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On the spectrum of one dimensional p-Laplacian for an eigenvalue problem with Neumann boundary conditions

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ABSTRACT: This work deals with an indefinite weight one dimensional eigenvalue problem of the p-Laplacian operator subject to Neumann boundary conditions. We are interested in some properties of the spectrum like simplicity, monotonicity and strict monotonicity with respect to the weight. We also aim the study of zeros points of eigenfunctions.

Contents

1. Introduction

This paper is devoted to the study of the following problem:

$$P_{Ne}(m,I)\begin{cases} -(|u'|^{p-2}u')' = \lambda m|u|^{p-2}u & \text{in } \ I =]a,b[, \\ u'(a) = u'(b) = 0, \end{cases}$$

where $a, b \in \mathbb{R}$ and m an indefinite weight function satisfying $mes(I^+) \neq 0$ where $I^+ = \{x \in I/m(x) > 0\}$.

It's well known that the spectrum of Neumann problem in dimension $N \geq 1$, considered in a smooth bounded domain with indefinite weight (cf. [7,8]) contains a sequence of nonnegative eigenvalues $(\lambda_n^{Ne}(m,I))_{n\in\mathbb{N}^*}$ given by:

$$\frac{1}{\lambda_n^{Ne}} = \sup_{K\in\Gamma_n} \min_K \frac{\int_I m(x)|u|^p}{\int_I |u'|^p}, \tag{1.1}$$

where $\Gamma_n = \{K \subset S \ / \ K$ compact symmetric and $\gamma(K) \geq n\}$ and S is the unit sphere of $W^{1,p}(I)$. The sequence $(\lambda_n^{Ne}(m,I))_{n\in\mathbb{N}^*}$ verify:
- $\lambda_n^{Ne}(m,I) \to +\infty, \ n \to +\infty.$

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• If m change its sign in I and $\int_I m(x)dx < 0$ then the first eigenvalue (cf. [8]) defined by

$$\lambda_1^{Ne}(m,I) = \inf\{\|u'\|_p^p \,;u \in W^{1,p}(I) \;/\; \int_I m(x)|u|^p dx = 1\} > 0 \tag{1.2}$$

is strictly positive, simple, isolated and any eigenfunction u_1 associated to $\lambda_1^{Ne}(m,I)$ is of constant sign in I, i.e u_1 has no zero in I.

The purpose of this paper is the study of some properties of the spectrum of $P_{Ne}(m,I)$. We prove the following assertions:
▷ Any eigenfunction corresponding to an eigenvalue λ_n^{Ne} has exactly $(n-1)$ unique zeros in I.
▷ Eigenvalues in the spectrum are simple.
▷ $\lambda_n^{Ne}(m,I)$, $n \geq 1$, satisfies the strict monotonicity property (SMP) with respect to the weight.

Since the 80's, the study of eigenvalue problems of the p-Laplacian operator subject to different kinds of boundary conditions has attracted the interest of several authors (cf. [1,6,15,13,7,8,3,10,5,11,2,12,16,4,9] and the references therein). In particular, we present here some results that motivated this work:
First, we mention the result in [8], that we consider as a principal key for the development of our results. It allowed us passing from one eigenvalue problem of Neumann, studied in I, to a Dirichlet problem in an extension of I. We state this result later in this paper (cf. lemma 2.1).
Other, thanks to this transformation, one can use the interesting results of A.Anane, O.Chakrone and M.Moussa [3] (see preliminary section, theorem 2.2), who have studied the spectrum of one dimensional p-Laplacian for Dirichlet problem, especially the property of strict monotonicity with respect to the domain. Furthermore, one cannot forget the important results in [8,4,9] like multiplicity property and monotonicity, that will be announced later, which were particularly useful for the proof of simplicity and for the investigation of zeros points.

This paper is organized as follow: In section 2, we collect some results which are necessary in what follows. Section 3 is concerned with the simplicity of eigenvalues. In section 4, we investigate the number of zeros points of eigenfunctions. Finally, in section 5, we prove the strict monotonicity property (SMP) with respect to the weight.

2. Preliminaries

Throughout this paper, we adopt the following notations:
• $(u, \lambda^{Ne}(m,I))$ will design a solution of problem $P_{Ne}(m,I)$ where $u \in W^{1,p}(I)$, $u \neq 0$.
• We denote by $\lambda^D(m,I)$ a solution of Dirichlet problem in I.
• $Z(u)$ the set of zero points associated to u defined by:

$$Z(u) = \{u \in I \;/\; u(x) = 0\}.$$

• To simplify, we sometimes denote $\lambda(m,I)$ by λ and $\lambda(m_{|A}, A)$ by $\lambda(m_{|A})$.

