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Integral Transform Method for Solving Time Fractional Systems and Fractional Heat Equation

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ABSTRACT: In the present paper, time fractional partial differential equation is considered, where the fractional derivative is defined in the Caputo sense. Laplace transform method has been applied to obtain an exact solution. The authors solved certain homogeneous and nonhomogeneous time fractional heat equations using integral transform. Transform method is a powerful tool for solving fractional singular Integro - differential equations and PDEs. The result reveals that the transform method is very convenient and effective.

Key Words: Caputo fractional derivative; Time fractional heat equation; Laplace transform; Singular integral equation.

Contents

1. Introduction and Definitions

Time fractional partial differential equations, obtained from the standard partial differential equations by replacing the integer order time derivative by a fractional derivative have been studied and treated in different contexts by several research workers. The fractional diffusion equation, the fractional wave equation, the fractional advection-dispersion equation, the fractional kinetic equation and other fractional PDEs have been studied and explicit solutions have been achieved by Mainardi, Pagnini and Saxena [1], Langlands [2], Mainardi, Pagnini and Gorenflo [3], Mainardi and Pagnini [4,5], Yu and Zhang [6], Liu, Anh, Turner and Zhang [7], Saichev and Zaslavsky [8], Saxena, Mathai and Haubold [9], Wyss [10], Schneider and Wyss [11] and several other research works can be found in the literature [15,16]. In these works, the techniques of using integral transforms were used to obtain the formal solutions of fractional PDEs. Integral transforms are extensively

2000 *Mathematics Subject Classification:* 26A33, 34A08, 34K37, 35R11, 44A10 used in solving boundary value problems and integral equations. The problem related to partial differential equations can be solved by using a special integral transform thus many authors solved the boundary value problems by using single Laplace transform. Laplace transform is very useful in applied mathematics, for instance for solving some differential equations and partial differential equations, and in automatic control, where it defines a transfer function. The left Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(x)}{(t-x)^{1-\alpha}} dx. $$

The left Caputo fractional derivatives of order $\alpha > 0$ $(n-1 < \alpha \leq n, n \in N)$ is defined by

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx. $$

Laplace transform of function $f(t)$ is given as

$$ L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt := F(s). $$

If $L\{f(t)\} = F(s)$, then $L^{-1}\{F(s)\}$ is given by

$$ f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, $$

where $F(s)$ is analytic in the region $Re(s) > c$. For $n-1 < \alpha \leq n$, one gets

$$ L\{{}_0^C D_t^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0). $$

Two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. $$

The simplest Wright function is given by the series

$$ W(\alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n + \beta)}, $$

when $\alpha, \beta, z \in C$. We have the following relationship

$$ L\{t^{\beta-1} E_{\alpha,\beta}(\pm a t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha \mp a} \, (Re(s) > |a|^{\frac{1}{\alpha}}). $$

Definition 1.1. *The Fresnel integrals are defined through the following integral representations*

$$Fresnelc(x) = \int_0^x \cos \xi^2 d\xi,$$

$$Fresnels(x) = \int_0^x \sin \xi^2 d\xi.$$

They arise in the description of near field Fresnel diffraction phenomena.

Theorem 1.2. ***(Schouten-Van der Pol Theorem)*** *Consider a function $f(t)$ which has the Laplace transform $F(s)$ which is analytic in the half-plane $Re(s) > s_0$. We can use this knowledge to find $g(t)$ whose Laplace transform $G(s)$ equals $F(\phi(s))$, where $\phi(s)$ is also analytic for $Re(s) > s_0$. This means that if*

$$G(s) = F(\phi(s)) = \int_0^\infty f(\tau) \exp(-\phi(s)\tau) d\tau$$

and

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\phi(s)) \exp(ts) ds,$$

then

$$g(t) = \int_0^\infty f(\tau) \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(-\phi(s)\tau) \exp(ts) ds \right) d\tau.$$

Proof: See [13]. □

2. Singular Integral Equations of Fractional Order

Laplace transform can be used to solve certain types of singular integral equations. The mathematical formulation of physical phenomena often involves Cauchy type, or more severe, singular integral equations. There are many applications in many important fields, like fracture mechanics, elastic contact problems, the theory of porous filtering contain integral and integro - differential equation with singular kernel.

Lemma 2.1. *The following class of Fredholm singular integral equation of second kind of the following type*

$$\phi(x) = f(x) + \lambda \int_0^\infty (\frac{x}{t})^{\frac{n}{2}} J_n(2\sqrt{xt}) \phi(t) dt, \ n = 0, 1, 2, .. \tag{2.1}$$

has the formal solution as

$$\phi(x) = \frac{f(x)}{1-\lambda^2} + \frac{\lambda}{(1-\lambda^2)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{s^{n+1}} F(\frac{1}{s}) ds. \tag{2.2}$$

Solution: Let $L(f(t)) = F(s)$ and $L(\phi(t)) = \Phi(s)$ be the Laplace transforms of $f(t)$ and $\phi(t)$, respectively, then by using the Laplace transform of (2.1) we have,

$$\Phi(s) = F(s) + \lambda \frac{1}{s^{n+1}} \Phi(\frac{1}{s}) \tag{2.3}$$

Now, in relation (2.3) we replace s with $\frac{1}{s}$, to obtain

$$\Phi(\frac{1}{s}) = F(\frac{1}{s}) + \lambda s^{n+1} \Phi(s). \tag{2.4}$$

Combination of (2.3) and (2.4) and calculation of $\Phi(s)$ leads to the following,

$$\Phi(s) = \frac{F(s) + \frac{\lambda}{s^{n+1}} F(\frac{1}{s})}{1 - \lambda^2}. \tag{2.5}$$

