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Stokes problem with the possibility of controlling the velocity in a L-shaped domain

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ABSTRACT: The movement is studied from a viscous and incompressible homogeneous fluid which crosses a field of the channel in the form of L, with the possibility to exert pressure of known difference between two opposite edges. We extend previous work in [1] which studies a problem of Stokes in the stationary case and with one parameter that characterizes the pressure difference between two sides in a specific domain (symmetric channel). We show existence, unicity and regularity of the solution of an evolution problem with four parameters that characterize the pressure difference between two opposite sides of our field.

Key Words: Stokes problem, Regularity of the solution, weak solution.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega = \Gamma = \bigcup_{i=1}^{8} \Gamma_i$, where $\Gamma_1 = \{0\}\times[0,1], \Gamma_2 = \{0\}\times[1,3], \Gamma_3 = [0,1]\times\{3\}, \Gamma_4 = \{1\}\times[1,3], \Gamma_5 = [1,5]\times\{1\}$, $\Gamma_6 = \{5\} \times [0,1]$, $\Gamma_7 = [1,5] \times \{0\}$ and $\Gamma_8 = [0,1] \times \{0\}$, see Figure 1. Given four real numbers $\lambda_1, \lambda_2, \lambda_3$ and λ_4, we consider the problem:

$$(S_1)\begin{cases} \text{Find } u = (u_1, u_2) \in V_1 \text{ such that} \\ \sum_{i=1}^{2} \int_\Omega \frac{\partial u_i}{\partial t} v_i + \sum_{i=1}^{2} \int_\Omega \nabla u_i \nabla v_i = \lambda_1 \int_0^1 v_1(5,y)dy + \lambda_2 \int_1^3 v_1(1,y)dy \\ +\lambda_3 \int_0^1 v_2(x,3)dx + \lambda_4 \int_1^5 v_2(x,1)dx \qquad \forall v = (v_1, v_2) \in V_1, \\ u_1(\mathbf{x}, 0) = a_{01}(\mathbf{x}) \qquad \text{a.e. } \mathbf{x} \text{ in} \Omega, \\ u_2(\mathbf{x}, 0) = a_{02}(\mathbf{x}) \qquad \text{a.e. } \mathbf{x} \text{ in} \Omega, \end{cases}$$

where V_1 is the closing of $\{v = (v_1, v_2) \in C^1([0,T]; H);\ div\ v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0,\ v_i|_{\Gamma_1} = v_i|_{\Gamma_6},\ v_i|_{\Gamma_2} = v_i|_{\Gamma_4},\ v_i|_{\Gamma_3} = v_i|_{\Gamma_8},\ v_i|_{\Gamma_5} = v_i|_{\Gamma_7}$ for $i = 1, 2\}$ in $C([0,T]; H)$, H is the closing of $\vartheta = \{u \in (\mathcal{D}(\Omega))^2;\ div\ v = 0\}$ in $(L^2(\Omega))^2$, where $\mathcal{D}$ consist of all functions in $C^\infty(\Omega)$ which have compact support in Ω and

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$a_0 = (a_{01}, a_{02}) \in H$.

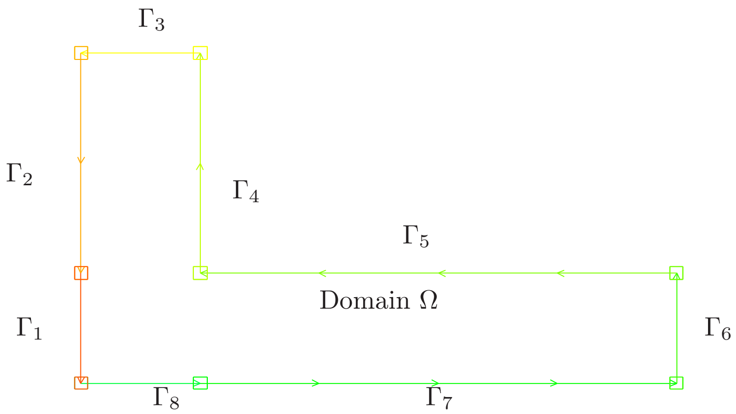

Figure 1: Vertical plane from the channel

As applications of this problem, we cite the different types of flows for example see [2,3]. This problem was studied by C. Amrouche, M. Batchi and J. Batina in the stationary case with only one parameter see [1], we extend the preceding work to a problem of evolution with four parameters. Our aim is, in first time to prove the existence, uniqueness and regularity of the solution, second time we show the equivalence between the variational problem, where the notion of pressure does not appear explicitly, and classic problem which highlights the pressure and these differences between the opposite sides of our field. We cite also a variety of works in the stationary case see [2,4].

This paper is organized as follows. In Section 2, we give preliminaries. In Section 3, we establish existence, unicity and regularity of the solution. In Section 4, we prove the equivalence between our variational problem and the classical problem associated.