In this section, we state some useful results. We begin by the following lemma which presents a transformation of the Neumann problem $P_{Ne}(m, I)$ to a Dirichlet problem in an extension of I (cf. [7] or [8])

Lemma 2.1. *[8,4] Let (u, λ^{Ne}) be a solution of problem $P_{Ne}(m, I)$. Assume that $Z(u) = \{z_1, z_2, ..., z_k\}$ then $(\bar{u}, \lambda^{Ne})$ is a solution of the following Dirichlet problem:*

$$P_D(\bar{m}_u, \bar{I}_u) \begin{cases} -(|u'|^{p-2}u')' = \lambda \bar{m}_u |u|^{p-2}u & in\ \bar{I}_u, \\ u(2a - z_1) = u(2b - z_k) = 0, \end{cases}$$

where $\bar{I}_u =]2a - z_1, 2b - z_k[$ and $\bar{u}$, $\bar{m}_u$ are defined as follows:

$$\bar{u}(x) = \begin{cases} u(2a - x) & if \quad x \in]2a - z_1, a[, \\ u(x) & if \quad x \in [a, b], \\ u(2b - x) & if \quad x \in]b, 2b - z_k[. \end{cases}$$

And

$$\bar{m}_u(x) = \begin{cases} m(2a - x) & if \quad x \in]2a - z_1, a[, \\ m(x) & if \quad x \in [a, b], \\ m(2b - x) & if \quad x \in]b, 2b - z_k[. \end{cases}$$

Consider the following Dirichlet problem:

$$P_D(m, I) \begin{cases} -(|u'|^{p-2}u')' = \lambda m |u|^{p-2}u & \text{in} \quad I =]a, b[, \\ u(a) = u(b) = 0, \end{cases}$$

where $m \in L^\infty(I)$ such that $mes(I^+) \neq 0$; $I^+ = \{x \in I / m(x) > 0\}$.
In [3], the authors showed the results in following theorem, which are essential for our work.

Theorem 2.2. *[3]*
Let $m \in M(I) = \{m \in L^\infty(I)/\ mes(\{x \in I, m(x) > 0\}) \neq 0\}$ such that: $m \not\equiv 0$ and $p \neq 2$. We have the following results:
1) Every eigenfunction corresponding to the k-th eigenvalue $\lambda_k^D(m, I)$ of the problem $P_D(m, I)$ has exactly $(k-1)$ zeros in $]a, b[$. Moreover, if any eigenfunction u corresponding to some eigenvalue $\lambda^D(m, I)$ has $(k-1)$ zeros in I then $\lambda^D(m, I) = \lambda_k^D(m, I)$.
2) For any k, $1 \le k \le n$, $\lambda_k^D(m, I)$ is simple and verify the strict monotonicity property with respect to the weight and the domain.
3) Let $m \in M(I)$, there exists a sequence $\lambda_k^D(m, I)$, $k = 1, 2, ...$ of eigenvalues associated to the problem $P_D(m, I)$ ordered as:

$$0 < \lambda_1^D(m, I) < \lambda_2^D(m, I) < ..... < \lambda_k^D(m, I) \to +\infty \ quand\ k \to +\infty.$$

4) For any integer n, $\lambda_n^D(m, I)$ can be written:

$$\frac{1}{\lambda_n^D(m, I)} = \sup_{F \in F_n} \inf_{F \cap S} \int_a^b m|v|^p dx,$$

where $F_n = \{F\ /\ F$ is a n dimensional subspace of $W_0^{1,p}(I)\}$.

The following propositions represents a multiplicity result corresponding to Neumann problem $P_{Ne}(m, I)$ and a caracterisation of the second eigenvalue λ_2^{Ne}.

Proposition 2.3. *(cf. [8])*
Let $n \geq 2$, the following result hold:

$$\text{If } \lambda_n^{Ne} = \lambda_{n+1}^{Ne} = \dots\dots\dots = \lambda_{n+q}^{Ne} \quad \text{then} \quad \gamma(K_{\lambda_n^{Ne}}) \geq q+1,$$

where

$$K_{\lambda_n^{Ne}} = \{u \in S \ / \ (u, \lambda_n^{Ne}) \text{ is a solution of } P_{Ne}(m)\}.$$

Proposition 2.4. *(cf. [8])*

$$\inf\{\lambda^{Ne} > \lambda_1^{Ne} / \ \lambda^{Ne} \text{ is an eigenvalue of } P_{Ne}(m)\} = \lambda_2^{Ne}. \tag{2.1}$$

Before closing this section, we prove the following result:

Proposition 2.5.
If (u, λ^{Ne}) is a solution of $P_{Ne}(m, I)$ then the set of zeros points $Z(u)$ of u is finite.

Proof: Let (u, λ^{Ne}) be a solution of $P_{Ne}(m, I)$. Suppose by contradiction that $Z(u)$ is infinite then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in I such that $u(x_n) = 0$, for every n. For a subsequence still denoted by $(x_n)_{n\in\mathbb{N}}$, we can assume that:

$$x_n \to x \text{ in } \ I; \ x_n \neq x, \ \forall n.$$

Since $u \in C^1(I)$ then :

$$u'(x) = \lim_{n\to\infty} \frac{u(x_n) - u(x)}{x_n - x} = 0.$$

However, according to the maximum principle of Vasquez, we obtain $u'(x) \neq 0$. This, is a contradiction; hence $Z(u)$ is finite. □

3. Simplicity of eigenvalues

This section is concerned with the simplicity of eigenvalues corresponding to $P_{Ne}(m, I)$.

