Bol. Soc. Paran. Mat. (3s.) v. 32 1 (2014): 291–300.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v32i1.20234

On a class of Kirchhoff type problems involving Hardy type potentials

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ABSTRACT: This article deals with the multiplicity of solutions for the following Kirchhoff type problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u &= \frac{\mu}{|x|^2}a(x)u + \lambda f(u) \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega,\end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $M : \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous and increasing function, $a : \Omega \to \mathbb{R}$ may change sign, $f : \mathbb{R} \to \mathbb{R}$ is continuous and sublinear at infinity, λ, μ are two parameters. Our proof is based on the three critical points theorem in [3].

Key Words: Kirchhoff type problem; Hardy type potential; Sublinear nonlinearity; Multiple solutions; Three critical points theorem.

Contents

1. Introduction and Preliminaries

In this article, we are concerned with a class of Kirchhoff type problems with Hardy type potential

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u &= \frac{\mu}{|x|^2}a(x)u + \lambda f(u) \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $M : \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous and increasing function with $\mathbb{R}_0^+ := [0, +\infty)$, the function $a : \Omega \to \mathbb{R}$ may change sign, λ is a positive parameter, $0 \leq \mu < \mu^\star$, where $\mu^\star = \left(\frac{N-2}{2}\right)^2$ is the best constant in the Hardy inequality, i.e.,

$$\int_\Omega \frac{|\varphi|^2}{|x|^2}\,dx \leq \frac{1}{\mu^\star}\int_\Omega |\nabla\varphi|^2\,dx \tag{1.2}$$

for all $\varphi \in C_0^\infty(\Omega)$, see [8].

Since the first equation in (1.1) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which

2000 *Mathematics Subject Classification:* 34B27, 35J60, 35B05.

depends on the average of itself, such as the population density, see [4]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.3}$$

presented by Kirchhoff in 1883, see [7]. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.3) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of thematerial, ρ is themass density, and P_0 is the initial tension.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [1,2,6,9,12,14,16], in which the authors have used different methods to get the existence of solutions for (1.1) in the case $\mu = 0$. In [11,17], Z. Zhang et al. studied the existence of nontrivial solutions and sign-changing solutions. In [5,8,10,13,15] the authors studied the existence of solutions for (1.1) in the case $M(t) \equiv 1$ and $a(x) \equiv 1$. Motivated by the papers mentioned above, in this work, we study the existence of solutions for Kirchhoff type problem (1.1) in which the function f is assumed to be sublinear at infinity. Our situation here is different from [1,2,16] in which the authors considered problem (1.1) in the case $\mu = 0$ and f is superlinear or assymptotically linear at infinity.

In order to state the main result of this paper, let us introduce the following assumptions for problem (1.1):

(A) $a : \overline{\Omega} \to \mathbb{R}$ is a continuous function, there exists $A_0 > 0$ such that $-A_0 \le a(x) \le A_0$ for a.e. $x \in \overline{\Omega}$;

(M_0) There exists $m_0 > 0$ such that

$$M(t) \ge m_0 \text{ for all } t \in \mathbb{R}_0^+;$$

(F_1) $f : \mathbb{R} \to \mathbb{R}$ is continuous and sublinear at infinity, i.e.,

$$\lim_{|t|\to+\infty} \frac{f(t)}{t} = 0$$

(F_2) f is superlinear at zero, i.e.,

$$\lim_{t\to 0} \frac{f(t)}{t} = 0;$$

(F_3) It holds that

$$\sup_{t\in\mathbb{R}} F(t) > 0,$$

where $F(t) = \int_0^t f(x,s)\,ds$.

Definition 1.1. *We say that $u \in H_0^1(\Omega)$ is a weak solution of problem (1.1) if*

$$M\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega \nabla u\cdot\nabla v\,dx - \mu\int_\Omega \frac{a(x)}{|x|^2}uv\,dx - \lambda\int_\Omega f(u)v\,dx = 0$$

for all $v \in C_0^\infty(\Omega)$.

