$$\frac{\partial \alpha}{\partial t} = \mathfrak{f}_1 \mathbf{T} + \mathfrak{f}_2 \mathbf{N} + \mathfrak{f}_3 \mathbf{B}, \tag{3.1}$$

where $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$ are smooth functions.

Letting the arclength variation be

$$s(u,t) = \int_0^u v du.$$

In the $\mathbb{E}^3$ the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0.$$

Definition 3.1. *The flow* $\frac{\partial \alpha}{\partial t}$ *in* $\mathbb{E}^3$ *are said to be inextensible if*

$$\frac{\partial}{\partial t} \left\| \frac{\partial \alpha}{\partial u} \right\| = 0. \tag{3.2}$$

Theorem 3.2. *Let* $\frac{\partial \alpha}{\partial t} = \mathfrak{f}_1 \mathbf{T} + \mathfrak{f}_2 \mathbf{N} + \mathfrak{f}_3 \mathbf{B}$ *be a smooth flow of* α. *The flow is inextensible if and only if*

$$\frac{\partial \mathfrak{f}_1}{\partial u} = \kappa \mathfrak{f}_2 v. \tag{3.3}$$

Proof: Assume that $\frac{\partial \alpha}{\partial u}$ is inextensible.Then,

$$\frac{\partial}{\partial t}\sigma(u,t)=\int_0^u \frac{\partial v}{\partial t}du=\int_0^u [\frac{\partial \mathfrak{f}_1}{\partial u}-\kappa \mathfrak{f}_2 v]du=0. \tag{3.4}$$

Substituting (3.2) in (3.4) complete the proof of the theorem. □

We now restrict ourselves to arc length parametrized curves. That is, $v=1$ and the local coordinate u corresponds to the curve arc length s. We require the following lemma.

Lemma 3.3.

$$\frac{\partial \mathbf{T}}{\partial t}=[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]\mathbf{N}+[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]\mathbf{B}, \tag{3.5}$$

where $\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_3$ *are smooth functions of time and arc length.*

Proof: Using definition of α, we have

$$\frac{\partial}{\partial t}\mathbf{T}=[\frac{\partial \mathfrak{f}_1}{\partial s}-\kappa \mathfrak{f}_2]\mathbf{T}+[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]\mathbf{N}+[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]\mathbf{B}. \tag{3.6}$$

Substituting (3.3) in (3.6), we obtain (3.5). This completes the proof. □

Now we give the characterization of evolution of curvature as below:

Theorem 3.4. *Let* $\frac{\partial \alpha}{\partial t}$ *be inextensible. Then, the evolution of curvature*

$$\frac{\partial \kappa}{\partial t}=\frac{\partial}{\partial s}[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]-\tau[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau],$$

where $\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_3$ *are smooth functions of time and arc length.*

Proof: Assume that $\frac{\partial \alpha}{\partial t}$ is inextensible in $\mathbb{E}^3$.

Thus it is easy to obtain that

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial}{\partial t}\mathbf{T} &= -\kappa[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]\mathbf{T} \\ &\quad +[\frac{\partial}{\partial s}[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]-\tau[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]]\mathbf{N} \\ &\quad +[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]\mathbf{B}.\end{aligned} \tag{3.7}$$

On the other hand, we have

$$\frac{\partial \kappa}{\partial t}=\frac{\partial}{\partial t}<\frac{\partial}{\partial s}\mathbf{T},\mathbf{N}>.$$

Also,

$$\frac{\partial \kappa}{\partial t} =< \frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{T}, \mathbf{N} > + < \frac{\partial}{\partial s}\mathbf{T}, \frac{\partial}{\partial t}\mathbf{N} > .$$

From definition of flow, we have

$$< \mathbf{N}, \frac{\partial}{\partial t}\mathbf{N} >=0.$$

Combining these we have

$$\frac{\partial \kappa}{\partial t} = \frac{\partial}{\partial s}[\mathfrak{f}_1\kappa + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3\tau] - \tau[\frac{\partial \mathfrak{f}_3}{\partial s} + \mathfrak{f}_2\tau].$$

Thus, we obtain the theorem. This completes the proof. □

From the above theorem, we have

Theorem 3.5.

$$\frac{\partial \mathbf{N}}{\partial t} = -[\mathfrak{f}_1\kappa + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3\tau]\mathbf{T} + \frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s} + \mathfrak{f}_2\tau] + \tau[\mathfrak{f}_1\kappa + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3\tau]]\mathbf{B}.$$