Proposition 3.1. *For any $k \neq 1$, λ_k^{Ne} associated to $P_{Ne}(m, I)$ is simple.*

Proof: If (u, λ^{Ne}) is a solution of $P_{Ne}(m, I)$ such that $\lambda^{Ne} \neq \lambda_1^{Ne}$ then $Z(u)$ is finite.
Assume that $Z(u) = \{z_1(u), z_2(u), \dots\dots, z_n(u)\}$ verify $z_1(u) < z_2(u) < \dots\dots\dots < z_n(u)$.
Then according to lemma 2.1, $(\bar{u}, \lambda^{Ne})$ is a solution of Dirichlet problem $P_D(\bar{m}_u, \bar{I}_u)$ where $\bar{I}_u =]2a - z_1(u), 2b - z_n(u)[$,

$$\bar{u}(x) = \begin{cases} u(2a - x) & if \quad x \in]2a - z_1(u), a[, \\ u(x) & if \quad x \in [a, b], \\ u(2b - x) & if \quad x \in]b, 2b - z_n(u)[, \end{cases}$$

and

$$\bar{m}_u(x) = \begin{cases} m(2a-x) & if \quad x \in]2a - z_1(u), a[, \\ m(x) & if \quad x \in [a,b], \\ m(2b-x) & if \quad x \in]b, 2b - z_n(u)[. \end{cases}$$

Using theorem 2.2, we conclude that $\lambda^{Ne} = \lambda^D_{n+1}(\bar{m}_u, \bar{I}_u)$.
Let (v, λ^{Ne}) be another solution of $P_{Ne}(m, I)$, $v \neq u$, such that: $Z(v)$=$\{z_1(v), z_2(v),$ ....., $z_k(v)\}$. Taking $\bar{I}_v =]2a - z_1(v), 2b - z_k(v)[$. We show firstly that $\bar{I}_u = \bar{I}_v = \bar{I}$. Indeed, If $z_1(u) < z_1(v)$ (the case $z_1(u) > z_1(v)$ can be treated similarly), then

$$]2a - z_1(u), z_1(u)[\subsetneq]2a - z_1(v), z_1(v)[.$$

Since $\lambda^D_1(m, I)$ verify the SMP with respect to the domain (cf. Theorem 2.2), we get

$$\begin{aligned} \lambda^D_1(\bar{m}_{u|]2a-z_1(u),z_1(u)[}) &= \lambda^D_1(\bar{m}_{v|]2a-z_1(u),z_1(u)[}) \\ &> \lambda^D_1(\bar{m}_{v|]2a-z_1(v),z_1(v)[}). \end{aligned}$$

However, it follows

$$\lambda^{Ne} = \lambda^D_1(\bar{m}_{u|]2a-z_1(u),z_1(u)[}) = \lambda^D_1(\bar{m}_{v|]2a-z_1(v),z_1(v)[}),$$

we obtain

$$\begin{aligned} \lambda^{Ne} &= \lambda^D_1(\bar{m}_{u|]2a-z_1(u),z_1(u)[}) \\ &> \lambda^D_1(\bar{m}_{v|]2a-z_1(v),z_1(v)[}) \\ &= \lambda^{Ne}. \end{aligned}$$

Contradiction! then $z_1(u) = z_1(v)$.
A similar argument, applied to z_n, leads to $z_n(u) = z_k(v)$. Thus

$$\bar{I}_u = \bar{I}_v = \bar{I} \text{ and } \bar{m}_u = \bar{m}_v = \bar{m}.$$

Consequently,

$$\lambda^{Ne} = \lambda^D_{n+1}(\bar{m}, \bar{I}) = \lambda^D_{k+1}(\bar{m}, \bar{I}).$$

So, making use of theorem 2.2 (cf. [3]), one obtain $n = k$, $Z(\bar{u}) = Z(\bar{v})$ and there exists $\alpha \in \mathbb{R}$ such that $\bar{u} = \alpha\bar{v}$, i.e $\{u \in W^{1,p}_0(\bar{I}) \,/\, (\bar{u}, \lambda^{Ne})$ a solution of $P_D(\bar{m}, \bar{I})\}$ is a vector space of dimension 1. It follows then that $Z(u) = Z(v)$ and $u = \alpha v$; what implies that $\{u \in W^{1,p}(I)/(u, \lambda^{Ne})$ a solution of $P_{Ne}(m, I)\}$ is a vector space of one dimension. Then λ^{Ne} is simple. □

Proposition 3.2. *The eigenvalues of $P_{Ne}(m, I)$ are ordered as follows*

$$0 \leq \lambda^{Ne}_1(m, I) < \lambda^{Ne}_2(m, I) < .... < \lambda^{Ne}_n(m, I) \to +\infty \ ; \ n \to +\infty.$$

Proof: If m changes its sign on I and verify $\int_I m(x)dx < 0$ then $\lambda^{Ne}_1(m, I) > 0$. Moreover, from proposition 2.4, we have $\lambda^{Ne}_1(m, I) < \lambda^{Ne}_2(m, I)$.
If $n \geq 2$, from the simplicity of $\lambda^{Ne}_n(m, I)$ and the multiplicity property (see proposition 2.3), we have $\lambda^{Ne}_n(m, I) < \lambda^{Ne}_{n+1}(m, I)$. Indeed, if not, one gets $\gamma(K_{\lambda^{Ne}_n}) \geq 2$. Which leads to a contradiction with the simplicity of λ^{Ne}_n. □

4. On zeros points of eigenfunctions

In this section, we investigate the number of zeros points of an eigenfunction associated to an eigenvalue λ^{Ne}. For this end, we use a recurrence argument.
The result is well known for $n = 1$.
For $n = 2$, we have the following result for $\lambda_2^{Ne}(m, I)$

Proposition 4.1. *Any eigenfunction associated to $\lambda_2^{Ne}(m, I)$ has a unique zero z_1 in I. Moreover, if (u, λ^{Ne}) is a solution of $P_{Ne}(m, I)$ satisfying $Z(u) = \{z\}$ then $\lambda^{Ne} = \lambda_2^{Ne}$*