Theorem 1.2. *Assume that the conditions (A), (M_0) and (F_1)-(F_3) are satisfied. Then there exists $\overline{\mu} > 0$ such that for any $\mu \in [0, \overline{\mu})$, there exist an open interval $\Lambda_\mu \subset (0, +\infty)$ and a real number $\delta_\mu > 0$ such that for every $\lambda \in \Lambda_\mu$ problem (1.1) has at least two distinct, nontrivial weak solutions in $H_0^1(\Omega)$ whose $H_0^1(\Omega)$-norms are less than δ_μ.*

It should be noticed that by the presence of the singular potential we cannot obtain the similar result for the p-Laplacian $-\Delta_p u$. This comes from the fact that the energy functional does not satisfy the Palais-Smale condition. Theorem 1.2 will be proved by using a recent result on the existence of at least three critical points by G. Bonanno [3]. For the reader's convenience, we describe it as follows.

Proposition 1.3 (See [3, Theorem 2.1])**.** *Let $(X, \|.\|)$ be a separable and reflexive real Banach space, $\mathcal{A}, \mathcal{F} : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\mathcal{A}(x_0) = \mathcal{F}(x_0) = 0$, $\mathcal{A}(x) \geq 0$ for all $x \in X$ and there exist $x_1 \in X$, $\rho > 0$ such that*

(i) $\rho < \mathcal{A}(x_1)$,

(ii) $\sup_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x) < \rho\frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)}$.

Further, put

$$\overline{a} = \frac{\xi\rho}{\rho\frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)} - \sup_{\{\mathcal{A}(x)<\rho\}}\mathcal{F}(x)}, \quad \text{with } \xi > 1,$$

and assume that the functional $\mathcal{A} - \lambda\mathcal{F}$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and

(iii) $\lim_{\|x\|\to\infty}[\mathcal{A}(x) - \lambda\mathcal{F}(x)] = +\infty$ *for every $\lambda \in [0, \overline{a}]$.*

Then, there exist an open interval $\Lambda \subset [0, \overline{a}]$ and a positive real number δ such that each $\lambda \in \Lambda$, the equation

$$D\mathcal{A}(u) - \lambda D\mathcal{F}(u) = 0$$

has at least three solutions in X whose $\|.\|$-norms are less than δ.

2. Multiple solutions

Let us define the functional $J_{\mu,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ by the following formula

$$\begin{aligned} J_{\mu,\lambda}(u) &= \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u|^2\,dx\right) - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u|^2\,dx - \lambda\int_\Omega F(u)\,dx, \\ &= \mathcal{A}(u) - \lambda\mathcal{F}(u), \end{aligned} \tag{2.1}$$

where

$$\mathcal{A}(u) = \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u|^2\,dx\right) - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u|^2\,dx,$$
$$\mathcal{F}(u) = \int_\Omega F(u)\,dx, \quad u \in H_0^1(\Omega), \quad \widehat{M}(t) = \int_0^t M(s)\,ds, \quad F(t) = \int_0^t f(s)\,ds. \tag{2.2}$$

In the rest of this paper, we denote by S_q the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, that is, $S_q\|u\|_{L^q(\Omega)} \le \|u\|$, where $\|.\|$ is the norm in $H_0^1(\Omega)$, $\|u\| = \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}}$.

Lemma 2.1. *There exists $\overline{\mu} > 0$ such that for any $\mu \in [0, \overline{\mu})$, the functional $J_{\mu,\lambda}$ is sequentially weakly lower semicontinuous on $H_0^1(\Omega)$.*

Proof: Let $\{u_m\}$ be a sequence that converges weakly to u in $H_0^1(\Omega)$. By the conditions (A) and (M_0), taking $\overline{\mu} = \frac{\mu^\star m_0}{A_0}$, then for each $0 \le \mu < \overline{\mu}$, using the same arguments as in the proof of [10, Theorem 3.2], we can obtain

$$\liminf_{m\to\infty}\left\{\frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u_m|^2\,dx\right) - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u_m|^2\,dx\right\} \ge \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u|^2\,dx\right) - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u|^2\,dx. \tag{2.3}$$