Proof: Using Frenet equations, we have

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{T} = \frac{\partial}{\partial t}(\kappa\mathbf{N}) = \frac{\partial \kappa}{\partial t}\mathbf{N} + \kappa\frac{\partial \mathbf{N}}{\partial t}. \tag{3.8}$$

Substituting (3.7) into Eq. (3.8) , we have

$$\begin{aligned}\frac{\partial \mathbf{N}}{\partial t} &= -[\mathfrak{f}_1\kappa + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3\tau]\mathbf{T} \\ &+\frac{1}{\kappa}[\frac{\partial}{\partial s}[\mathfrak{f}_1\kappa + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3\tau] - \tau[\frac{\partial \mathfrak{f}_3}{\partial s} + \mathfrak{f}_2\tau] - \frac{\partial \kappa}{\partial t}]\mathbf{N} \\ &+\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s} + \mathfrak{f}_2\tau] + \tau[\mathfrak{f}_1\kappa + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3\tau]]\mathbf{B}.\end{aligned}$$

On the other hand, using above equation we have

$$\frac{\partial \mathbf{N}}{\partial t} = -[\mathfrak{f}_1\kappa + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3\tau]\mathbf{T} + \frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s} + \mathfrak{f}_2\tau] + \tau[\mathfrak{f}_1\kappa + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3\tau]]\mathbf{B},$$

which completes the proof. □

Now we give the characterization of evolution of torsion as below:

Theorem 3.6. *Let* $\frac{\partial \alpha}{\partial t}$ *be inextensible. Then, the evolution of curvature*

$$\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s} + \mathfrak{f}_2\tau] + \tau[\mathfrak{f}_1\kappa + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3\tau]]] + \kappa[\frac{\partial \mathfrak{f}_3}{\partial s} + \mathfrak{f}_2\tau],$$

where $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$ *are smooth functions of time and arc length.*

Proof:

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial}{\partial t}\mathbf{N} &= -\frac{\partial}{\partial s}[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]\mathbf{T}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau] \\ &\quad +\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]\mathbf{B}-[\kappa[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau] \\ &\quad +\tau[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]]\mathbf{N}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{N} &= -\frac{\partial}{\partial t}\kappa\mathbf{T}+\frac{\partial}{\partial t}\tau\mathbf{B} \\ &\quad -\kappa[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]\mathbf{N}-\kappa[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]\mathbf{B}+\tau\frac{\partial}{\partial t}\mathbf{B}\end{aligned}$$

Also,

$$\frac{\partial \tau}{\partial t}=-<\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{B},\mathbf{N}>-<\frac{\partial}{\partial s}\mathbf{B},\frac{\partial}{\partial t}\mathbf{N}>.$$

From definition of flow, we have

$$<\mathbf{B},\frac{\partial}{\partial t}\mathbf{B}>=0.$$

Thus, we obtain the theorem. The proof of theorem is completed. □

Theorem 3.7. *Let* $\frac{\partial \alpha}{\partial t}$ *be inextensible. Then,*

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= \frac{1}{\tau}[\frac{\partial \kappa}{\partial t}-\frac{\partial}{\partial s}[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]\mathbf{T} \\ &\quad -[[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]]\mathbf{N},\end{aligned} \tag{3.9}$$

where $\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_3$ *are smooth functions of time and arc length.*

Proof: Using Frenet equations, we have

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= \frac{1}{\tau}[\frac{\partial \kappa}{\partial t}-\frac{\partial}{\partial s}[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]\mathbf{T}+\frac{1}{\tau}[\kappa[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau] \\ &\quad -\frac{\partial \tau}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]]\mathbf{B} \\ &\quad -\frac{1}{\tau}[\tau[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]]\mathbf{N}.\end{aligned} \tag{3.10}$$

From definition of flow, we have

$$<\mathbf{B},\frac{\partial}{\partial t}\mathbf{B}>=0. \tag{3.11}$$

Substituting (3.11) to (3.10), we have (3.9) as desired. This completes the proof. □

Theorem 3.8. *Let* $\dfrac{\partial \alpha}{\partial t}$ *be inextensible. Then,*

$$\begin{aligned}&\kappa[\frac{1}{\tau}[\frac{\partial \kappa}{\partial t}-\frac{\partial}{\partial s}[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]\\ =\ & 2\frac{\partial}{\partial s}[[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]],\end{aligned}$$

where $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$ *are smooth functions of time and arc length.*