Proof: If $(u, \lambda_2^{Ne}(m))$ is a solution of $P_{Ne}(m, I)$. Assume that u has q zeros points in I, $q \geq 2$.
Let $Z(u) = \{z_1, z_2, ..., z_q\}$ with $z_1 < z_2, ... < z_q$ and define the functionals $(\varphi_i)_{0 \leq i \leq q}$ as follows :

$$\varphi_0(x) = \begin{cases} u(x) & if \quad a \leq x \leq z_1, \\ 0 & if \quad z_1 < x \leq b, \end{cases}$$

$$\varphi_1(x) = \begin{cases} u(x) & if \quad z_1 \leq x \leq z_2, \\ 0 & \qquad\qquad if \ not, \end{cases}$$

$$\vdots$$

$$\vdots$$

$$\varphi_q(x) = \begin{cases} u(x) & if \quad z_q \leq x \leq b, \\ 0 & \qquad\qquad if \ not. \end{cases}$$

Taking $K_q =< \varphi_0, \varphi_1, \ldots, \varphi_q > \cap S$ so that $\gamma(K_q) = q + 1$, one can easily verify the following equality:

$$\frac{1}{\lambda_2^{Ne}(m)} = \min_{K_q} \frac{\int_a^b m|u|^p dx}{\int_a^b |u'|^p dx},$$

which implies

$$\frac{1}{\lambda_{q+1}^{Ne}(m)} = \sup_{K \in \Gamma_{q+1}} \min_{K} \frac{\int_a^b m|u|^p dx}{\int_a^b |u'|^p dx} \geq \frac{1}{\lambda_2^{Ne}(m)},$$

and consequently $\lambda_{q+1}^{Ne}(m) \leq \lambda_2^{Ne}(m)$.
On the other hand, one has $\lambda_2^{Ne}(m) \leq \lambda_3^{Ne}(m) \leq \ldots \leq \lambda_{q+1}^{Ne}(m)$, from where we get

$$\lambda_2^{Ne}(m) = \lambda_3^{Ne}(m) = ... = \lambda_{q+1}^{Ne}(m).$$

So, making use of multiplicity property (cf. proposition 2.3), one has $\gamma(K_{\lambda_2^{Ne}}) \geq q \geq 2$. Which contradicts the fact that $\lambda_2^{Ne}(m)$ is simple. Hence $q = 1$ and the conclusion then follows: " λ_2^{Ne} has a unique zero in I".

It remains to prove that if (v, λ^{Ne}) is a solution of $P_{Ne}(m, I)$ which has one zero in I, i.e $Z(v) = \{z\}$ then $\lambda^{Ne}(m, I) = \lambda_2^{Ne}(m, I)$.
We proved that for any eigenfunction u corresponding to $\lambda_2^{Ne}(m)$, $Z(u) = \{z_1\}$ (uniqueness of the zero). So, assume that $z_1 < z$.
Since $(\bar{v}, \lambda^{Ne})$ (resp. $(\bar{u}, \lambda_2^{Ne}(m))$) is a solution of Dirichlet problem $P_D(\bar{m}_v, \bar{I}_v)$ (resp. $P_D(\bar{m}_u, \bar{I}_u)$), then using the SMP of λ_1^D with respect to the domain. One deduces firstly that

$$\begin{array}{rcl} \lambda^{Ne}(m, I) = \lambda_1^D(\bar{m}_{v|]2a-z,z[}) & < & \lambda_1^D(\bar{m}_{v|]2a-z_1,z_1[}) \\ & = & \lambda_1^D(\bar{m}_{u|]2a-z_1,z_1[}) \\ & = & \lambda_2^{Ne}(m, I). \end{array}$$

i.e $\lambda^{Ne}(m, I) < \lambda_2^{Ne}(m, I)$.
Secondly, since $]z, 2b - z[\subsetneq]z_1, 2b - z_1]$ then

$$\begin{array}{rcl} \lambda_2^{Ne}(m, I) & = & \lambda_2^D(\bar{m}_u, \bar{I}_u) \\ & = & \lambda_1^D(\bar{m}_{u|]z_1,2b-z_1[}) \\ & < & \lambda_1^D(\bar{m}_{u|]z,2b-z[}) = \lambda_1^D(\bar{m}_{v|]z,2b-z[}) \\ & = & \lambda_1^D(\bar{m}_{v|]z,2b-z[}) \\ & = & \lambda^{Ne}(m, I). \end{array}$$

Which leads to the required result. □

In order to show that the proposition result remains true for $n > 2$, we use a recurrence argument. Assume that for any k, $1 \leq k \leq n$, the following hypothesis holds:
Recurrence hypothesis For any eigenfunction u corresponding to $\lambda_k^{Ne}(m, I)$, there exists unique z_i; $1 \leq i \leq k-1$ such that $Z(u) = \{z_1, z_2, \ldots\ldots., z_{k-1}\}$. Moreover, if (v, λ^{Ne}) is a solution of $P_{Ne}(m, I)$ that has $(k-1)$ zeros points in I then $\lambda^{Ne} = \lambda_k^{Ne}(m, I)$.

In the following proposition, we prove the result for $\lambda_{n+1}^{Ne}(m, I)$.

