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A unifying approach to the difference operators and their applications

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abstract: In the present paper, we introduce the idea of difference operators Δ^α and $\Delta^{(\alpha)}(\alpha \in \mathbb{R})$ and establish certain results which have several applications in Functional as well as Numerical analysis. Indeed, the operator Δ^α generalizes several difference operators defined by Kızmaz [1], Et [2], Et and Çolak [3], Malkowsky and Parashar [4], Et [5], Malkowsky et al. [6], Baliarsingh [7] and many others.

Key Words: Difference operators Δ^α and $\Delta^{(\alpha)}$; Product of two sequence spaces.

Contents

1. Introduction and definitions

By $\Gamma(p)$, we denote the Gamma function of a real number p and $p \notin \{0, -1, -2, -3, \dots\}$. By the definition, it can be expressed as an improper integral i.e.

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt. \tag{1.1}$$

From the equation (1.1), we observe that

(i) For any natural number p, $\Gamma(p+1) = p!$.

(ii) For any real number p and $p \notin \{0, -1, -2, -3 \dots\}$, $\Gamma(p+1) = p\Gamma(p)$.

(iii) For particular cases, we have $\Gamma(1) = \Gamma(2) = 1,\ \Gamma(3) = 2!,\ \Gamma(4) = 3!$ ....

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Let w be the space all real valued sequences. For a real number α and $x \in w$, we define difference operators $\Delta^{\alpha}, \Delta^{(\alpha)}, \Delta^{-\alpha}$ and $\Delta^{(-\alpha)}$ as follows:

$$(\Delta^{\alpha}x)_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}x_{k+i}, \tag{1.2}$$

$$(\Delta^{(\alpha)}x)_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}x_{k-i}, \tag{1.3}$$

$$(\Delta^{-\alpha}x)_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(1-\alpha)}{i!\Gamma(1-\alpha-i)}x_{k+i}, \tag{1.4}$$

$$(\Delta^{(-\alpha)}x)_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(1-\alpha)}{i!\Gamma(1-\alpha-i)}x_{k-i}. \tag{1.5}$$

We assume throughout that the summation defined in (1.2-1.5) are convergent for $x \in w$. In particular, for $\alpha = \frac{1}{2}$, we obtain that

- $\Delta^{1/2}x_k = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \frac{7}{256}x_{k+5} - \frac{21}{1024}x_{k+6} + \dots,$

- $\Delta^{(1/2)}x_k = x_k - \frac{1}{2}x_{k-1} - \frac{1}{8}x_{k-2} - \frac{1}{16}x_{k-3} - \frac{5}{128}x_{k-4} - \frac{7}{256}x_{k-5} - \frac{21}{1024}x_{k-6} + \dots,$

- $\Delta^{-1/2}x_k = x_k + \frac{1}{2}x_{k+1} + \frac{3}{8}x_{k+2} + \frac{5}{16}x_{k+3} + \frac{35}{128}x_{k+4} + \frac{63}{256}x_{k+5} + \frac{231}{1024}x_{k+6} + \dots,$

- $\Delta^{(-1/2)}x_k = x_k + \frac{1}{2}x_{k-1} + \frac{3}{8}x_{k-2} + \frac{5}{16}x_{k-3} + \frac{35}{128}x_{k-4} + \frac{63}{256}x_{k-5} + \frac{231}{1024}x_{k-6} + \dots.$

