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Null Parallel p-Equidistant B-Scrolls *

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abstract: In this paper, null parallel p-equidistant B-scrolls are defined in 3-dimensional Minkowski space R_1^3. We prove necessary and sufficient conditions for these B-scrolls to be equivalent of their Cartan frames. The relations between matrices of the shape operators and the algebraic invariants (Gaussian, mean curvatures, principal curvatures) of these B-scrolls are shown. Besides we give the relations between second Gaussian curvatures, mean curvatures and the distribution parameters of non-developable null parallel p-equidistant B-scrolls. Finally, an example is given related to the null parallel p-equidistant B-scrolls in R_1^3.

Key Words: Ruled surface, Minkowski, B-Scroll, Null, Parallel p-Equidistant.

Contents

1. Basic Concepts

The Minkowski 3-space R_1^3 is the Euclidean 3-space R^3 provided with the standard flat metric

$$h = -dx_1^2 + dx_2^2 + dx_3^2 \tag{1.1}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of R_1^3. Since h is an indefinite metric, recall that a vector $v \in R_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $h(v, v) > 0$ or $v = 0$, timelike if $h(v, v) < 0$ and null(lightlike), if $h(v, v) = 0$ and $v \neq 0$. The norm of a vector $v \in R_1^3$ is defined as $\|v\| = \sqrt{|h(v, v)|}$. Therefore, v is a unit vector if $h(v, v) = \pm 1$. Furthermore, vectors v and w are said to be orthogonal if $h(u, w) = 0$, [11]. For any vectors $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in R_1^3$, the Lorentzian product $v \wedge w$ of v and w is defined as [1]

$$v \wedge w = (v_2 w_3 - v_3 w_2, v_1 w_3 - v_3 w_1, v_2 w_1 - v_1 w_2). \tag{1.2}$$

Similarly an arbitrary curve $\alpha = \alpha(s)$ in R_1^3 can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null (lightlike), respectively. If $h(\alpha'(s), \alpha'(s)) = \pm 1$ then α is a unit speed curve and s is arc-length parameter of α, [10]. Let M be a surface in 3-dimensional Minkowski space R_1^3. The surface M is called a timelike surface if the induced metric on the

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surface is a Lorentzian metric. Therefore, the normal of this timelike surface is a spacelike vector, [4]. Let us suppose that $h(P_1, P_2) = 0$. The following table is valid for the plane π which is spanned by the vectors P_1 and P_2 according to being the vectors P_1 and P_2 spacelike, timelike or null vectors, [8].

P_2 \ P_1	spacelike	timelike	null
spacelike	π spacelike	π timelike	π null
timelike	π timelike	—	—
null	π null	—	—

Table 1

Let M be a Lorentz manifold and α be a null curve on the manifold M in R_1^3. The frame in R_1^3 is positively oriented triple (ℓ, n, u) of vectors which satisfies the following conditions

$$\begin{array}{ll} h(\ell,\ell) = h(n,n) = 0, & h(\ell,n) = -1 \\ h(\ell,u) = h(n,u) = 0, & h(u,u) = 1 \end{array} \tag{1.3}$$

A null frame for a null curve $\alpha(s)$ is a frame field $F(\ell(s), n(s), u(s))$ such that $\frac{d\alpha}{ds}$ is positive scalar multiple of ℓ, [9]. In this situation, (α, F) couple is called framed null curve. Frames of null curves are not unique. Moreover, frames are changed under reparametrization of a curve. Therefore, the curve and the frame must be given together. The Frenet formulas of α with respect to the frame F are given by [9]:

$$\begin{aligned} \frac{d\ell}{ds} &= k\ell + \kappa u, \\ \frac{dn}{ds} &= -kn + \tau u, \\ \frac{du}{ds} &= \tau\ell + \kappa u. \end{aligned} \tag{1.4}$$