Upon using complex inversion formula, relation (2.5) leads to the following,

$$\phi(x) = \frac{f(x)}{1-\lambda^2} + \frac{\lambda}{(1-\lambda^2)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{s^{n+1}} F(\frac{1}{s}) ds.$$

Example 2.2. *Solve the following singular integral equation*

$$\phi(x) = \cosh x + \lambda \int_0^\infty \sqrt{\frac{x}{t}} J_1(2\sqrt{xt}) \phi(t) dt.$$

Solution: Laplace-transform of the above integral equation, leads to the following

$$\phi(x) = \frac{\cosh x}{1-\lambda^2} + \frac{\lambda}{(1-\lambda^2)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{s(s^2-1)} ds,$$

thus, the final solution is

$$\phi(x) = \frac{(1-\lambda)\cosh x + \lambda}{(1-\lambda^2)}.$$

Lemma 2.3. *The following fractional Fredholm singular integro-differential equation of the form,*

$${}_0^C D_x^\alpha \phi(x) = f(x) + \lambda \int_0^\infty (\frac{x}{t})^{\frac{n}{2}} J_n(2\sqrt{xt}) \phi(t) dt, \tag{2.6}$$

where $\phi(0) = 0$ and $0 < \alpha \leq 1$, has the formal solution as

$$\phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s^{-\alpha} F(s) + \frac{\lambda}{s^{n+1}} F(\frac{1}{s})}{1-\lambda^2} e^{st} ds. \tag{2.7}$$

Solution: Let $L(\phi(t)) = \Phi(s)$ and $L(f(t)) = F(s)$ be the Laplace transforms of $\phi(t)$ and $f(t)$, respectively, then by using the Laplace transform of (2.6) we have the following relation,

$$s^{\alpha}\Phi(s) = F(s) + \lambda \frac{1}{s^{n+1}}\Phi(\frac{1}{s}), \tag{2.8}$$

now, in relation (2.8) we replace s by $\frac{1}{s}$, to obtain

$$s^{-\alpha}\Phi(\frac{1}{s}) = F(\frac{1}{s}) + \lambda s^{n+1}\, \Phi(s). \tag{2.9}$$

Combination of (2.8) and (2.9) and calculation of $\Phi(s)$ leads to the following,

$$\Phi(s) = \frac{s^{-\alpha}F(s) + \frac{\lambda}{s^{n+1}}F(\frac{1}{s})}{1-\lambda^2}. \tag{2.10}$$

Upon using complex inversion formula, relation (2.10) leads to the following,

$$\phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{s^{-\alpha}F(s) + \frac{\lambda}{s^{n+1}}F(\frac{1}{s})}{1-\lambda^2} e^{st} ds.$$

Example 2.4. *Solve the following fractional singular integral equation*

$${}_0^C D_x^{\alpha}\phi(x) = \sin x + \int_0^{\infty} (\frac{x}{t}) J_2(2\sqrt{xt})\phi(t)dt,$$

where $\phi(0) = 0$ *and* $0 < \alpha \leq 1$.

Solution: Upon using relation (2.11) leads to the following

$$\begin{aligned}
\phi(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{1-\lambda^2}\left(\frac{1}{s^{\alpha}(s^2+1)} + \frac{\lambda}{s(s^2+1)}\right) e^{xs} ds. \\
&= \frac{1}{1-\lambda^2}\left({}_0I_x^{\alpha} \sin x + \lambda(1-\cos x)\right).
\end{aligned}$$

Special case: When $\alpha = 0.5$, we get

$$\begin{aligned}
\phi(x) &= \frac{1}{1-\lambda^2}\left({}_0I_x^{\frac{1}{2}} \sin x + \lambda(1-\cos x)\right) \\
&= \frac{\sqrt{2}\sin x Fresnelc(\sqrt{\frac{2x}{\pi}}) - \sqrt{2}\cos x Fresnels(\sqrt{\frac{2x}{\pi}})}{1-\lambda^2} + \frac{\lambda}{1-\lambda^2}(1-\cos x).
\end{aligned}$$

3. A Method for Obtaining Inverse Laplace Transform

Let us recall a theorem developed by Bobylev and Cercignani [13,14] concerning the inversion of multivalued transforms that are analytic everywhere in the $s-$ plane except along the negative real axis. The theorem is as follows:

Theorem 3.1. ***(Bobylev - Cercignani Theorem)*** *Let $f(t)$ denote a real-valued function, where its Laplace transform $F(s)$ exists. Let $F(s)$ satisfy the following hypothesis:*

1) $F(s)$ is a multi valued function which has no singularities in the cut $s-$ plane. The branch cut lies along the negative real axis $(-\infty, 0]$.

2) $F^*(s) = F(s^*)$, where the star denotes the complex conjugate.

3) $F^{\pm}(\eta) = \lim_{\phi \to \pi^-} F(\eta e^{\pm \phi i})$ and $F^+(\eta) = (F_-(\eta))^*$.