It is natural to show that the operators $\Delta^{\alpha}, \Delta^{(\alpha)}, \Delta^{-\alpha}$ and $\Delta^{(-\alpha)}$ can be expressed as triangles as follows :

$$\Delta^{\alpha} = \begin{pmatrix} 1 & -\alpha & \frac{\alpha(\alpha-1)}{2!} & -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} & \cdots \\ 0 & 1 & -\alpha & \frac{\alpha(\alpha-1)}{2!} & \cdots \\ 0 & 0 & 1 & -\alpha & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Delta^{(\alpha)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -\alpha & 1 & 0 & 0 & \cdots \\ \frac{\alpha(\alpha-1)}{2!} & -\alpha & 1 & 0 & \cdots \\ -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} & \frac{\alpha(\alpha-1)}{2!} & -\alpha & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Delta^{-\alpha} = \begin{pmatrix} 1 & \alpha & \frac{\alpha(\alpha+1)}{2!} & \frac{\alpha(\alpha+1)(\alpha+2)}{3!} & \cdots \\ 0 & 1 & \alpha & \frac{\alpha(\alpha+1)}{2!} & \cdots \\ 0 & 0 & 1 & \alpha & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Delta^{(-\alpha)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \alpha & 1 & 0 & 0 & \cdots \\ \frac{\alpha(\alpha+1)}{2!} & \alpha & 1 & 0 & \cdots \\ \frac{\alpha(\alpha+1)(\alpha+2)}{3!} & \frac{\alpha(\alpha+1)}{2!} & \alpha & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

In our observations, the following special cases which are included in the generalizations of the operators Δ^{α} and $\Delta^{(\alpha)}$:

(i) If $\alpha = 1$, then the operator Δ^{α} reduces to Δ and $(\Delta x)_k = x_k - x_{k+1}$, defined by Kızmaz [1].

(ii) If $\alpha = m \in$, then the operator Δ^{α} reduces to Δ^m and $(\Delta^m x)_k = \sum_{i=0}^{m}(-1)^m \binom{m}{i} x_{k+i}$, defined by Et and Çolak [2].

(iii) If $\alpha = 1$, then the operator $\Delta^{(\alpha)}$ reduces to $\Delta^{(1)}$ and $(\Delta^{(1)}x)_k = x_k - x_{k-1}$, defined by Malkowsky an Parashar [3].

(iv) If $\alpha = m \in$, then the operator $\Delta^{(\alpha)}$ reduces to $\Delta^{(m)}$ and $(\Delta^{(m)}x)_k = \sum_{i=0}^{m}(-1)^m \binom{m}{i} x_{k-i}$, studied by Malkowsky et al. [4] and Et [5].

Recently, different classes of difference sequences have been introduced and their different properties including topological structures, duals, and matrix transformations have been studied by Tripathy [8], Et and Basarir [9], Dutta and Baliarsingh [10,12], Mursaleen [11], Tripathy et al. [13,15], Asma and Çolak [14] and many others (see [16]-[23]). In this article, we unify most of the difference operators studied by earlier authors and extend their results in a more general and comprehensive way.

2. Main results

In this section, we state some interesting results concerning the linearity property of the difference operators $\Delta^\alpha, \Delta^{(\alpha)}, \Delta^{-\alpha}$ and $\Delta^{(-\alpha)}$. Also, we discuss certain relations among these operators.

Theorem 2.1. *The operators $X : w \to w$ for $X \in \{\Delta^\alpha, \Delta^{(\alpha)}, \Delta^{-\alpha}, \Delta^{(-\alpha)}\}$ are linear over $\mathbb{C}$.*

Proof: Proof is trivial, hence omitted. □

Theorem 2.2. *If α and β are two real numbers, then*

(i) $\Delta^\alpha \circ \Delta^\beta \equiv \Delta^\beta \circ \Delta^\alpha \equiv \Delta^{\alpha+\beta}$.

(ii) $\Delta^{(\alpha)} \circ \Delta^{(\beta)} \equiv \Delta^{(\beta)} \circ \Delta^{(\alpha)} \equiv \Delta^{(\alpha+\beta)}$.

Proof: Proof follows from Theorem 2.1, so we omit the details. □

Theorem 2.3. *If α is a real number, then*

(i) $\Delta^\alpha \circ \Delta^{-\alpha} \equiv \Delta^{-\alpha} \circ \Delta^\alpha \equiv Id$.