The functions k, κ and τ are called the curvature functions of α. There always exists a parameter s of α such that $k = 0$ in (1.5). This parameter is called a distinguished parameter of α [6]. Then the Frenet formula of α can be written by

$$\begin{aligned} \frac{d\ell}{ds} &= \kappa u, \\ \frac{dn}{ds} &= \tau u, \\ \frac{du}{ds} &= \tau\ell + \kappa n. \end{aligned} \tag{1.5}$$

Here, ℓ is the tangent vector (rather than its direction) and u is analogous to the principal normal in the standard Frenet frame for a curve in E^3, [9]. The null frame $F(\ell(s), n(s), u(s))$ is called Cartan frame of α. A parametrized null curve parametrized by distinguished parameter s together with its Cartan frame is called a Cartan framed null curve [6]. In addition to that

$$\ell \wedge n = u, \quad n \wedge u = -n, \quad \ell \wedge u = \ell, \tag{1.6}$$

A ruled surface is a surface swept out by a straight line Y moving along a curve α. The various positions of the generating line Y are called the rulings of the surface. Such a surface has a parametrization in ruled form as follows:

$$\varphi(s, v) = \alpha(s) + vY(s). \tag{1.7}$$

We call α to be the base curve and Y to be the director vector. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface. The remaining ruled surfaces are called skew surfaces. If there exists a common perpendicular to two preceding rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction [4], [11].

In 3-dimensional Minkowski space R_1^3 if the base curve α and the director vector Y are chosen as a null curve and a null line, respectively, then the ruled surface is called null scroll and denoted by M. It is easily seen that the null scroll M is a timelike surface. Especially when $\kappa(s) \neq 0$ and $\tau(s) = cons\tan t$, the null scroll M is called a B-scroll and parametric equation is given as follows [7]:

$$\varphi(s, v) = \alpha(s) + vn(s)$$

B-scrolls were first introduced by Graves [5] and used to classify the codimension one isometric immersion between Lorentz surfaces. Then some authors studied and developed the geometry of B-scroll [13].

Let M be a B-scroll. Then the tangent planes along a ruling of M coincide if and only if $\tau = 0$, [3].

The striction curve and drall (distribution parameter) of non-developable B-scroll in R_1^3, respectively, is given as follows [3]:

$$\beta(s) = \alpha(s) - \frac{h(\alpha'(s), n'(s))}{\|n'(s)\|} n(s) \tag{1.8}$$

and

$$\lambda = -\frac{\det(\ell, n, n')}{\|n'(s)\|}. \tag{1.9}$$

Let M be a surface in R_1^3. If D and N be Levi-Civita connection and the unit normal vector field on M, respectively then the shape operator of M which is obtained from N is defined by

$$S(X) = -D_X N, \qquad \forall X \in \chi(M) \tag{1.10}$$

where $\chi(M)$ is the space of the vector fields of M, [11].
Let $S(P)$ is the shape operator of M at the point P. Therefore,

$$\begin{aligned} K : M &\to IR \\ P &\to K(P) = \det S(P) \end{aligned} \tag{1.11}$$

the function is called as Gaussian curvature function and also the value of $K(P)$ is the Gaussian curvature of M at the point P. Similarly

$$\begin{aligned} H : M &\to IR \\ P &\to H(P) = \tfrac{trS(P)}{boyM} \end{aligned} \tag{1.12}$$

the function is called as mean curvature function and also the value of $H(P)$ is called as mean curvature of M at the point P. If M is a non-developable surface and E, F, G are the coefficients of the first fundamental form I, then E, F, G are as follows [12]:

$$E = h(\varphi_s, \varphi_s), \qquad F = h(\varphi_s, \varphi_v), \qquad G = h(\varphi_v, \varphi_v). \tag{1.13}$$

If we take $D = EG - F^2$, then the coefficients of the second fundamental form e, f and g of M are given as

$$e = \frac{h(\varphi_{ss}, \varphi_s \wedge \varphi_v)}{D}, \quad f = \frac{h(\varphi_{sv}, \varphi_s \wedge \varphi_v)}{D}, \quad g = \frac{h(\varphi_{vv}, \varphi_s \wedge \varphi_v)}{D}. \tag{1.14}$$