4) $F(s) = o(1)$ as $|s| \to \infty$ and $F(s) = o(\frac{1}{|s|})$ as $|s| \to 0$, uniformly in any sector $|\arg(s)| < \pi - \eta$, $0 < \eta < \pi$.

5) There exists $\varepsilon > 0$, such that for every $\pi - \varepsilon < \phi \le \pi$, $\frac{F(re^{\pm\phi i})}{1+r} \in L_1(R^+)$ and $|F(re^{\pm \phi i})| < a(r)$, where $a(r)$ does not depend on ϕ and $a(r)e^{-\delta r} \in L_1(R^+)$ for any $\delta > 0$.

then

$$f(t) = \frac{1}{\pi}\int_0^\infty Im(F^-(\eta))e^{-t\eta}d\eta.$$

We now present some interesting applications of the above theorem.

Lemma 3.2. *The following relationship holds true*

$$L^{-1}\left\{\frac{1}{s^\alpha(\sqrt{s}+b)}\exp\left(-x\sqrt{\frac{s^\alpha}{s^\alpha+a}}\right)\right\} = \frac{1}{\pi}\int_0^\infty Im(F^-(\eta))e^{-t\eta}d\eta,$$

where $0 < \alpha < 1$, $a, b > 0$ and

$$Im(F^-(\eta)) = \frac{e^{-x\sqrt{\frac{\eta^\alpha}{\rho}}\cos(\frac{\pi\alpha-\theta}{2})}}{\eta^\alpha(\eta+b^2)}\times$$

$$\left\{\sqrt{\eta}\cos\left(x\sqrt{\frac{\eta^\alpha}{\rho}}\sin(\frac{\pi\alpha-\theta}{2})+\pi\alpha\right)+b\sin\left(x\sqrt{\frac{\eta^\alpha}{\rho}}\sin(\frac{\pi\alpha-\theta}{2})+\pi\alpha\right)\right\}.$$

Proof: Clearly, $F(s)$ satisfies all of the conditions listed in the Theorem 3.1, So that

$$F^-(\eta) = \lim_{\phi\to\pi} F(\eta e^{-\phi i}) = \frac{1}{\eta^\alpha e^{-\pi\alpha i}(-\sqrt{\eta}i+b)}\exp\left(-x\frac{\eta^{\frac{\alpha}{2}}e^{-\frac{\pi\alpha}{2}i}}{\sqrt{\eta^\alpha e^{-\pi\alpha i}+a}}\right).$$

Now, we have

$$F^-(\eta) = \frac{e^{\pi\alpha i}(\sqrt{\eta}i + b)}{\eta^\alpha(\eta + b^2)} \exp\left(-x\sqrt{\frac{\eta^\alpha}{\rho}} e^{-\frac{\pi\alpha}{2}i} e^{\frac{\theta i}{2}}\right),$$

where

$$\rho = \sqrt{\eta^{2\alpha} + 2a\eta^\alpha \cos \pi\alpha + a^2},$$

$$\theta = \tan^{-1}\left(\frac{\eta^\alpha \sin \alpha\pi}{\eta^\alpha \cos \alpha\pi + a}\right) (0 < \theta < \pi).$$

Thus

$$F^-(\eta) = \frac{e^{\pi\alpha i}(\sqrt{\eta}i + b)}{\eta^\alpha(\eta + b^2)} \exp\left(-x\sqrt{\frac{\eta^\alpha}{\rho}}(\cos(\frac{\theta - \pi\alpha}{2}) + i\sin(\frac{\theta - \pi\alpha}{2}))\right),$$

as a consequence, we get

$$Im(F^-(\eta)) = \frac{e^{-x\sqrt{\frac{\eta^\alpha}{\rho}}\cos(\frac{\pi\alpha - \theta}{2})}}{\eta^\alpha(\eta + b^2)} \times$$

$$\left\{\sqrt{\eta}\cos\left(x\sqrt{\frac{\eta^\alpha}{\rho}}\sin(\frac{\pi\alpha - \theta}{2}) + \pi\alpha\right) + b\sin\left(x\sqrt{\frac{\eta^\alpha}{\rho}}\sin(\frac{\pi\alpha - \theta}{2}) + \pi\alpha\right)\right\}.$$

Therefore, the inverse Laplace transform is as following

$$f(t) = \frac{1}{\pi}\int_0^\infty Im(F^-(\eta))e^{-t\eta} d\eta.$$

□

Problem 3.3: Let us consider the following partial fractional differential equation

$$\frac{\partial}{\partial x}\{\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\} + a\frac{\partial^\beta u(x,t)}{\partial t^\beta} + b\frac{\partial u(x,t)}{\partial x} = 0,$$

where $0 < \alpha < 1$, $0 < \beta \leq 1$, $0 < x < \infty$, $t, a, b > 0$ with the boundary conditions

$$u(0,t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \lim_{x\to\infty} |u(x,t)| < \infty,$$

and the initial conditions $u(x,0) = u_x(x,0) = 0$.