(ii) $\Delta^{(\alpha)} \circ \Delta^{(-\alpha)} \equiv \Delta^{(-\alpha)} \circ \Delta^{(\alpha)} \equiv Id$,

where Id is the identity operator in w.

Proof: (i) The proof of this theorem is divided into two parts. First we prove the theorem for any positive integer α which can be obtained by using inductive principle. Suppose $x \in w$ and for $\alpha = 1$, we have

$$\begin{aligned}(\Delta \circ \Delta^{-1}x)_k &= (\Delta(\Delta^{-1}x))_k \\ &= \Delta(x_k + x_{k+1} + x_{k+2} + \dots) \\ &= x_k - x_{k+1} + x_{k+1} - x_{k+2} + x_{k+2} - x_{k+3} + x_{k+3} - x_{k+4} + \dots \\ &= x_k.\end{aligned}$$

This shows that $\Delta \circ \Delta^{-1} \equiv Id$ in w. By principle of induction one can establish $\Delta^r \circ \Delta^{-r} \equiv Id$ in w. Similarly, for a fraction α, we can show that $\Delta^\alpha \circ \Delta^{-\alpha} \equiv Id$ in w.

(ii) In view of the proof of (i), that of (ii) is similar, so we omit it. □

Theorem 2.4. *For a positive integer* α *and* $x \in w$,

(i) $(\Delta^{\alpha} x)_k = (-1)^{\alpha} (\Delta^{(\alpha)} x)_{k+\alpha}$.

(ii) $(\Delta^{(\alpha)} x)_k = (-1)^{\alpha} (\Delta^{\alpha} x)_{k-\alpha}$

Proof: (i) We prove the theorem by induction principle. For $\alpha = 1$ and $x \in w$, we have

$$(\Delta x)_k = x_k - x_{k+1} = (-1)(x_{k+1} - x_k) = (\Delta^{(1)} x)_{k+1}.$$

This completes the Basis step. Let us assume that the theorem is true for a natural number r, i.e. $(\Delta^r x)_k = (-1)^r (\Delta^{(r)} x)_{k+r}$. Now, we take

$$\begin{aligned}
(\Delta^{r+1} x)_k &= (\Delta(\Delta^r x))_k \\
&= \Delta(\Delta^r x)_k \\
&= \Delta((-1)^r (\Delta^{(r)} x)_{k+r}), \text{ (by the assumption).} \\
&= (-1)^r (\Delta^{(r)} x)_{k+r} - (-1)^r (\Delta^{(r)} x)_{k+r+1}, \\
&= (-1)^{r+1} [(\Delta^{(r)} x)_{k+r+1} - \Delta^{(r)} x)_{k+r}] = (-1)^{r+1} (\Delta^{(r+1)} x)_{k+r+1}, \\
&\text{(by Theorem 2.2).}
\end{aligned}$$

This completes the proof.
(ii) The proof is similar to that of (i). □

Theorem 2.5. *For any real* α *and* $x \in w$, *we have*

$$((\Delta^{\alpha} + \Delta^{-\alpha}) x)_k = 2x_k + \sum_{i=1}^{\infty} \frac{(\alpha^+)_{i-1} + (-1)^i (\alpha^-)_{i-1}}{i} x_{k+i},$$

where $(\alpha^+)_{i-1} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+i-1)$ *and* $(\alpha^-)_{i-1} = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha - i + 1)$.

Proof: The proof is straightforward from the definition, so we omit it. □

Let $x = (x_k)$ and $y = (y_k)$ be two sequences in w. We define the product of x and y as $xy = (x_k y_k)$. Now, the first forward and backward differences of xy are given by $\Delta(xy) = (x_k y_k - x_{k+1} y_{k+1})$ and $\Delta^{(1)}(xy) = (x_k y_k - x_{k-1} y_{k-1})$, respectively. The basic objective of this part is to find the α-th difference of product sequence xy where α is a positive integer. So, we state the following theorems.