On the other hand, by (F_1), there exists a constant $C_1 > 0$, such that

$$|f(t)| \le C_1(1 + |t|), \text{ for all } t \in \mathbb{R}. \tag{2.4}$$

From (2.4) and the Hölder inequality, we get

$$\begin{aligned}
&\left|\int_\Omega F(u_m)\,dx - \int_\Omega F(u)\,dx\right| \\
&\le \int_\Omega |F(u_m) - F(u)|\,dx \\
&\le \int_\Omega |f(u + \theta_m(u_m - u))||u_m - u|\,dx \\
&\le C_1\int_\Omega (1 + |u + \theta_m(u_m - u)|)|u_m - u|\,dx \\
&\le C_1\left[\Big(meas(\Omega)\Big)^{\frac{1}{2}} + \|u + \theta_m(u_m - u)|\|_{L^2(\Omega)}\right]\|u_m - u\|_{L^2(\Omega)}, \quad \theta_m \in (0,1),
\end{aligned} \tag{2.5}$$

which shows that

$$\lim_{m\to\infty}\int_\Omega F(u_m)\,dx = \int_\Omega F(u)\,dx. \tag{2.6}$$

From relations (2.3) and (2.6), we conclude that

$$\liminf_{m\to\infty} J_{\mu,\lambda}(u_m) \geq J_{\mu,\lambda}(u)$$

and thus, the functional $J_{\mu,\lambda}$ is sequentially weakly lower semi-continuous in $H_0^1(\Omega)$. □

Lemma 2.2. *For every $\mu \in [0,\overline{\mu})$ and $\lambda \in \mathbb{R}$, the functional $J_{\mu,\lambda}$ is coercive and satisfies the Palais-Smale condition.*

Proof: Let us fix $\lambda \in \mathbb{R}$, arbitrary. By (F_1), there exists $\delta = \delta(\lambda) > 0$, such that

$$|f(t)| \leq S_2^2 \left(m_0 - \frac{\mu A_0}{\mu^\star} \right) (1+|\lambda|)^{-1}|t| \text{ for all } |t| > \delta.$$

Integrating the above inequality we have

$$|F(t)| \leq \frac{S_2^2}{2} \left(m_0 - \frac{\mu A_0}{\mu^\star} \right) (1+|\lambda|)^{-1}|t|^2 + \max_{|s|\leq\delta} |f(s)||t| \text{ for all } t \in \mathbb{R}. \tag{2.7}$$

Hence, by (A) and (1.2), it follows from the continuous embeddings and the Hölder inequality that

$$\begin{aligned} J_{\mu,\lambda}(u) &= \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u|^2\,dx\right) - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u|^2\,dx - \lambda\int_\Omega F(u)\,dx \\ &\geq \frac{m_0}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\mu A_0}{2}\int_\Omega \frac{|u|^2}{|x|^2}\,dx - |\lambda|\int_\Omega |F(u)|\,dx \\ &\geq \frac{1}{2}\Big(m_0 - \frac{\mu A_0}{\mu^\star}\Big)\int_\Omega |\nabla u|^2\,dx - \frac{|\lambda| S_2^2}{2(1+|\lambda|)}\Big(m_0 - \frac{\mu A_0}{\mu^\star}\Big)\int_\Omega |u|^2\,dx \qquad (2.8) \\ &\qquad - |\lambda|\max_{|t|\leq\delta}|f(t)|\int_\Omega |u|\,dx \\ &\geq \frac{1}{2(1+|\lambda|)}\Big(m_0 - \frac{\mu A_0}{\mu^\star}\Big)\|u\|^2 - \max_{|t|\leq\delta}|f(t)|\frac{|\lambda|}{S_1}(meas(\Omega))^{\frac{1}{2}}\|u\|. \end{aligned}$$

Since $\overline{\mu} = \frac{\mu^* m_0}{A_0} > 0$, we deduce that for each $\mu \in [0,\overline{\mu})$ and $\lambda \in \mathbb{R}$, the functional $J_{\mu,\lambda}$ is coercive.