2. Preliminaries

Let us denote by V the closing of ϑ in $(H^1(\Omega))^2$. We consider the following Banach spaces $L^2(0,T;V)$ and $C([0,T];H)$ with the norms $\|u\|_{L^2(0,T;V)} = \left(\int_0^T \|u(t)\|_V^2 dt\right)^{\frac{1}{2}}$ and $\|u\|_{C([0,T];H)} = \sup\limits_{t\in[0,T]} \|u(t)\|_H$ respectively.

Proposition 2.1. *If $u \in W(a,b;V,V')$, then for all v in V, we have*

$$\frac{d}{dt}(u(.),v)_{V,V} = \langle u'(.),v\rangle_{\mathcal{D}'(]a,b[),V} \ \textit{in}\ \mathcal{D}'(]a,b[),$$

where $W(a,b;X,Y) = \{u \in L^2(a,b;X);\ u' \in L^2(a,b;Y)\}$ *is a Hilbert space with the norm* $\|u\|_W = \left(\int_a^b (|u(t)|_X^2 + |u'(t)|_Y^2)dt\right)^{\frac{1}{2}}$ *(see [3]).*

Proof: Let $\varphi \in \mathcal{D}(]a,b[)$, we have $\int_a^b \langle u'(t), v\rangle \varphi(t)dt = \int_a^b \langle u'(t)\varphi(t), v\rangle dt$. Since $u' \in L^2(a,b;V')$, we deduce that the function $t \to \langle u'(t), v\rangle$ is in $L^2(a,b)$ for all $v \in V$.
In the same way, we have

$$\begin{aligned}\int_a^b \langle u'(t)\varphi(t), v\rangle dt &= \langle \int_a^b u'(t)\varphi(t)dt, v\rangle = -\langle \int_a^b u(t)\varphi'(t)dt, v\rangle \\ &= -\int_a^b \langle u(t)\varphi'(t), v\rangle dt = -\int_a^b (u(t), v)\varphi'(t)dt.\end{aligned}$$

Thus $\int_a^b \langle u'(t), v\rangle \varphi(t)dt = -\int_a^b (u(t), v)\varphi'(t)dt = \int_a^b \frac{d}{dt}(u(t), v)\varphi(t)dt$, and therefore $\langle u'(.), v\rangle = \frac{d}{dt}(u(.), v)$. □

3. Existence, uniqueness and regularity of solution

Theorem 3.1. *If the solution of the problem* (S_1) *exists, it is necessarily unique.*

Proof: Let u_1 and u_2 be two solutions of the problem (S_1). Put $w = u_1 - u_2$,

$$(\partial_t w, v) + ((w, v)) = 0 \ \forall v \in V_1 \text{ and } w(0) = 0,$$

where $(\partial_t w, v) = \sum_{i=1}^{2} \int_\Omega \frac{\partial w_i}{\partial t} v_i$, $((w, v)) = \sum_{i=1}^{2} \int_\Omega \nabla w_i \nabla v_i$. Then

$$\int_0^T (\partial_t w, v)dt + \int_0^T ((w, v))dt = 0.$$

Let us take $v = w\chi_{(0,s)}(t)$, $s \in (0,T)$, thus $\int_0^s (\partial_t w, w)dt + \int_0^s ((w, w))dt = 0$.
Hence

$$\int_0^s (\partial_t w, w)dt = -\int_0^s ((w, w))dt = -\int_0^s \|w(t)\|_V^2 dt \leq 0.$$

Consequently

$$\frac{1}{2}\|w(s)\|_H^2 - \frac{1}{2}\|w(0)\|_H^2 = \frac{1}{2}\|w(s)\|_H^2 \leq 0,$$

and this shows that $w = 0$. □

The problem (S_1) becomes

$$\begin{cases} \text{Find } u \in V_1 \text{ such that} & \\ \frac{\partial}{\partial t}(u, v) + ((u, v)) = \langle \lambda_1 e_1, v\rangle_{\Gamma_6} + \langle \lambda_2 e_1, v\rangle_{\Gamma_4} & \\ +\langle \lambda_3 e_2, v\rangle_{\Gamma_3} + \langle \lambda_4 e_2, v\rangle_{\Gamma_5} & \forall v \in V_1, \\ u(\mathbf{x}, 0) = a_0(\mathbf{x}) & \text{a.e. } \mathbf{x} \text{ in } \Omega, \end{cases}$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.
We are looking for an approximate solution (u_m) of the form:

$$u_m(t) = \sum_{i=1}^{m} g_i(t) w_i, \tag{3.1}$$

where $\{w_i\}$ is a Hilbertienne basis of H and $g_i \in C([0, T]), g_i(0) = g_{i0} = (u_0, w_i)$.
We have $\forall j = 1, \ldots, m$

$$\begin{aligned} \frac{\partial}{\partial t}(u_m(t), w_j) + ((u_m(t), w_j)) = \langle \lambda_1 e_1, w_j \rangle_{\Gamma_6} + \langle \lambda_2 e_1, w_j \rangle_{\Gamma_4} \\ + \langle \lambda_3 e_2, w_j \rangle_{\Gamma_3} + \langle \lambda_4 e_2, w_j \rangle_{\Gamma_5}. \end{aligned} \tag{3.2}$$