Theorem 2.6. (Leibnitz Theorem). *Let* $\alpha = n$ *be a positive integer and* $x, y \in w$, *then*

$$((\Delta^n) xy)_k = x_k \Delta^n y_k + n \Delta x_k \Delta^{n-1} y_{k+1} + \frac{n(n-1)}{2!} \Delta^2 x_k \Delta^{n-2} y_{k+2} + \cdots + \Delta^n x_k y_{k+n},$$

Proof: This proceeds by induction on natural numbers n, the result being trivial for $n = 0$ and reducing for $n = 1$ to the well-known rule for differentiating a product (once). Suppose $n = 1$ and $x, y \in w$, we obtain that

$$x_k \Delta y_k + \Delta x_k y_{k+1} = x_k(y_k - y_{k+1}) + (x_k - x_{k+1})y_{k+1} = x_k y_k - x_{k+1}y_{k+1} = (\Delta(xy))_k.$$

Let us assume that the theorem holds for a positive integer r which can be stated as

$$((\Delta^r)xy)_k = \binom{r}{0} x_k \Delta^r y_k + \binom{r}{1} \Delta x_k \Delta^{r-1} y_{k+1} + \binom{r}{2} \Delta^2 x_k \Delta^{r-2} y_{k+2} + \dots + \binom{r}{r} \Delta^r x_k y_{k+r},$$

Now,

$$\begin{aligned}
((\Delta^{r+1})xy)_k &= \binom{r}{0}\Delta(x_k \Delta^r y_k) + \binom{r}{1}\Delta(\Delta x_k \Delta^{r-1} y_{k+1}) + \binom{r}{2}\Delta(\Delta^2 x_k \Delta^{r-2} y_{k+2}) \\
&\quad + \dots + \binom{r}{r}\Delta(\Delta^r x_k y_{k+r}) \\
&= \binom{r}{0} x_k \Delta^{r+1} y_k + \left[\binom{r}{0} + \binom{r}{1}\right] \Delta x_k \Delta^r y_{k+1} \\
&\quad + \left[\binom{r}{1} + \binom{r}{2}\right] \Delta^2 x_k \Delta^{r-1} y_{k+2} + \dots \\
&\quad + \left[\binom{r}{r-1} + \binom{r}{r}\right] \Delta^r x_k \Delta y_{k+r} + \binom{r}{r} \Delta^{r+1} x_k y_{k+r+1} \\
&= \binom{r+1}{0} x_k \Delta^{r+1} y_k + \binom{r+1}{1} \Delta x_k \Delta^r y_{k+1} \\
&\quad + \binom{r+1}{2} \Delta^2 x_k \Delta^{r-1} y_{k+2} + \dots \\
&\quad + \binom{r+1}{r} \Delta^r x_k \Delta y_{k+r} + \binom{r+1}{r+1} \Delta^{r+1} x_k y_{k+r+1}
\end{aligned}$$

This leads to the completion of the proof. □

3. Application to the numerical analysis

In this section, we discuss some numerical applications of the difference operators $\Delta^\alpha, \Delta^{(\alpha)}, \Delta^{-\alpha}$ and $\Delta^{(-\alpha)}$. In fact, these calculations are often used in finding interpolating polynomial, numerical differentiation and integration of a function where we write $f = (f_k)$ as a sequence of functional values of $f(x)$ at $x_1, x_2, x_3, \dots$. The well known Newton's forward and backward interpolation formula for $f(x)$ can be explained with the help of these operators. Now, we consider some particular cases.

Corollary 3.1. *(i) If* $x = (1, 1, 1, \dots)$, *then* $(\Delta^{\alpha} x)_k = (\Delta^{(\alpha)} x)_k = 0$.