Next, let $\{u_m\}$ be a sequence in $H_0^1(\Omega)$, such that

$$J_{\mu,\lambda}(u_m) \to c < \infty \text{ and } J'_{\mu,\lambda}(u_m) \to 0 \text{ in } H^{-1}(\Omega) \text{ as } m \to \infty, \tag{2.9}$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

Since $J_{\mu,\lambda}$ is coercive, the sequence $\{u_m\}$ is bounded in $H_0^1(\Omega)$. Then, there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, that converges weakly to some $u \in H_0^1(\Omega)$ and $\{u_m\}$ converges strongly to u in $L^2(\Omega)$.

We will prove that for any $u, v \in H_0^1(\Omega)$,

$$\Phi(u,v) \geq m_0\|u-v\|^2, \tag{2.10}$$

where m_0 is given by (M_0) and

$$\begin{aligned}\Phi(u,v) &= M\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega \nabla u\cdot(\nabla u-\nabla v)\,dx\\ &\quad -M\left(\int_\Omega |\nabla v|^2\,dx\right)\int_\Omega \nabla v\cdot(\nabla u-\nabla v)\,dx.\end{aligned}$$

Indeed, using the Cauchy inequality we have

$$\nabla u\cdot\nabla v \le |\nabla u||\nabla v| \le \frac{1}{2}(|\nabla u|^2+|\nabla v|^2)$$

or

$$\nabla u\cdot(\nabla u-\nabla v) \ge \frac{1}{2}(|\nabla u|^2-|\nabla v|^2). \tag{2.11}$$

Hence, because $M(t)$ is increasing, it implies that

$$\begin{aligned}\Phi(u,v) &= \left\{M\left(\int_\Omega |\nabla u|^2\,dx\right)-M\left(\int_\Omega |\nabla v|^2\,dx\right)\right\}\int_\Omega \nabla u\cdot(\nabla u-\nabla v)\,dx\\ &\quad + M\left(\int_\Omega |\nabla v|^2\,dx\right)\int_\Omega |\nabla u-\nabla v|^2\,dx\\ &\ge \frac{1}{2}\left\{M\left(\int_\Omega |\nabla u|^2\,dx\right)-M\left(\int_\Omega |\nabla v|^2\,dx\right)\right\}\left[\int_\Omega |\nabla u|^2\,dx-\int_\Omega |\nabla v|^2\,dx\right]\\ &\quad + M\left(\int_\Omega |\nabla v|^2\,dx\right)\int_\Omega |\nabla u-\nabla v|^2\,dx\\ &\ge m_0\|u-v\|^2.\end{aligned} \tag{2.12}$$

Now, from (2.9), (2.10) and the Hardy inequality, we find that

$$\begin{aligned}o(1) &= (J'(u_m)-J'(u))(u_m-u)\\ &= \frac{1}{2}M\left(\int_\Omega |\nabla u_m|^2\,dx\right)\int_\Omega \nabla u_m\cdot(\nabla u_m-\nabla u)\,dx\\ &\quad -\frac{1}{2}M\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega \nabla u\cdot(\nabla u_m-\nabla u)\,dx\\ &\quad -\frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^2}|u_m-u|^2\,dx-\lambda\int_\Omega \big(f(u_m)-f(u)\big)(u_m-u)\,dx\\ &\ge \frac{m_0}{2}\|u_m-u\|^2-\frac{\mu A_0}{2\mu^*}\|u_m-u\|^2-\lambda\int_\Omega \big(f(u_m)-f(u)\big)(u_m-u)\,dx\\ &= \frac{1}{2}\left(m_0-\frac{\mu A_0}{\mu^*}\right)\|u_m-u\|^2-\lambda\int_\Omega \big(f(u_m)-f(u)\big)(u_m-u)\,dx.\end{aligned} \tag{2.13}$$

On the other hand, by the Hölder inequality,

$$\begin{aligned} &\left|\int_\Omega \Big(f(u_m)-f(u)\Big)(u_m-u)\,dx\right| \\ &\le C_1\int_\Omega (2+|u_m|+|u|)|u_m-u|\,dx \qquad (2.14)\\ &\le C_1\left[2\Big(meas(\Omega)\Big)^{\frac12}+\|u_m\|_{L^2(\Omega)}+\|u\|_{L^2(\Omega)}\right]\|u_m-u\|_{L^2(\Omega)}, \end{aligned}$$

which approaches 0 as $m\to\infty$.