Proposition 4.2. *For any eigenfunction u corresponding to $\lambda_{n+1}^{Ne}(m, I)$, there exists a family of unique $(z_i)_{1\leq i\leq n}$ such that $Z(u) = \{z_1, z_2, \ldots\ldots., z_n\}$. Moreover, if (v, λ^{Ne}) is a solution of $P_{Ne}(m, I)$ which has n zeros points in I then $\lambda^{Ne} = \lambda_{n+1}^{Ne}(m, I)$.*

Proof: Typically, as for $\lambda_2^{Ne}(m, I)$. Let u be an eigenfunction corresponding to $\lambda_{n+1}^{Ne}(m, I)$ which has q zeros in I: $Z(u) = \{z_1, z_2, \ldots\ldots\ldots\ldots., z_q\}$.
To prove that $q = n$, we distinguish two cases.
Case 1: If $\mathbf{q > n}$. We construct a compact symmetric K_q such that $\gamma(K_q) = q+1$. Indeed, taking $K_q =< \varphi_0, \varphi_1, \ldots, \varphi_q > \cap S$ where $(\varphi_i)_{1\leq i\leq q}$ are defined as above

and S the unit sphere of $W^{1,p}(I)$, one has $\gamma(K_q) = q+1$ and we verify easily that

$$\begin{aligned}\frac{1}{\lambda_{n+1}^{Ne}(m)} &= \min_{K_{q+1}} \frac{\int_a^b m|u|^p dx}{\int_a^b |u'|^p dx} \\ &\geq \sup_{K\in\Gamma_q} \min_{K} \frac{\int_a^b m|u|^p dx}{\int_a^b |u'|^p dx} \\ &= \frac{1}{\lambda_{q+1}^{Ne}(m)} ,\end{aligned}$$

i.e

$$\lambda_{q+1}^{Ne}(m) \leq \lambda_{n+1}^{Ne}(m).$$

However, since $\lambda_{n+1}^{Ne}(m) \leq \lambda_{n+2}^{Ne}(m) \leq \ldots\ldots \leq \lambda_{q+1}^{Ne}(m)$, then $\lambda_{n+1}^{Ne}(m) = \lambda_{n+2}^{Ne}(m) = \ldots\ldots = \lambda_{q+1}^{Ne}(m)$.
Using the multiplicity property (cf. Proposition 2.3), one deduces that $\gamma(K_{\lambda_{n+1}^{Ne}}) \geq q-n+1 \geq 2$, where $K_{\lambda_{n+1}^{Ne}} = \{u \in S \ / \ (u, \lambda_{n+1}^{Ne})$ is a solution of $P_{Ne}(m,I)\}$, which contradicts the fact that λ_{n+1}^{Ne} is simple.
Case 2: If q < n then $q \leq n-1$. From the recurrence hypothesis, u has q zeros in I and so it correspond to the $(q+1)$-th eigenvalue of $P_{Ne}(m,I)$ and verify $\lambda_{n+1}^{Ne}(m) = \lambda_{q+1}^{Ne}(m)$.
Since $q < n$ then $\lambda_{n+1}^{Ne}(m) = \lambda_{q+1}^{Ne}(m) \leq \lambda_n^{Ne}(m)$. Contradiction! Hence $q = n$ and the uniqueness of $(z_i)_{1\leq i\leq n}$ holds.

It remains now to prove that if (v, λ^{Ne}) is a solution of $P_{Ne}(m,I)$ with n zeros points in I then $\lambda^{Ne} = \lambda_{n+1}^{Ne}(m,I)$.
Let u be an eigenfunction associated to $\lambda_{n+1}^{Ne}(m,I)$ such that $Z(u) = \{z_1, z_2, \ldots\ldots, z_n\}$. Suppose that $Z(v) = \{z_1', z_2', \ldots\ldots, z_n'\}$. We know that $(\bar{u}, \lambda_{n+1}^{Ne}(m,I))$ (resp. $(\bar{v}, \lambda^{Ne})$) is a solution of $P_D(\bar{m}_u, \bar{I}_u)$ (resp. $P_D(\bar{m}_v, \bar{I}_v)$) where $\bar{I}_u =]2a-z_1, 2b-z_n[$, $\bar{I}_v =]2a-z_1', 2b-z_n'[$ and $\bar{m}_u$ (resp. $\bar{m}_v$) is defined on $\bar{I}_u$ (resp. $\bar{I}_v$) as above. Then

$$\lambda_{n+1}^{Ne}(m,I) = \lambda_{n+1}^{D}(\bar{m}_u, \bar{I}_u) \ \text{ and } \ \lambda^{Ne}(m,I) = \lambda_{n+1}^{D}(\bar{m}_v, \bar{I}_v). \tag{4.1}$$

Suppose now that $z_1 < z_1'$ then $]2a-z_1, z_1[\subsetneq]2a-z_1', z_1'[$. Firstly, by the SMP of λ_1^D with respect to the weight, one gets :

$$\begin{aligned}\lambda^{Ne}(m,I) &= \lambda_{n+1}^{D}(\bar{m}_v, \bar{I}_v) \\ &= \lambda_1^{D}(\bar{m}_{v|]2a-z_1',z_1'[}) \\ &< \lambda_1^{D}(\bar{m}_{v|]2a-z_1,z_1[}) \\ &< \lambda_1^{D}(\bar{m}_{u|]2a-z_1,z_1[}) \\ &= \lambda_{n+1}^{Ne}(m,I)\end{aligned}$$

i.e $\lambda^{Ne}(m,I) < \lambda_{n+1}^{Ne}(m,I)$.
Secondly, for z_n and z_n', we distinguish 3 cases :