Besides, the second Gaussian curvature of the surface M is given as, [12].

$$K_{II} = \frac{1}{(|eg| - f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{vv} + f_{sv} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_v \\ f_v - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_s \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}. \tag{1.15}$$

Let M be a non-developable surface in R_1^3. If K and H denote the mean curvature and the Gaussian curvature of the surface M, respectively, then the second fundamental form II can be written as

$$II = L_{ij} dx_i dx_j.$$

So, the second mean curvature H_{II} of M is given by

$$H_{II} = H - \frac{1}{2\sqrt{|\det II|}} \sum_{ij} \frac{\partial}{\partial u^i} \left(\sqrt{|\det II|} L^{ij} \frac{\partial}{\partial u^j} \left(\ln \sqrt{|K|} \right) \right) \tag{1.16}$$

where L^{ij} is the inverse of the matrix representation L_{ij} of the second fundamental form and u^1, u^2 correspond to the parameters s and v, respectively, [12].
Throughout this paper, we consider n and α as a null vector and a null curve, respectively.

2. Null Parallel p-equidistant B-scrolls in R_1^3

Let α be a null curve and $\{\ell(s), n(s), u(s)\}$ be a Cartan frame of the null curve α. If the null vector $n(s)$ moves along the null curve α, then the B-scroll is given by the following parametrization

$$\varphi(s,v) = \alpha(s) + v\,n(s) \tag{2.1}$$

where, the curve α and the vector $n(s)$ are called base curve and the director vector of B-scroll, respectively. This B-scroll is denoted by M $(\kappa(s) \neq 0, \quad \tau(s) = constant$, see [7]). Considering equation (2.1) we have

$$\varphi_s = \ell(s) + v\tau u(s), \varphi_v = n(s). \tag{2.2}$$

Therefore, the unit normal vector N_0 of B-scroll M is

$$N_0 = \frac{\varphi_s \wedge \varphi_v}{\|\varphi_s \wedge \varphi_v\|} = u(s) + v\tau n(s). \tag{2.3}$$

It is obvious that the unit normal vector of M is spacelike vector. It means that the B-scroll M is a timelike surface.
The planes corresponding to the sub-spaces $Sp\{\ell, n\}$, $Sp\{n, u\}$ and $Sp\{u, \ell\}$ along the base curve α of B-scroll M are called central plane, polar plane and asymptotic plane, respectively. From table 1, the central plane is timelike, the polar and the asymptotic planes are null planes.
Let α^* be another null curve with Cartan frame
$F^* = (\ell^*(s^*), n^*(s^*), u^*(s^*))$ so that

$$\begin{array}{ll} \ell^*(s^*) = \frac{d\alpha^*}{ds^*}, & h(n^*, n^*) = h(\ell^*, \ell^*) = 0, \\ h(\ell^*, u^*) = h(n^*, u^*) = 0, & h(\ell^*, n^*) = -1, \quad h(u^*, u^*) = 1. \end{array} \tag{2.4}$$

In addition to that, Frenet formulas of α^* with respect to Cartan frame $F^*(\ell^*, n^*, u^*)$ are

$$\begin{aligned} \frac{d\ell^*}{ds^*} &= \kappa^* u^* \\ \frac{dn^*}{ds^*} &= \tau^* u^* \\ \frac{du^*}{ds^*} &= \tau^* \ell^* + \kappa^* n^*. \end{aligned} \tag{2.5}$$

Let M^* be a B-scroll. Thus, the B-scroll M^* is parametrically given by as

$$\varphi^*(s^*, v^*) = \alpha^*(s^*) + v^* n^*(s^*). \tag{2.6}$$

Definition 2.1. *Let M and M^* be two B-scrolls in R_1^3 and p be the distance between the asymptotic planes of M and M^*. If*

1)*The generator vectors of M and M^* are parallel,*

2)*The distance p is constant, then the pair of B-scrolls M and M^* are called the null parallel p-equidistant B-scrolls in R_1^3.*

Thus, the parametric representation of null parallel p-equidistant B-scrolls are given as follows

$$\begin{array}{l} M : \varphi(s,v) = \alpha(s) + vn(s) \\ M^* : \varphi^*(s^*,v^*) = \alpha^*(s^*) + v^* n^*(s^*) \,. \end{array} \tag{2.7}$$

Throughout this paper, the base curves of null parallel p-equidistant B-scrolls M and M^* are also striction curves.