From (2.13), (2.14) and the fact that $0\le\mu<\overline{\mu}=\frac{\mu^* m_0}{A_0}$, we deduce that u_m converges strongly to u in $H_0^1(\Omega)$. □

Lemma 2.3. *For each $\mu\in[0,\overline{\mu})$ we have*

$$\lim_{\rho\to 0^+}\frac{\sup\{\mathcal{F}(u):\ \mathcal{A}(u)<\rho\}}{\rho}=0,$$

where the functionals $\mathcal{A}$ and $\mathcal{F}$ are given by (2.2).

Proof: By (F_2), for an arbitrary small $\epsilon>0$, there exists $\delta>0$, such that

$$|f(t)|\le\frac{\epsilon S_2^2}{2}\left(m_0-\frac{\mu A_0}{\mu^*}\right)|t| \text{ for all } |t|<\delta.$$

Combining the above inequality with (2.4) we get

$$|F(t)|\le\frac{\epsilon S_2^2}{4}\left(m_0-\frac{\mu A_0}{\mu^*}\right)|t|^2+C_\delta|t|^q \text{ for all } t\in\mathbb{R}, \qquad (2.15)$$

where $q\in\left(2,\frac{2N}{N-2}\right)$, and C_δ is a constant that does not depend on t.

Next, for each $\rho>0$, we define the sets

$$B_\rho^1=\left\{u\in H_0^1(\Omega):\quad \mathcal{A}(u)<\rho\right\},$$

$$B_\rho^2=\left\{u\in H_0^1(\Omega):\quad \left(m_0-\frac{\mu A_0}{\mu^*}\right)\|u\|^2<2\rho\right\}.$$

By (1.2), the conditions (A) and (M_0) we have $B_\rho^1\subset B_\rho^2$. Moreover, using (2.15), it follows that for any $u\in B_\rho^2$,

$$\mathcal{F}(u)\le\frac{\epsilon}{4}\left(m_0-\frac{\mu A_0}{\mu^*}\right)\|u\|^2+C_\delta S_q^{-q}\|u\|^q. \qquad (2.16)$$

Since $0\in B_\rho^1$ and $J_{\mu,\lambda}(0)=0$, we have $0\le\sup_{u\in B_\rho^1}J_{\mu,\lambda}(u)$. On the other hand, if $u\in B_\rho^2$, then

$$\|u\|\le\left(m_0-\frac{\mu A_0}{\mu^*}\right)^{\frac{-1}{2}}(2\rho)^{\frac12}.$$

Now, using (2.16), we deduce that

$$
\begin{aligned}
0 \leq \frac{\sup_{u\in B^1_\rho}\mathcal{F}(u)}{\rho} &\leq \frac{\sup_{u\in B^2_\rho}\mathcal{F}(u)}{\rho} \\
&\leq \frac{\epsilon}{2} + C_\delta S_q^{-q}\left(m_0 - \frac{\mu A_0}{\mu^*}\right)^{\frac{-q}{2}}(2\rho)^{\frac{q}{2}-1}.
\end{aligned} \tag{2.17}
$$

Since $q > 2$, letting $\rho \to 0^+$, because $\epsilon > 0$ is arbitrary, we get the conclusion. □

Proof: [Proof of Theorem 1.2] In order to prove Theorem 1.2, we shall apply Proposition 1.3 by choosing $X = H^1_0(\Omega)$ as well as $\mathcal{A}$ and $\mathcal{F}$ as in (2.2). Now, we shall check all assumptions of Proposition 1.3. Indeed, we have $\mathcal{A}(0) = \mathcal{F}(0) = 0$ and since $-A_0 \leq a(x) \leq A_0$ for all $x \in \overline{\Omega}$, we deduce from (1.2) that for any $0 \leq \mu < \overline{\mu}$, $\mathcal{A}(u) \geq 0$ for any $u \in H^1_0(\Omega)$.