Solution: The Laplace transform of the fractional PDE and boundary and initial conditions yields

$$U(x,s) = \frac{1}{s^\alpha}\exp\left(-\frac{axs^\beta}{s^\alpha + b}\right).$$

$U(x,s)$ satisfies all of the conditions listed in the Theorem 3.1, therefore

$$
\begin{aligned}
U^{-}(x,\eta) &= \lim_{\phi\to\pi} U(x,\eta e^{-\phi i}) = \frac{1}{\eta^{\alpha}e^{-\pi\alpha i}} \exp\left(-\frac{ax\eta^{\beta}e^{-\pi\beta i}}{\eta^{\alpha}e^{-\pi\alpha i}+b}\right) \\
&= \frac{e^{\pi\alpha i}}{\eta^{\alpha}} \exp\left(-\frac{ax\eta^{\beta}e^{-\pi\beta i}(\eta^{\alpha}e^{\pi\alpha i}+b)}{\rho}\right),
\end{aligned}
$$

where $\rho = \eta^{2\alpha} + 2b\eta^{\alpha}\cos\pi\alpha + b^2$. Then

$$
Im(U^{-}(x,\eta)) = \frac{1}{\eta^{\alpha}} \exp\left\{-\frac{abx\eta^{\beta}\cos\beta\pi + ax\eta^{\alpha+\beta}\cos(\beta-\alpha)\pi}{\rho}\right\}
$$

$$
\times \sin\left\{\pi\alpha + \frac{abx\eta^{\beta}\sin\beta\pi + ax\eta^{\alpha+\beta}\sin(\beta-\alpha)\pi}{\rho}\right\}.
$$

Finally, the solution is obtained as

$$
u(x,t) = \frac{1}{\pi}\int_0^{\infty} Im(U^{-}(x,\eta))e^{-t\eta}d\eta.
$$

4. Fractional PDE with Moving Boundary

In fractional PDE problems, we showed that Laplace transforms are particularly - useful when the boundary conditions are time dependent. Consider now the case when one of the boundaries is moving. This type of problem arises in combustion problems where the boundary moves due to the burning of the fuel [13].

Problem 4.1: Let us solve the following time-fractional heat equation

$$
\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = a^2\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{4.1}
$$

where $0<\alpha\le 1$, $\beta t<x<\infty$, $t>0$ and subject to the boundary conditions

$$
u(x,t)\Big|_{x=\beta t} = \frac{1}{\sqrt{\pi t}}\exp(-\frac{1}{4t}), \lim_{x\to\infty}|u(x,t)|<\infty,
$$

and the initial condition

$$
u(x,0) = 0\,(0<x<\infty).
$$

Solution: By introducing the new coordinate $\eta = x-\beta t$, then the problem can be reformulated as

$$
\frac{\partial^{\alpha}w(\eta,t)}{\partial t^{\alpha}} - \beta\frac{\partial}{\partial\eta}\left({}_0I_t^{1-\alpha}w(\eta,t)\right) = a^2\frac{\partial^2 w(\eta,t)}{\partial\eta^2}, \tag{4.2}
$$

where $0<\eta<\infty$, $t>0$ and subject to the boundary conditions

$$
w(0,t) = \frac{1}{\sqrt{\pi t}}\exp(-\frac{1}{4t}), \lim_{\eta\to\infty}|w(\eta,t)|<\infty,
$$

and the initial condition

$$w(\eta, 0) = 0\,(0 < \eta < \infty).$$

Taking the Laplace transform of the equation (4.2), we obtain

$$\frac{\partial^2 W(\eta, s)}{\partial \eta^2} + \frac{\beta}{a^2 s^{1-\alpha}} \frac{\partial W(\eta, s)}{\partial \eta} - \frac{s^\alpha}{a^2} W(\eta, s) = 0 \tag{4.3}$$

with

$$W(0, s) = \frac{e^{-\sqrt{s}}}{\sqrt{s}}, \lim_{\eta\to\infty} |W(\eta, t)| < \infty.$$

The solution to the equation (4.3) is

$$W(\eta, s) = \frac{e^{-\sqrt{s}}}{\sqrt{s}} \exp\left(-\frac{\beta\eta}{2a^2 s^{1-\alpha}} - \frac{\eta}{a s^{1-\alpha}} \sqrt{s^{2-\alpha} + \frac{\beta^2}{4a^2}} \right).$$

Case 1: If $\alpha = 1$, then the solution is obtained as [13]

$$u(x, t) = e^{-\frac{\beta(x-\beta t)}{2a^2}} \int_0^t \frac{e^{-\frac{1}{4(t-\tau)}}}{\sqrt{\pi(t-\tau)}} \phi(x - \beta\tau, \tau) d\tau,$$

where

$$\phi(x-\beta\tau, \tau) = \frac{1}{2}\left(e^{-\frac{\beta(x-\beta\tau)}{2a^2}} erfc(\frac{x-\beta\tau}{2a\sqrt{\tau}} - \frac{\beta\sqrt{\tau}}{2a}) + e^{\frac{\beta(x-\beta\tau)}{2a^2}} erfc(\frac{x-\beta\tau}{2a\sqrt{\tau}} + \frac{\beta\sqrt{\tau}}{2a}) \right).$$