(ii) If $x = (1, 0, 1, 0, \dots)$, *then* $(\Delta^{\alpha} x)_k = (\Delta^{(\alpha)} x)_k = (-1)^k 2^{\alpha-1}$.

(iii) If $x = (\frac{1}{2^k})$, *then* $(\Delta^{\alpha} x)_k = (\Delta^{(\alpha)} x)_k = \frac{1}{2^{\alpha+k}}$. *In particular,* $(\Delta^{-1} x)_k = \frac{1}{2^{k-1}}$.

(iv) If $x = (a^k)$ *for* $|a| < 1$, *then* $(\Delta^{\alpha} x)_k = (\Delta^{(\alpha)} x)_k = a^k (1-a)^{\alpha}$ *and* $(\Delta^{-\alpha} x)_k = (\Delta^{(-\alpha)} x)_k = \frac{a^k}{(1-a)^{\alpha}}$.

Theorem 3.2. (Newton's forward difference formula). *Let* $f = (f_1, f_2, f_3, \dots)$ *be the sequence of the functional values of* $f(x)$ *at* $x = (x_1, x_2, x_3, \dots)$ *(equally spaced with common difference* $h > 0$*), then the interpolating polynomial* $p(x)$ *at a point* $z = x_1 + th$ *can be expressed as*

$$p(z) = f_1 + t(\Delta^{(1)} f)_2 + t(t-1)\frac{(\Delta^{(2)} f)_3}{2!} + t(t-1)(t-2)\frac{(\Delta^{(3)} f)_4}{3!} + \dots, \quad \text{where } t = \frac{z - x_1}{h}.$$

Theorem 3.3. (Newton's backward difference formula). *Let* $f = (f_1, f_2, f_3, \dots, f_n, \dots)$ *be the sequence of the functional values of* $f(x)$ *at* $x = (x_1, x_2, x_3, \dots, x_n, \dots)$ *(equally spaced with common difference* $h > 0$*), then the interpolating polynomial* $p(x)$ *at a point* $z = x_n + ht$, $(n \in)$ *can be expressed as*

$$p(z) = f_n + t(\Delta^{(1)} f)_n + t(t+1)\frac{(\Delta^{(2)} f)_n}{2!} + t(t+1)(t+2)\frac{(\Delta^{(3)} f)_n}{3!} + \dots, \quad \text{where } t = \frac{z - x_n}{h}.$$

Example 3.4. *Let* $f = (10, 28, 62, 118, 202, 320, \dots)$ *be the given sequence of the functional values of* $f(x)$ *at* $x = (1, 2, 3, 4, 5, 6, \dots)$, *then clearly,* $h = 1, t = (z-1), (\Delta^{(1)} f)_2 = 18, (\Delta^{(2)} f)_3 = 16, (\Delta^{(3)} f)_4 = 6$ *and* $(\Delta^{(4)} f)_5 = (\Delta^{(5)} f)_6 = 0 \dots$. *By Theorem 3.2, the interpolating polynomial of* $f(x)$ *is given by*

$$p(z) = 10 + (z-1)18 + (z-1)(z-2)8 + (z-1)(z-2)(z-3) = z^3 + 2z^2 + 5z + 2.$$

Again for backward difference formula, let us take $n = 6$ *and also* $h = 1, t = z-6, (\Delta^{(1)} f)_6 = 118, (\Delta^{(2)} f)_6 = 34, (\Delta^{(3)} f)_6 = 6$ *and* $(\Delta^{(4)} f)_6 = (\Delta^{(5)} f)_6 = 0 \dots$. *By Theorem 3.3, the interpolating polynomial of* $f(x)$ *is given by*

$$p(z) = 320 + (z-6)118 + (z-6)(z-5)17 + (z-6)(z-5)(z-4) = z^3 + 2z^2 + 5z + 2.$$

Conclusion: The main objective of the present investigation is to obtain certain results involving the generalized fractional and integral difference operators which have several applications in the field of analysis and applied sciences.

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