Theorem 2.2. *Let M and M^* be two null parallel p-equidistant B-scrolls in R_1^3. The Cartan frames $\{\ell, n, u\}$ and $\{\ell^*, n^*, u^*\}$ of the base curves $\alpha(s)$ and $\alpha^*(s^*)$ are equivalent at the corresponding points in M and M^*, respectively if and only if $\tau^* = \tau \frac{ds}{ds^*}$ and $\kappa^* = \kappa \frac{ds}{ds^*}$.*

Proof: Firstly, suppose that the Cartan frames $\{\ell, n, u\}$ and $\{\ell^*, n^*, u^*\}$ of base curves of $\alpha(s)$ and $\alpha^*(s^*)$ are equivalent at the corresponding points of M and M^*, respectively.
This means that

$$\ell = \ell^*, \qquad n = n^*, \qquad u = u^*. \tag{2.8}$$

From equation (2.5) we reach

$$\kappa^* = h\left(\frac{d\ell^*}{ds^*}, u^*\right).$$

Considering the last equation together with hypothesis we easily see that

$$\kappa^* = \kappa\left(\tfrac{ds}{ds^*}\right). \tag{2.9}$$

Using equation (2.5) and following similar way, we get

$$\tau^* = \tau\frac{ds}{ds^*}. \tag{2.10}$$

Conversely, let the relationship between the curvature and torsion of M and M^* be $\kappa^* = \kappa \frac{ds}{ds^*}$ and $\tau^* = \tau \frac{ds}{ds^*}$, respectively.
Since M and M^* are null parallel p-equidistant B-scrolls, the generator vectors n and n^* of M and M^*, respectively are parallel. Therefore, we can choose

$$n^* = n. \tag{2.11}$$

From the last equation, we have

$$\frac{dn}{ds} = \frac{dn^*}{ds^*}\cdot\frac{ds^*}{ds} \tag{2.12}$$

Substituting the equations (1.6) and (2.5) into the last equation, by routine calculation, one can obtain

$$u^* = u$$

In addition to that, we can write from the last equation

$$\tfrac{du}{ds} = \tfrac{du^*}{ds^*}\cdot\tfrac{ds^*}{ds} \tag{2.13}$$

Using the equation (1.6) and (2.5) together with the equation (2.13), it is seen that

$$\ell^* = \ell.$$

This completes the proof.

Theorem 2.3. *In R_1^3, let M and M^* be null parallel p-equidistant B-scrolls, S and S^* be the matrices corresponding to the shape operators of M and M^*, respectively. Then, there is a relation between the matrices S and S^* as follows*

$$S^* = \left(\frac{ds}{ds^*}\right) S.$$

Proof: Let us find the matrices of S and S^*, which is correspond to shape operators of null parallel p-equidistant B-scrolls M and M^*, respectively. From equations (1.6) and (2.3), we obtain

$$S\left(\varphi_v\right) = -D_{\varphi_v} N_0 = -\frac{dN_0}{dv} = -\tau n(s) \tag{2.14}$$

and

$$S\left(\varphi_s\right) = -D_{\varphi_s} N_0 = -\frac{dN_0}{ds} = -\tau\ell(s) - \kappa(s)n(s) - v\tau^2 u(s). \tag{2.15}$$