Case 2: If $\alpha \neq 1$, we apply the Bobylev-Cercignani Theorem 3.1. Therefore

$$\begin{aligned} W^-(\eta, \xi) &= \lim_{\phi\to\pi} W(\eta, \xi e^{-\phi i}) \\ &= \frac{\exp(-\sqrt{\xi} e^{-\frac{\pi\alpha}{2} i})}{\sqrt{\xi} e^{-\frac{\pi}{2} i}} \exp\Bigg(-\frac{\beta\eta e^{\pi(1-\alpha)i}}{2a^2 \xi^{1-\alpha}} \\ &\qquad - \frac{\eta e^{\pi(1-\alpha)i}}{a\xi^{1-\alpha}} \sqrt{\xi^{2-\alpha} e^{-\pi(2-\alpha)i} + \frac{\beta^2}{4a^2}} \Bigg) \\ &= \frac{\exp(-\sqrt{\xi} e^{-\frac{\pi}{2} i})}{\sqrt{\xi} e^{-\frac{\pi}{2} i}} \exp\left(-\frac{\beta\eta e^{\pi(1-\alpha)i}}{2a^2 \xi^{1-\alpha}} - \frac{\eta e^{\pi(1-\alpha)i}}{a\xi^{1-\alpha}} \sqrt{\rho} e^{\frac{\theta i}{2}} \right), \end{aligned}$$

where

$$\rho = \sqrt{\xi^{4-2\alpha} + \frac{\beta^2}{2a^2} \xi^{2-\alpha} \cos(2-\alpha)\pi + \frac{\beta^4}{16a^4}},$$

$$\theta = \tan^{-1}\left(-\frac{\xi^{2-\alpha}\sin(2-\alpha)\pi}{\xi^{2-\alpha}\cos(2-\alpha)\pi + \frac{\beta^2}{4a^2}}\right)\ (0 < \theta < \pi).$$

Hence

$$W^{-}(\eta,\xi) = \frac{i\exp(\sqrt{\xi}i)}{\sqrt{\xi}}\exp\left(-\frac{\beta\eta e^{\pi(1-\alpha)i}}{2a^2\xi^{1-\alpha}} - \frac{\eta e^{\pi(1-\alpha)i}}{a\xi^{1-\alpha}}\sqrt{\rho}e^{\frac{\theta i}{2}}\right),$$

that yields

$$Im(W^{-}(\eta,\xi)) =$$
$$\frac{1}{\sqrt{\xi}}\exp\left(-\frac{\beta\eta\cos(1-\alpha)\pi}{2a^2\xi^{1-\alpha}} - \frac{\eta\sqrt{\rho}}{a\xi^{1-\alpha}}\cos(\frac{\theta}{2}+(1-\alpha)\pi)\right)$$
$$\times\cos\left(\frac{\beta\eta\sin(1-\alpha)\pi}{2a^2\xi^{1-\alpha}} + \frac{\eta\sqrt{\rho}}{a\xi^{1-\alpha}}\sin(\frac{\theta}{2}+(1-\alpha)\pi) - \sqrt{\xi}\right).$$

The formal solution is as following,

$$u(x,t) = \frac{1}{\pi}\int_0^\infty Im(W^{-}(x-\beta\xi,\xi))e^{-t\xi}d\xi.$$

5. A Problem of Fractional Order System with Three Variables

In this section, the authors considered certain non-homogeneous time fractional system of heat equations which is a generalization to the problem of heat transferring from metallic bar through the surrounding media studied by V.A. Ditkin, P.A. Prudnikov [12]. In this work, only the Laplace transformation is considered as it is easily understood and being popular among engineers and scientists. The basic goal of this work has been to employ the Laplace transform method for studying the above mentioned problem. The goal has been achieved by formally deriving exact analytical solution.

Problem 5.1: We consider the following system of fractional order

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \lambda a\frac{\partial v}{\partial r}\bigg|_{r=a} + 1 - \gamma u \tag{5.1}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \beta v \tag{5.2}$$

where $0 < \alpha \le 1$, $t > 0$, $-l \le x \le l$, $r \ge a$ and $\beta,\gamma \in R$ with the boundary conditions

$$u(x,0) = v(x,r,0) = 0, u(-l,t) = u(l,t) = 0$$

and

$$v(x,a,t) = u(x,t),\ \lim_{r\to\infty} v(x,r,t) = 0.$$

Solution: By taking the Laplace transform of relation (5.2), we have

$$r^2 V_{rr} + rV_r + (i\sqrt{s^\alpha - \beta})^2 r^2 V = 0.$$

Let us assume that $L\{v(x,r,t)\} = V(x,r,s)$, then one has

$$V(x,r,s) = c_1 J_0(i\sqrt{s^\alpha - \beta}r) + c_2 Y_0(i\sqrt{s^\alpha - \beta}r),$$

where J_0 and Y_0 are Bessel functions of the first and second kind of order zero, respectively. Using the fact that $\lim\limits_{r\to\infty} v(x,r,t) = 0$, we get

$$V(x,r,s) = c_1 J_0(i\sqrt{s^\alpha - \beta}r).$$

But $v(x,a,t) = u(x,t)$, therefore

$$V(x,r,s) = U(x,s)\frac{J_0(i\sqrt{s^\alpha - \beta}r)}{J_0(i\sqrt{s^\alpha - \beta}a)},$$

where $L\{u(x,t)\} = U(x,s)$. On the other hand, one has

$$\left.\frac{\partial V}{\partial r}\right|_{r=a} = U(x,s)\left(-i\sqrt{s^\alpha - \beta}\frac{J_1(i\sqrt{s^\alpha - \beta}a)}{J_0(i\sqrt{s^\alpha - \beta}a)}\right).$$