Proof: Differentiating Eq. (3.9) with respect to s,

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \mathbf{B}}{\partial t} &= [\frac{\partial}{\partial s}[\frac{1}{\tau}[\frac{\partial \kappa}{\partial t}-\frac{\partial}{\partial s}[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]\\ &\quad+\kappa[[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]]]\mathbf{T}\\ &\quad+[\kappa[\frac{1}{\tau}[\frac{\partial \kappa}{\partial t}-\frac{\partial}{\partial s}[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]\\ &\quad-\frac{\partial}{\partial s}[[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]]]\mathbf{N}\\ &\quad-\tau[[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]]\mathbf{B}.\end{aligned}$$

Also,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{B} &= -\frac{\partial}{\partial t}\tau\mathbf{N}-\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]\mathbf{T}\\ &\quad-\tau\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]\mathbf{B}.\end{aligned}$$

Hence, the proof is complete. □

Corollary 3.9.

$$\begin{aligned}-\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau] &= \frac{\partial}{\partial s}[\frac{1}{\tau}[\frac{\partial \kappa}{\partial t}-\frac{\partial}{\partial s}[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]\\ &\quad+\kappa[[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\frac{\partial \mathfrak{f}_3}{\partial s}+\mathfrak{f}_2\tau]+\tau[\mathfrak{f}_1\kappa+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3\tau]]]],\end{aligned}$$

where $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$ *are smooth functions of time and arc length.*

Proof: It is obvious from Theorem (3.8). □

Example 3.10. *The helix is parametrized by*

$$\gamma(u,t) = (A(t)\cos(u), A(t)\sin(u), B(t)u), \tag{3.12}$$

where A*,* B *are functions only of time. The arc-length derivative is*

$$\left(A^2+B^2\right)^{\frac{1}{2}}\frac{\partial}{\partial s} = \frac{\partial}{\partial u}$$

and the evolution of γ is explicity given by

$$(\frac{\partial A}{\partial t}\cos(u), \frac{\partial A}{\partial t}\sin(u), \frac{\partial B}{\partial t}u) = \frac{1}{A^2+B^2}(-A\cos(u), -A\sin(u), 0).$$

Hence

$$\frac{\partial A}{\partial t} = -A\left(A^2+B^2\right)^{-1}, \quad \frac{\partial B}{\partial t} = 0$$

and solitions are given by

$$\frac{A(t)^2}{2} + B^2\log(A(t)) = -t + \frac{A(0)^2}{2} + B^2\log(A(0)).$$

Note that, for positive B, $A(t)$ converges to, but never reaches, zero.

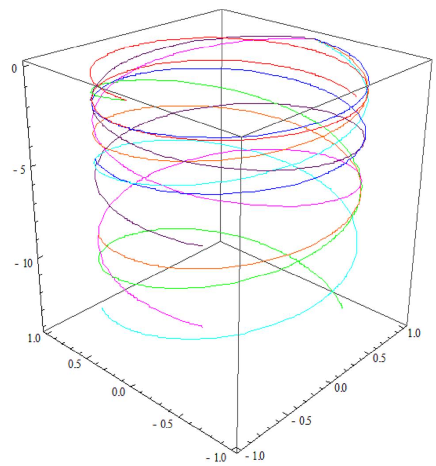
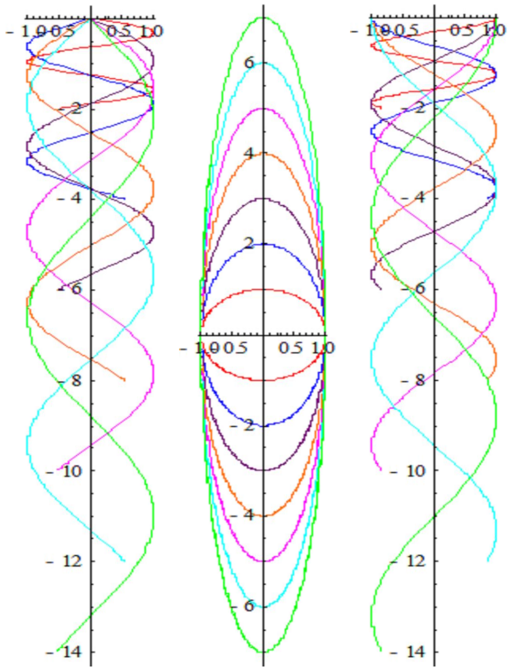


Figure 1: *The equation Eq. (3.12) is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Green at the time $t = 1$, $t = 1.2$, $t = 1.4$, $t = 1.6$, $t = 1.8$, $t = 2$, $t = 2.2$, respectively.*



(a) (b) (c)

Figure 2: *Projections of γ to yz, xy, xz planes are illustrated in (a), (b), (c), respectively.*

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