Considering the equations (2.14) and (2.15) together with the equation (2.2), we reach

$$S\left(\varphi_v\right) = -\tau\varphi_v + 0\varphi_s$$

and

$$S\left(\varphi_s\right) = -\kappa(s)\varphi_v - \tau\varphi_s.$$

In this case, the matrix which is correspond to shape operators of M as follows,

$$S = \begin{bmatrix} -\tau & 0 \\ -\kappa(s) & -\tau \end{bmatrix}. \tag{2.16}$$

Using the equations (2.3) and (2.5) and following similar way, we can get

$$S^*\left(\varphi^*_{v^*}\right) = -\tau^*\, \varphi^*_{v^*} - 0\, \varphi^*_{s^*}$$

and

$$S^*\left(\varphi^*_{s^*}\right) = -\kappa^*\left(s^*\right) \varphi^*_{v^*} - \tau^*\, \varphi^*_{s^*}$$

Thus, the matrix corresponding to shape operator of M^* is

$$S^* = \begin{bmatrix} \tau^* & 0 \\ -\kappa^*\left(s^*\right) & -\tau^* \end{bmatrix}.$$

Substituting the equations (2.9) and (2.10) into the last equation, we obtain

$$S^* = \begin{bmatrix} \left(\frac{ds}{ds^*}\right)\tau & 0 \\ -\left(\frac{ds}{ds^*}\right)\kappa\left(s\right) & -\left(\frac{ds}{ds^*}\right)\tau \end{bmatrix}. \tag{2.17}$$

From the equations (2.16) and (2.17), it is obvious that

$$S^* = \left(\frac{ds}{ds^*}\right) S. \tag{2.18}$$

Theorem 2.4. *Let M and M^* be null parallel p-equidistant B-scrolls in R_1^3.*
i) *If K and K^* denote Gaussian curvatures of M and M^*, respectively. In this case, there is following relation between K and K^*:*

$$K^* = K \left(\frac{ds}{ds^*}\right)^2.$$

ii) *The mean curvature of M and M^* is denoted by H and H^*, respectively. Therefore, there is a relation between H and H^* as follows:*

$$H^* = H \left(\frac{ds}{ds^*}\right).$$

Proof: i) To calculate the Gaussian curvatures K and K^* of M and M^*, respectively, considering the equations (1.12),(2.16) and (2.17) we have

$$K = \tau^2 \tag{2.19}$$

and

$$K^* = {\tau^*}^2. \tag{2.20}$$

From equations (2.10), (2.19) and (2.20), we obtain the relation between K and K^* as,

$$K^* = K \left(\frac{ds}{ds^*}\right)^2. \tag{2.21}$$

ii) Now, we compute the mean curvatures H and H^* of M and M^*, respectively. Using the equations (1.13), (2.16) and (2.17), it is easy to see that the mean curvatures H and H^* are given by

$$H = -\tau \tag{2.22}$$

and

$$H^* = -\tau^* \tag{2.23}$$

respectively. Considering the equations (2.10) and (2.22) together with the last equation, it is clear that

$$H^* = H \left(\frac{ds}{ds^*}\right). \tag{2.24}$$

Theorem 2.5. *In R_1^3, let M and M^* be null parallel p-equidistant B-scrolls and k_i and k_i^*, $1 \leq i \leq 2$ be the principal curvatures of M and M^*, respectively. In this case, the relations between k_i and k_i^* are as,*

$$k_i^* = k_i \left(\frac{ds}{ds^*}\right), 1 \leq i \leq 2.$$

Proof: Since the principal curvatures of M are the roots of the characteristic polynomial which is given by "$\det(S - kI_2) = 0$", we have

$$k_1 = k_2 = -\tau. \tag{2.25}$$

Similarly, the principal curvatures of M^* are

$$k_1^* = k_2^* = -\tau^*. \tag{2.26}$$

Thus, from the equation (2.10) together with the equations (2.25) and (2.26), we reach

$$k_i^* = k_i \left(\frac{ds}{ds^*}\right), \qquad 1 \leq i \leq 2 \tag{2.27}$$

Theorem 2.6. *In R_1^3, let us assume that M and M^* are non-developable null parallel p-equidistant B-scrolls, λ and λ^* are the distribution parameters of M and M^*, respectively. Then the following relation satisfy*

$$\lambda^* = \lambda \left(\frac{ds^*}{ds}\right).$$

Proof: Taking into consideration the equations (1.10) and (2.10) the following relation can be obtained between the distribution parameters λ and λ^* of M and M^*, respectively.