Taking the Laplace transform of relation (5.1), we will have

$$s^\alpha U = U_{xx} - i\lambda a\sqrt{s^\alpha - \beta}\frac{J_1(i\sqrt{s^\alpha - \beta}a)}{J_0(i\sqrt{s^\alpha - \beta}a)}U + \frac{1}{s} - \gamma U,$$

so that

$$U_{xx} - h(s)U = -\frac{1}{s}, \tag{5.3}$$

where

$$h(s) = s^\alpha + \gamma + i\lambda a\sqrt{s^\alpha - \beta}\frac{J_1(i\sqrt{s^\alpha - \beta}a)}{J_0(i\sqrt{s^\alpha - \beta}a)}.$$

Differential equation (5.3) has the following solution

$$U(x,s) = c_1\cosh(\sqrt{h(s)}x) + c_2\sinh(\sqrt{h(s)}x) + \frac{1}{sh(s)}.$$

By using the boundary conditions $u(-l,t) = u(l,t) = 0$, we obtain

$$U(x,s) = \frac{1}{sh(s)}\left(1 - \frac{\cosh(\sqrt{h(s)}x)}{\cosh(\sqrt{h(s)}l)}\right).$$

For simplicity, let us assume that

$$F(x,h(s)) = 1 - \frac{\cosh(\sqrt{h(s)}x)}{\cosh(\sqrt{h(s)}l)},$$

thus

$$U(x,s) = \frac{F(x,h(s))}{sh(s)}.$$

Now, if

$$L_t\{\phi(x,t)\} = \frac{F(x,s)}{s}, L_t\{\psi(\xi,t)\} = \frac{e^{-\xi h(s)}}{s},$$

therefore

$$u(x,t) = L_t^{-1}\{U(x,s)\} = L^{-1}\{\frac{F(x,h(s))}{sh(s)}\} = \int_0^\infty \psi(\xi,t)\phi(x,\xi)d\xi.$$

Finally, we get

$$\begin{aligned}
\phi(x,t) &= L_t^{-1}\{\frac{F(x,s)}{s}\} = L_t^{-1}\left\{\frac{1}{s}(1-\frac{\cosh(\sqrt{s}x)}{\cosh(\sqrt{s}l)})\right\} \\
&= = 1 - L_t^{-1}\left\{\frac{\cosh(\sqrt{s}x)}{s\cosh(\sqrt{s}l)}\right\} = 1 - L_t^{-1}\left\{e^{\sqrt{s}(x-l)}\frac{1+e^{-2\sqrt{s}x}}{s(1+e^{-2\sqrt{s}l})}\right\} \\
&= 1 - \sum_{n=0}^{\infty} L_t^{-1}\left\{\frac{\exp(-((2n+1)l-x)\sqrt{s})}{s} - \frac{\exp(-((2n+1)l+x)\sqrt{s})}{s}\right\} \\
&= 1 - \sum_{n=0}^{\infty}\left(erfc(\frac{(2n+1)l-x}{2\sqrt{t}}) - erfc(\frac{(2n+1)l+x}{2\sqrt{t}})\right).
\end{aligned}$$

On the other hand

$$h(s) = s^\alpha + \gamma + i\lambda a\sqrt{s^\alpha-\beta}\frac{J_1(i\sqrt{s^\alpha-\beta}a)}{J_0(i\sqrt{s^\alpha-\beta}a)},$$

hence

$$\begin{aligned}
\psi(\xi,t) &= L_t^{-1}\{\frac{1}{s}\exp(-\xi h(s))\} \\
&= L_t^{-1}\left\{e^{-\xi\gamma}\frac{e^{-\xi s^\alpha}}{s}\exp\left(-i\xi\lambda a\sqrt{s^\alpha-\beta}\frac{J_1(i\sqrt{s^\alpha-\beta}a)}{J_0(i\sqrt{s^\alpha-\beta}a)}\right)\right\} \\
&= e^{-\xi(\gamma+\beta)}L_t^{-1}\left\{\frac{\sqrt{s^\alpha-\beta}}{s}\frac{e^{-\xi(s^\alpha-\beta)}}{\sqrt{s^\alpha-\beta}}\exp\left(-i\xi\lambda a\sqrt{s^\alpha-\beta}\frac{J_1(i\sqrt{s^\alpha-\beta}a)}{J_0(i\sqrt{s^\alpha-\beta}a)}\right)\right\}.
\end{aligned}$$