1^st case: $\mathbf{z_n = z'_n}$. In this case $]z'_1, 2b - z'_n[\subsetneq]z_1, 2b - z_n[$. Since λ_n^D verify the SMP with respect to the weight then

$$\begin{array}{rcl} \lambda_{n+1}^{Ne}(m, I) & = & \lambda_{n+1}^{D}(\bar{m}_u, \bar{I}_u) \\ & = & \lambda_n^D(\bar{m}_{u|]z_1, 2b-z_n[}) \\ & < & \lambda_n^D(\bar{m}_{u|]z'_1, 2b-z'_n[}) \\ & = & \lambda_n^D(\bar{m}_{v|]z'_1, 2b-z'_n[}) \\ & = & \lambda_{n+1}^{D}(\bar{m}_v, \bar{I}_v) \\ & = & \lambda^{Ne}(m, I). \end{array}$$

i.e $\lambda^{Ne}(m, I) > \lambda_{n+1}^{Ne}(m, I)$.
2^nd case: $\mathbf{z_n < z'_n}$. Since $]z'_n, 2b - z'_n[\subsetneq]z_n, 2b - z_n[$ then

$$\begin{array}{rcl} \lambda_{n+1}^{Ne}(m, I) & = & \lambda_{n+1}^{D}(\bar{m}_u, \bar{I}_u) \\ & = & \lambda_1^D(\bar{m}_{u|]z_n, 2b-z_n[}) \\ & < & \lambda_1^D(\bar{m}_{u|]z'_n, 2b-z'_n[}) \\ & = & \lambda_1^D(\bar{m}_{v|]z'_n, 2b-z'_n[}) \\ & = & \lambda_{n+1}^{D}(\bar{m}_v, \bar{I}_v) \\ & = & \lambda^{Ne}(m, I). \end{array}$$

Consequently: $\lambda^{Ne}(m, I) > \lambda_{n+1}^{Ne}(m, I)$.
3^rd case: $\mathbf{z_n > z'_n}$. In this case, $]z'_1, z'_n[\subsetneq]z_n, z_n[$; $n > 1$ then

$$\begin{array}{rcl} \lambda_{n+1}^{Ne}(m, I) & = & \lambda_{n+1}^{D}(\bar{m}_u, \bar{I}_u) \\ & = & \lambda_{n-1}^D(\bar{m}_{u|]z_1, z_n[}) \\ & < & \lambda_{n-1}^D(\bar{m}_{u|]z'_1, z'_n[}) \\ & = & \lambda_{n-1}^D(\bar{m}_{v|]z'_1, z'_n[}) \\ & = & \lambda_{n+1}^{D}(\bar{m}_v, \bar{I}_v) \\ & = & \lambda^{Ne}(m, I). \end{array}$$

i.e $\lambda^{Ne}(m, I) > \lambda_{n+1}^{Ne}(m, I)$.
Finally, according to previous results, one deduces the desired result:

$$\lambda^{Ne}(m, I) = \lambda_{n+1}^{Ne}(m, I).$$

□

5. Strict monotonicity with respect to the weight

In this section, we look for the strict monotonicity property of eigenvalues with respect to the weight.
Note that for $\lambda_1^{Ne}(m, I)$, the SMP with respect to the weight holds (see a detailed proof in [8] for $m \in L^\infty$ or [10] for a weight function in L^r).

Remark 5.1. *1. In [9], the authors showed an important result concerning the strict monotonicity property of the second eigenvalue for the p-laplacian problem with weight: " If m_1, m_2 are two weight functions in L^∞ such that $m_1(x) \leq m_2(x)$ a.e and $meas(\{x \in I/m_1(x) = m_2(x)\} < C$ where C a well defined constant then $\lambda_2(m_1) > \lambda_2(m_2)$.*

2. In [4], A.Anane and A.Dakkak established a nonexistence result for an assymmetric one dimensional p-Laplacian under different kinds of boundary conditions (Dirichlet, Neumann and periodic one). They proved that if $m \in L^\infty(I)$ verify $\lambda_k \leq m \leq \lambda_{k+1}$ a.e in I and $\lambda_k < m < \lambda_{k+1}$ on some subset of nonzero measure, then 1 is not an eigenvalue of the p-Laplacian.

For $\lambda_2^{Ne}(m, I)$, we state the following result:

Proposition 5.2. *$\lambda_2^{Ne}(m, I)$ verify the strict monotonicity property with respect to the weight.*

Proof: Let $m, m' \in L^\infty(I)$ such that $mes\{x \in I/m(x) > 0\} \neq 0$ and $mes(\{x \in I/m'(x) > 0\}) \neq 0$.
Assume that $m(x) \leq m'(x)$ a.e in I and $m < m'$ in some subset of nonzero measure. Let us consider $(u, \lambda_2^{Ne}(m, I))$ (resp. $(v, \lambda_2^{Ne}(m', I))$) a solution of $P_{Ne}(m, I)$ (resp. $P_{Ne}(m', I)$) such that $Z(u) = \{z_1\}$ (resp. $Z(v) = \{z_1'\}$). We know that $\bar{u}$ (resp. $\bar{v}$) is a solution of Dirichlet problem $P_D(\bar{m},]2a - z_1, 2b - z_1[)$ (resp. $P_D(\bar{m'},]2a - z_1', 2b - z_1'[)$) associated with $\lambda_2^D(\bar{m},]2a - z_1, 2b - z_1[)$ (resp. $\lambda_2^D(\bar{m'},]2a - z_1', 2b - z_1'[)$) and $Z(\bar{u}) = \{z_1\}$ and $Z(\bar{v}) = \{z_1'\}$.