$$\lambda^* = \frac{1}{\tau^*} = \frac{1}{\tau}\frac{ds^*}{ds} = \lambda\frac{ds^*}{ds}. \tag{2.28}$$

Theorem 2.7. *Let M and M^* be non-developable null parallel p-equidistant B-scrolls in R_1^3.*
i) *There is the relation between the second Gaussian curvatures K_{II} and K_{II}^* of M and M^* as follows*

$$K_{II}^* = K_{II}\left(\frac{ds}{ds^*}\right)$$

ii) *There is the following relation between H_{II} and H_{II}^*,*

$$H_{II}^* = H_{II}\left(\frac{ds}{ds^*}\right)$$

where H_{II} and H_{II}^* are the second mean curvatures of M and M^*, respectively.

Proof: Suppose that M and M^* are non-developable null parallel p-equidistant B-scrolls in R_1^3. Now we compute the coefficients of the first fundamental form E, F and G and the second fundamental form e, f and g of M. Using the equations (1.14), (1.15) and (2.2), we obtain that

$$E = v^2\tau^2, \quad F = -1, \quad G = 0 \tag{2.29}$$

Also

$$e = \kappa(s) - v^2\tau^3, \quad f = \tau, \quad g = 0. \tag{2.30}$$

Following similar way, the first and second fundamental forms of non-developable null parallel B-scroll M^* are as

$$E^* = v^{*2}, \quad F^* = -1, \quad G^* = 0 \tag{2.31}$$

and

$$e^* = \kappa^* (s^*) - v^{*2} \tau^{*3}, \quad f^* = \tau^*, \quad g^* = 0. \tag{2.32}$$

respectively.

i) Substituting e, f, g into the second Gaussian curvature matrix form, K_{II} which is given by equation (1.16), we get

$$K_{II} = -\tau$$

Similarly, we see that

$$K_{II}^* = -\tau^*.$$

From the equation (2.10) and the last two equations, it is easily to see that

$$K_{II}^* = K_{II} \left(\frac{ds}{ds^*} \right). \tag{2.33}$$

ii) Taking into consideration the equations (2.30), (2.32), (1.12) and (1.13) with the equation (1.17), we obtain

$$H_{II} = H$$

and

$$H_{II}^* = H^*.$$

From theorem 2.3, we have

$$H_{II}^* = H_{II} \left(\frac{ds}{ds^*} \right)$$

The results in the study are confirmed by the following example.

Example 2.8. *In R_1^3, let us assume that the null parallel p-equidistant B-scrolls M and M^* are parametrically given by*

$$\varphi(s, v) = (s, \cos s, \sin s) + v \left(\frac{1}{2}, \frac{1}{2} \sin s, -\frac{1}{2} \cos s \right)$$

and

$$\varphi^* (s^*, v^*) = (s^* + 2, \cos s^* + 2, \sin s^* + 2) + v^* \left(\frac{1}{2}, \frac{1}{2} \sin s^*, -\frac{1}{2} \cos s^* \right)$$

respectively(Figure 1 and Figure 2). In this case, using the equations (1.6) and (2.5) we find

$$\kappa^* = \kappa = 1, \quad \tau^* = \tau = -\frac{1}{2},$$

Thus, substituting the last equations into the equations (2.16) and (2.17), the matrices corresponding to shape operators of M and M^* are to be

$$S^* = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix} = S$$

In addition to that, from the equations (1.16) and (1.17), there is a relation between the second Gauss curvature and the second mean curvature as follows, respectively

$$K^*_{II} = K_{II} = \frac{1}{2}, \qquad H^*_{II} = H_{II} = \frac{1}{2}.$$

Lastly, considering the equations (2.16) and (2.17), the principal curvatures of M and M^* are as