Case 1: For $\alpha = 1$, we have

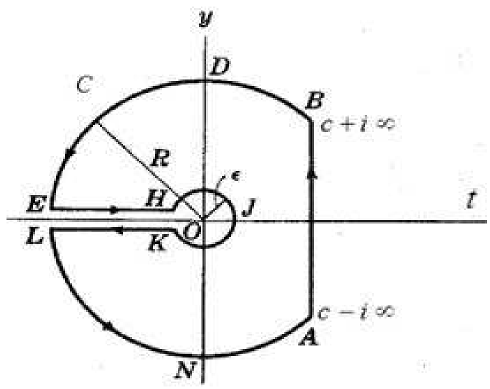
$$\begin{aligned} f_1(\xi,t) &= L_t^{-1}\left\{\frac{e^{-\xi(s-\beta)}}{\sqrt{s-\beta}}\exp\left(-i\xi\lambda a\sqrt{s-\beta}\frac{J_1(i\sqrt{s-\beta}a)}{J_0(i\sqrt{s-\beta}a)}\right)\right\} \\ &= e^{\beta t}L_t^{-1}\left\{\frac{e^{-\xi s}}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)\right\} \\ &= e^{\beta t}H(t-\xi)L_t^{-1}\left\{\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)\right\}\Bigg|_{\,t\to t-\xi}, \end{aligned}$$

where $H(t)$ is Heaviside unit step function. By complex inversion formula, the inverse Laplace transform is given by

$$L_t^{-1}\left\{\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)\right\} =$$
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds.$$

The integrand has a branch point at the origin and it is thus necessary to choose a contour which does not enclose the origin. We deform the Bromwich contour so that the circular arc BDE is terminated just short of the horizontal axis and the arc LNA starts just below the horizontal axis. In between the contour follows a path EH which is parallel to the axis, followed by a circular arc HJK enclosing the origin and a return section KL parallel to the axis meeting the arc LNA (see figure). As there are no poles inside this contour C, we have

$$\int_C \frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds = 0.$$



Now on BDE and LNA

$$\left|\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)\right| \le \frac{1}{\sqrt{s}},$$

so that the integrals over these arcs tend to zero as $R \to \infty$.

Over the circular arc HJK as its radius$\varepsilon \to 0$, we have $s = \varepsilon e^{i\theta}$, $\pi \le \theta \le -\pi$. Thus

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{HJK} \frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right) e^{ts}ds = 0.$$

Along EH, $s = ze^{i\pi}$, $\sqrt{s} = i\sqrt{z}$ and as s goes from $-R$ to $-\varepsilon$, z goes from R to ε. Hence

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{EH} \frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right) e^{ts}ds =$$

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{-R}^{-\varepsilon} \frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right) e^{ts}ds =$$

$$\int_0^{\infty} \frac{1}{i\sqrt{z}}\exp\left(-\xi\lambda a\sqrt{z}\frac{J_1(\sqrt{z}a)}{J_0(\sqrt{z}a)}\right) e^{-tz}dz.$$

Similarly, along KL, $s = ze^{-i\pi}$, $\sqrt{s} = -i\sqrt{z}$ and as s goes from $-\varepsilon$ to $-R$, z goes from ε to R. So

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{KL} \frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right) e^{ts}ds =$$

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{-\varepsilon}^{-R} \frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right) e^{ts}ds =$$

$$\int_0^{\infty} \frac{1}{i\sqrt{z}}\exp\left(-\xi\lambda a\sqrt{z}\frac{J_1(\sqrt{z}a)}{J_0(\sqrt{z}a)}\right) e^{-tz}dz.$$

Consequently

$$\frac{1}{2\pi i}\int_C \frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right) e^{ts}ds = \frac{1}{2\pi i}\int_{AB} ds + \frac{1}{2\pi i}\int_{BDE} ds$$

$$+\frac{1}{2\pi i}\int_{EH} ds + \frac{1}{2\pi i}\int_{HJK} ds + \frac{1}{2\pi i}\int_{KL} ds + \frac{1}{2\pi i}\int_{LNA} ds = 0.$$

So that, the final result is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds =$$

$$\frac{1}{\pi}\int_0^{\infty}\frac{1}{\sqrt{z}}\exp\left(-\xi\lambda a\sqrt{z}\frac{J_1(\sqrt{z}a)}{J_0(\sqrt{z}a)}\right)e^{-tz}dz.$$

Thus we obtain

$$\begin{aligned} f_1(\xi,t) &= L_t^{-1}\left\{\frac{e^{-\xi(s-\beta)}}{\sqrt{s-\beta}}\exp\left(-i\xi\lambda a\sqrt{s-\beta}\frac{J_1(i\sqrt{s-\beta}a)}{J_0(i\sqrt{s-\beta}a)}\right)\right\} \\ &= \frac{1}{\pi}e^{\beta t}H(t-\xi)\int_0^{\infty}\frac{1}{\sqrt{z}}\exp\left(-\xi\lambda a\sqrt{z}\frac{J_1(\sqrt{z}a)}{J_0(\sqrt{z}a)}\right)e^{-(t-\xi)z}dz. \end{aligned}$$

In case of $0<\alpha<1$, we get

$$\begin{aligned} f_2(\xi,t) &= L_t^{-1}\left\{\frac{e^{-\xi(s^{\alpha}-\beta)}}{\sqrt{s^{\alpha}-\beta}}\exp\left(-i\xi\lambda a\sqrt{s^{\alpha}-\beta}\frac{J_1(i\sqrt{s^{\alpha}-\beta}a)}{J_0(i\sqrt{s^{\alpha}-\beta}a)}\right)\right\} \\ &= \frac{1}{t}\int_0^{\infty}f_1(\xi,\tau)W(-\alpha,0;-\tau t^{-\alpha})d\tau\,. \end{aligned}$$