We proceed here similarly as in [3]. So, we distinguish three cases:
1^{st} case: If $\mathbf{z_1 = z_1' = z}$ Then, we have $\bar{m}$, $\bar{m'}$ are defined in $\bar{I} =]2a - z, 2b - z[$ and satisfy $\bar{m}(x) \leq \bar{m'}(x)$ a.e in $\bar{I}$ and $\bar{m} < \bar{m'}$ in some subset of nonzero measure of $\bar{I}$.
Since λ_2^D verify the SMP, then

$$\begin{array}{rcl} \lambda_2^{Ne}(m', I) & = & \lambda_2^D(\bar{m'}, \bar{I}_z) \\ & < & \lambda_2^D(\bar{m}, \bar{I}_z) \\ & = & \lambda_2^{Ne}(m, I) \end{array}$$

The result then follows.
2^{nd} case: $\mathbf{z_1 < z_1'}$. In this case, $\bar{m}_u(x) \leq \bar{m'}_v(x)$ a.e in $]2a - z_1, z_1[$ and $]2a - z_1, z_1[\subsetneq]2a - z_1', z_1'[$ then by using the monotonicity property and the SMP with respect to the weight of λ_1^D, we obtain

$$\begin{array}{rcl} \lambda_2^{Ne}(m, I) & = & \lambda_1^D(\bar{m}_{u|]2a-z_1,z_1[}) \\ & \geq & \lambda_1^D(\bar{m'}_{v|]2a-z_1,z_1[}) \\ & > & \lambda_1^D(\bar{m'}_{v|]2a-z_1',z_1'[}) \\ & = & \lambda_2^{Ne}(m', I). \end{array}$$

The result then holds.
3^{rd} case: $\mathbf{z_1' < z_1}$. Since $\bar{m}_u(x) \leq \bar{m'}_v(x)$ a.e in $]z_1, 2b - z_1[$ and $]z_1, 2b -$

$z_1[\subsetneq]z_1', 2b - z_1'[$ then similarly to the second case, we deduce that $\lambda_2^{Ne}(m, I) > \lambda_2^{Ne}(m', I)$. □

Now, we have to prove that the above result is valid for $\lambda_n^{Ne}(m, I), n > 2$.

Proposition 5.3. *For $n > 2$, $\lambda_n^{Ne}(m, I)$ verify the SMP with respect to the weight.*

Proof: Let $m, m' \in L^\infty(I)$ such that $mes\{x \in I/m(x) > 0\} \neq 0$ and $mes\{x \in I/m'(x) > 0\} \neq 0$.
Suppose that $m(x) \leq m'(x)$ a.e in I and $m < m'$ in some subset of nonzero measure. Let $(u, \lambda_n^{Ne}(m, I))$ (resp. $(v, \lambda_n^{Ne}(m', I))$) a solution of $P_{Ne}(m, I)$ (resp. $P_{Ne}(m', I)$) such that $Z(u) = \{z_1, z_2, ..., z_{n-1}\}$ (resp. $Z(v) = \{z_1', z_2', ..., z_{n-1}'\}$).
$\bar{u}$ (resp. $\bar{v}$) is a solution of Dirichlet problem $P_D(\bar{m}_u, \bar{I}_u)$ (resp. $P_D(\bar{m'}_v, \bar{I}_v)$) corresponding to $\lambda_n^D(\bar{m}_u, \bar{I}_u)$ (resp. $\lambda_n^D(\bar{m'}_v, \bar{I}_v)$) where $\bar{m}_u$ (resp. $\bar{m'}_v$) is defined on $\bar{I}_u =]2a - z_1, 2b - z_{n-1}[$ (resp. $\bar{I}_v =]2a - z_1', 2b - z_{n-1}'[$). As previously, we have $Z(\bar{u}) = Z(u)$ (resp. $Z(\bar{v}) = Z(v)$) and $\lambda_n^{Ne}(m, I) = \lambda_n^D(\bar{m}_u, \bar{I}_u)$ (resp. $\lambda_n^{Ne}(m', I) = \lambda_n^D(\bar{m'}_v, \bar{I}_v)$).
To prove the result, we use mainly the SMP with respect to the weight and domain of Dirichlet problem eigenvalues. We distinguish three cases for z_1 and z_1': ($z_1 = z_1' = z$), ($z_1 < z_1'$) and ($z_1 > z_1'$). For each case, we study three sub-cases related to z_{n-1} and z_{n-1}'.
1st case: $\mathbf{z_1 = z_1' = z}$.
(**1**) **If $\mathbf{z_{n-1} = z_{n-1}' = z'}$.** By the definition of $\bar{m}_u$ and $\bar{m'}_v$, one has $\bar{m}_u(x) \leq \bar{m'}_v(x)$ a.e in $\bar{I} = \bar{I}_u = \bar{I}_v$ and $\bar{m}_u < \bar{m'}_v$ on some subset of nonzero measure. Then

$$\begin{aligned}\lambda_n^{Ne}(m, I) &= \lambda_n^D(\bar{m}_u, \bar{I})\\ &> \lambda_n^D(\bar{m'}_v, \bar{I})\\ &= \lambda_n^{Ne}(m', I)\end{aligned}$$