$$k^*_i = k_i = 1, \qquad 1 \leq i \leq 2$$

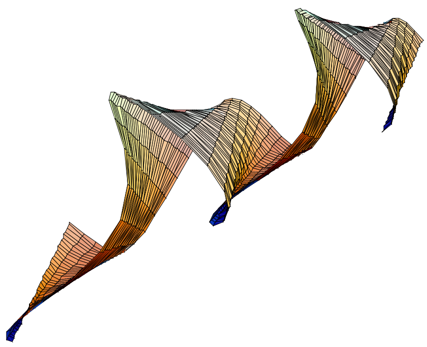

Figure 1. $\varphi(s,v) = (s, \cos s, \sin s) + v\left(\frac{1}{2}, \frac{1}{2}\sin s, -\frac{1}{2}\cos s\right)$ null parallel p-equidistant B-scroll

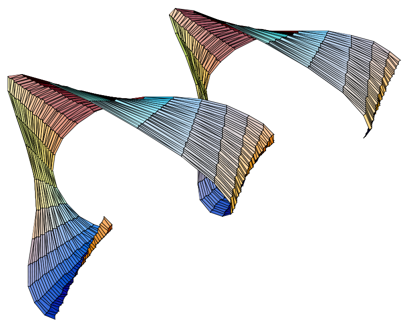

Figure 2. $\varphi^*(s^*, v^*) = (s^* + 2, \cos s^* + 2, \sin s^* + 2) + v^*\left(\frac{1}{2}, \frac{1}{2}\sin s^*, -\frac{1}{2}\cos s^*\right)$ null parallel p-equidistant B-scroll

References

1. Akutagawa K. and Nishikawa S., *The Gauss Map and Spacelike Surfaces with Prescribed Mean Curvature in Minkowski space*, Tohoku Math. J. 42, 67-82, (1990).
2. Balgetir H., Bektas M. and Ergut M., *Null Scrolls In The 3-Dimensional Lorentzian Space*, Balkan Society of Geometers, Geometry Balkan Press 5, No. 1, 1-5, (2003).
3. Balgetir H., Bektas M. and Ergut M., *On The B-Scrolls In The Three Dimensional Lorentzian Space* L^3, Kragujevac J. Math. 27, 163-174, (2005).
4. Beem J. K. and Ehrlich P. E., *Global Lorentzian Geometry*, Marcel Dekker Inc., New York, (1981).
5. Blair D. E. and Koufogiorgos Th., *Ruled Surfaces with Vanishing Second Gaussian Curvature*, Monatshefte Für Mathematik 113, No. 3, 177-181, (1992).
6. Duggal K. L. and Bejancu A., *Lightlike Submanifolds of Semi Riemannian Manifolds and Applications*, Kluwer Academic Publishers, Dorecht, (1996).
7. Fenghui J. and Zhong H. H., *On Lorentzian Surfaces with* $H^2 = K$ *In The Minkowski Space*, J. Math. Anal. Appl. 334, 54-58, (2007).
8. Ferrandez A, Gimenez A. and Lucas P., *Relativistic Particles and The Geometry of 4-D Null Curves*, Journal of geometry and Physics 57, 2124-2135, (2007).
9. Graves L., *Codimension One Isometric Immersions Between Lorentz Spaces*, Trans. Amer. Math. Soc. 252, 367-392, (1979).
10. Ikawa T., *On Curves and Submanifolds in an Indefinite-Riemannian Manifold*, Tsukuba J. Math. 9, No. 2, 353-371, (1985).
11. O'neil B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press Inc., (1983).
12. Sodsiri W., *Ruled Surfaces of Weingarten Type in Minkowski 3-space*, K. U. Leuven, Doctorate Thesis, (2005).
13. Spivak M., *A Comprehensive Introduction to Differential Geometry*, Vol. III, Publish or Perish, Berkeley, 272-297, (1979).

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