Finally, for $0<\alpha\le 1$,

$$\begin{aligned} f_3(t) &= L_t^{-1}\left\{\frac{\sqrt{s^{\alpha}-\beta}}{s}\right\} = L_t^{-1}\left\{s^{\frac{\alpha}{2}-1}(1-\beta s^{-\alpha})^{\frac{1}{2}}\right\} \\ &= \sum_{n=0}^{\infty}(-\beta)^n\begin{pmatrix}\frac{1}{2}\\ n\end{pmatrix}L_t^{-1}\{s^{-\alpha n+\frac{\alpha}{2}-1}\} = \sum_{n=0}^{\infty}(-\beta)^n\begin{pmatrix}\frac{1}{2}\\ n\end{pmatrix}\frac{t^{\alpha n-\frac{\alpha}{2}}}{\Gamma(\alpha n-\frac{\alpha}{2}+1)}. \end{aligned}$$

Consequently

$$\begin{aligned} \psi(\xi,t) &= L_t^{-1}\{\frac{1}{s}\exp(-\xi h(s))\} \\ &= \begin{cases} e^{-\xi(\gamma+\beta)}\int_0^t f_2(\xi,\eta)f_3(t-\eta)d\eta : 0<\alpha<1 \\ e^{-\xi(\gamma+\beta)}\int_0^t f_1(\xi,\eta)f_3(t-\eta)d\eta : \alpha=1 \end{cases}. \end{aligned}$$

Therefore, we obtain $u(x,t)$ as follows

$$
\begin{aligned}
u(x,t) &= \int_0^\infty \psi(\xi,t)\phi(x,\xi)d\xi \\
&= \begin{cases}
\int_0^\infty e^{-\xi(\gamma+\beta)}\left(\int_0^t f_2(\xi,\eta)f_3(t-\eta)d\eta\right) \\
\times\left(1-\sum_{n=0}^\infty\left(erfc(\frac{(2n+1)l-x}{2\sqrt{\xi}})-erfc(\frac{(2n+1)l+x}{2\sqrt{\xi}})\right)\right)d\xi : 0<\alpha<1 \\
\int_0^\infty e^{-\xi(\gamma+\beta)}\left(\int_0^t f_1(\xi,\eta)f_3(t-\eta)d\eta\right) \\
\times\left(1-\sum_{n=0}^\infty\left(erfc(\frac{(2n+1)l-x}{2\sqrt{\xi}})-erfc(\frac{(2n+1)l+x}{2\sqrt{\xi}})\right)\right)d\xi : \alpha=1
\end{cases}
\end{aligned}
$$

$$
V(x,r,s) = U(x,s)\frac{J_0(i\sqrt{s^\alpha-\beta}r)}{J_0(i\sqrt{s^\alpha-\beta}a)}.
$$

If $\alpha = 1$, we have

$$
g_1(r,t) = L_t^{-1}\left\{\frac{J_0(i\sqrt{s-\beta}r)}{J_0(i\sqrt{s-\beta}a)}\right\} = \frac{2}{a^2}\sum_{k=0}^\infty \frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}\exp(-(\frac{\lambda_k^2}{a^2}-\beta)t),
$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are roots of $J_0(i\sqrt{s-\beta}a)$. For $0 < \alpha < 1$,

$$
\begin{aligned}
g_2(r,t) &= L_t^{-1}\left\{\frac{J_0(i\sqrt{s^\alpha-\beta}r)}{J_0(i\sqrt{s^\alpha-\beta}a)}\right\} \\
&= \frac{2}{a^2 t}\sum_{k=0}^\infty \frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}\int_0^\infty \exp(-(\frac{\lambda_k^2}{a^2}-\beta)\tau)W(-\alpha,0;-\tau t^{-\alpha})d\tau \\
&= \frac{2}{a^2 t}\sum_{k=0}^\infty \frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}L\left\{W(-\alpha,0;-\tau t^{-\alpha}),\tau\to s\right\}\Bigg|_{s=\frac{\lambda_k^2}{a^2}-\beta} \\
&= \frac{2}{t}\sum_{k=0}^\infty \frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}\left(\sum_{n=0}^\infty \frac{(-a^2)^n t^{-\alpha n}}{\Gamma(-\alpha n)(\lambda_k^2-a^2\beta)^{n+1}}\right) \\
&= \frac{2}{t}\sum_{k=0}^\infty \frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)(\lambda_k^2-a^2\beta)}E_{-\alpha,0}(-\frac{a^2 t^{-\alpha}}{\lambda_k^2-a^2\beta}).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
v(x,r,t) &= L_t^{-1}\{V(x,r,s)\} = L_t^{-1}\left(U(x,s)\frac{J_0(i\sqrt{s^\alpha-\beta}r)}{J_0(i\sqrt{s^\alpha-\beta}a)}\right) \\
&= \begin{cases} \int_0^t u(x,\eta)g_2(r,t-\eta)d\eta : 0<\alpha<1 \\ \\ \int_0^t u(x,\eta)g_1(r,t-\eta)d\eta : \alpha=1 \end{cases} .
\end{aligned}
$$

6. Conclusion

The paper is devoted to study and application of Laplace transform. The main purpose of this work is to develop a method for finding formal solution of certain Fredholm fractional singular integral equations of second kind, analytic solution of the time fractional heat equation and system of partial fractional differential equations, which is a generalization to certain types of problems in the literature .We hope that it will also benefit many researchers in the disciplines of applied mathematics, mathematical physics and engineering.

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