(**2**) **If $\mathbf{z_{n-1} < z_{n-1}'}$.** Then $\bar{m}_u(x) \leq \bar{m'}(x)$ a.e in $]2a-z, z_{n-1}[$ and $]2a-z, z_{n-1}[\subsetneq]2a - z, z_{n-1}'[$ which implies the following result:

$$\begin{aligned}\lambda_n^{Ne}(m, I) &= \lambda_{n-1}^D(\bar{m}_{u|]2a-z,z_{n-1}[})\\ &\geq \lambda_{n-1}^D(\bar{m'}_{v|]2a-z,z_{n-1}[})\\ &> \lambda_{n-1}^D(\bar{m'}_{v|]2a-z,z_{n-1}'[})\\ &= \lambda_n^D(\bar{m'}_v, \bar{I}_v)\\ &= \lambda_n^{Ne}(m', I)\end{aligned}$$

(**3**) **If $\mathbf{z_{n-1} > z_{n-1}'}$.** We obtain

$$\begin{aligned}\lambda_n^{Ne}(m, I) &= \lambda_1^D(\bar{m}_{u|]z_{n-1},2b-z_{n-1}[})\\ &\geq \lambda_1^D(\bar{m'}_{v|]z_{n-1},2b-z_{n-1}[})\\ &> \lambda_1^D(\bar{m'}_{v|]z_{n-1}',2b-z_{n-1}'[})\\ &= \lambda_n^{Ne}(m', I)\end{aligned}$$

$\mathbf{2^{nd}}$ **case:** $\mathbf{z_1 < z'_1}$.
$(\mathbf{1})$ **If** $\mathbf{z_{n-1} = z'_{n-1} = z'}$. Like the $\mathbf{1^{st}}$ case: $(\mathbf{2})$, we have

$$\begin{array}{lll} \lambda_n^{Ne}(m,I) & = & \lambda_{n-1}^{D}(\bar{m}_{u|]2a-z_1,z'[}) \\ & \geq & \lambda_{n-1}^{D}(\bar{m'}_{v|]2a-z_1,z'[}) \\ & > & \lambda_{n-1}^{D}(\bar{m'}_{v|]2a-z'_1,z'[}) \\ & = & \lambda_n^{Ne}(m',I) \end{array}$$

$(\mathbf{2})$ **If** $\mathbf{z_{n-1} < z'_{n-1}}$. In this case, $\bar{m}_u(x) \leq \bar{m'}_v(x)$ a.e in $]2a-z_1, z_1[$ and $]2a-z_1, z_1[\subsetneq]2a-z'_1, z'_1[$ then

$$\begin{array}{lll} \lambda_n^{Ne}(m,I) & = & \lambda_1^{D}(\bar{m}_{u|]2a-z_1,z_1[}) \\ & \geq & \lambda_1^{D}(\bar{m'}_{v|]2a-z_1,z_1[}) \\ & > & \lambda_1^{D}(\bar{m'}_{v|]2a-z'_1,z'_1[}) \\ & = & \lambda_n^{Ne}(m',I) \end{array}$$

$(\mathbf{3})$ **If** $\mathbf{z_{n-1} > z'_{n-1}}$. This case can be proved exactly like the second case: $(\mathbf{2})$.
$\mathbf{3^{rd}}$ **case:** $\mathbf{z'_1 < z_1}$.
$(\mathbf{1})$ **If** $\mathbf{z_{n-1} = z'_{n-1} = z'}$. Since $\bar{m}_u(x) \leq \bar{m'}_v(x)$ a.e in $]z_1, 2b-z'[$ and $]z_1, 2b-z'[\subsetneq]z'_1, 2b-z'_1[$ then

$$\begin{array}{lll} \lambda_n^{Ne}(m,I) & = & \lambda_{n-1}^{D}(\bar{m}_{u|]z_1,2b-z'[}) \\ & \geq & \lambda_{n-1}^{D}(\bar{m'}_{v|]z_1,2b-z'[}) \\ & > & \lambda_{n-1}^{D}(\bar{m'}_{v|]z'_1,2b-z'[}) \\ & = & \lambda_n^{Ne}(m',I) \end{array}$$

$(\mathbf{2})$ **If** $\mathbf{z_{n-1} < z'_{n-1}}$. One has: $\bar{m}_u(x) \leq \bar{m'}_v(x)$ a.e in $]z_1, z_{n-1}[$, $]z_1, z_{n-1}[\subsetneq]z'_1, z'_{n-1}[$ and $n > 2$, so

$$\begin{array}{lll} \lambda_n^{Ne}(m,I) & = & \lambda_{n-2}^{D}(\bar{m}_{u|]z_1,z_{n-1}[}) \\ & \geq & \lambda_{n-2}^{D}(\bar{m'}_{v|]z_1,z_{n-1}[}) \\ & > & \lambda_{n-2}^{D}(\bar{m'}_{v|]z'_1,z'_{n-1}[}) \\ & = & \lambda_n^{Ne}(m',I) \end{array}$$

$(\mathbf{3})$ **If** $\mathbf{z_{n-1} > z'_{n-1}}$. The same proof as for first case: $(\mathbf{3})$.
Finally, $\lambda_n^{Ne}(m,I) > \lambda_n^{Ne}(m',I)$; $\forall n > 2$. The proof is complete. □

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