From (F_3), let $t_0 \in \mathbb{R}$ be such that $F(t_0) > 0$. Also choose $R_0 > 0$ such a way that $R_0 < \text{dist}(0, \partial\Omega)$. For $\sigma \in (0, 1)$, we define the function u_σ by

$$
u_\sigma(x) = \begin{cases} 0, & \text{for } x \in \mathbb{R}^N \backslash B_{R_0}(0), \\ t_0, & \text{for } x \in B_{\sigma R_0}(0), \\ \frac{t_0}{R_0(1-\sigma)}(R_0 - |x|) & \text{for } x \in B_{R_0}(0)\backslash B_{\sigma R_0}(0), \end{cases}
$$

where $B_r(0)$ denotes the N-dimensional open ball with center 0 and radius $r > 0$, and $|.|$ denotes the usual Euclidean norm in $\mathbb{R}^N$. It is clear that $u_\sigma \in H^1_0(\Omega)$. From the definition of u_σ, simple computations show that

$$
\|u_\sigma\|^2 = t_0^2(1-\sigma)^{-2}(1-\sigma^N)\omega_N R_0^{N-2}
$$

and

$$
\begin{aligned}
\mathcal{F}(u_\sigma) &= \int_{B_{\sigma R_0}(0)} F(u_\sigma)\,dx + \int_{B_{R_0}(0)\backslash B_{\sigma R_0}(0)} F(u_\sigma)\,dx \\
&\geq \left[F(t_0)\sigma^N - \max_{|t|\leq R_0}|F(t)|(1-\sigma)^N\right] R_0^N \omega_N,
\end{aligned}
$$

where ω_N is the volume of the unit ball $B_1(0)$. If we choose $\sigma \in (0, 1)$ close enough to 1, says σ_0, then the right-hand side of the last inequality becomes strictly positive. By Lemma 2.3, we can choose $\rho_0 \in (0, 1)$ such that

$$
\rho_0 < \left(m_0 - \frac{\mu A_0}{\mu^*}\right)\|u_{\sigma_0}\|^2 \leq \mathcal{A}(u_{\sigma_0})
$$

and

$$
\begin{aligned}
\frac{\sup_{\mathcal{A}(u)<\rho_0}\mathcal{F}(u)}{\rho_0} &< \frac{\left[F(t_0)\sigma_0^N - \max_{|t|\leq R_0}|F(t)|(1-\sigma_0)^N\right] R_0^N\omega_N}{2\mathcal{A}(u_{\sigma_0})} \\
&< \frac{\mathcal{F}(u_{\sigma_0})}{\mathcal{A}(u_{\sigma_0})}.
\end{aligned}
$$

Now, in Proposition 1.3, we choose $x_0 = 0$, $x_1 = u_{\sigma_0}$, $\xi = 1 + \rho_0$ and

$$\overline{a} = \overline{a}_\mu = \frac{1 + \rho_0}{\frac{\mathcal{F}(u_{\delta_0})}{\mathcal{A}(u_{\delta_0})} - \frac{\sup_{\mathcal{A}(u) < \rho_{\sigma_0}} \mathcal{F}(u)}{\rho_0}} > 0.$$

For any $\mu \in [0, \overline{\mu})$, taking into account the above lemmas, all assumptions of Proposition 1.3 are verified. Then there exist an open interval $\Lambda_{\overline{\mu}} \subset [0, \overline{a}]$ and a number $\delta_{\overline{\mu}}$, such that for each $\lambda \in \Lambda_{\overline{\mu}}$, the equation $D\mathcal{A}(u) - \lambda D\mathcal{F}(u) = 0$ has at least three solutions in $H_0^1(\Omega)$ whose $H_0^1(\Omega)$-norms are less than $\delta_{\overline{\mu}}$. By (F_2), $f(0) = 0$, one of them may be the trivial one, so problem (1.1) has at least two non-trivial weak solutions with the required properties. □

Acknowledgments

The author would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

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