Put $V_m = \{w_1, w_2, \ldots, w_m\}$. By replacing (3.1) in (3.2), we obtain

$$\frac{\partial}{\partial t}(\sum_{i=1}^{m} g_i(t) w_i, w_j) + ((\sum_{i=1}^{m} g_i(t) w_i, w_j)) = \gamma_j, \tag{3.3}$$

where $\gamma_j = \langle \lambda_1 e_1, w_j \rangle_{\Gamma_6} + \langle \lambda_2 e_1, w_j \rangle_{\Gamma_4} + \langle \lambda_3 e_2, w_j \rangle_{\Gamma_3} + \langle \lambda_4 e_2, w_j \rangle_{\Gamma_5}$.
Thus

$$\frac{\partial}{\partial t}\left[\sum_{i=1}^{m} g_i(t)(w_i, w_j)\right] + \sum_{i=1}^{m} g_i((w_i, w_j)) = \gamma_j. \tag{3.4}$$

So $g_j'(t) + \frac{1}{\alpha_j} g_j(t) = \gamma_j$, where α_j are the eigenvalues of $\psi : H \to V, f \mapsto u$, with u is the unique solution of the problem

$$u \in V; ((u, v)) = (f, v) \ \forall v \in V.$$

We have the following results: $\psi(w_i) = \alpha_i w_i$ for all $i \geq 1$, all the eigenvalues are strictly positive and $\alpha_i \to 0$ when $i \to \infty$, $\{w_i\}$ is an orthogonal system in V and we have $((w_i, w_j)) = \frac{1}{\alpha_i}\delta_{ij}$, where δ_{ij} is the kronocker symbol. We obtain

$$(E) \begin{cases} g_j'(t) + \frac{1}{\alpha_j} g_j(t) = \gamma_j & \forall j = 1, \ldots, m \ a.e \ t \in (0, T), \\ g_j(0) = g_{j0} \end{cases}$$

which admits a unique solution. Consequently $u_m = \sum\limits_{i=1}^{m} g_i(t) w_i$ is the solution of the approximate problem and in the same way as previously, we show the unicity of the solution u_m.

Theorem 3.2. *If $u_0 \in V_1$, the approximate solution u_m satisfies*

there exist $c, c' > 0$ such that $\|u_m'\|_{L^2(0,T;H)} \leq c$ and $\|u_m\|_{C([0,T];V)} \leq c'$.

Proof: The function $u_m(x, t) = \sum\limits_{i=1}^{m} g_i(t) w_i$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t}(u_m, v) + ((u_m, v)) &= \langle \lambda_1 e_1, v \rangle_{\Gamma_6} + \langle \lambda_2 e_1, v \rangle_{\Gamma_4} + \langle \lambda_3 e_2, v \rangle_{\Gamma_3} + \langle \lambda_4 e_2, v \rangle_{\Gamma_5} \\ &:= b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma) \ \forall v \in V. \end{aligned}$$

Then $(u'_m, v) + ((u_m, v)) = b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma)$. For $v(t) = u'_m(t) = \sum\limits_{i=1}^{m} g'_i(t) w_i \in V$, we have

$$(u'_m, u'_m) + ((u_m, u'_m)) = b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, u'_m, \Gamma) \text{ for a.e. } t \in (0, T). \tag{3.5}$$

Each term of (3.5) is integrable, indeed

$$(u'_m, u'_m) = \sum_{i=1}^{m} |g'_i(t)|^2 \in L^1(0, T)$$

because $g'_j(t) = \gamma_j - \frac{1}{\alpha_j} g_j(t)$ and $g'_j(t) \in L^2(0, T)$ $(g_j \in C([0, T]))$.

$$((u_m, u'_m)) = ((\sum_{i=1}^{m} g_i w_i, \sum_{i=1}^{m} g_j w_j)) = \sum_{i=1}^{m} \frac{1}{\alpha_i} g_i g'_i \in L^1(0, T).$$

$$b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, u'_m, \Gamma) = \sum_{i=1}^{m} g'_i b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, w_i, \Gamma) \in L^1(0, T).$$

We deduce that

$$\begin{aligned} \int_0^s \|u'_m(t)\|_H^2 dt + \int_0^s ((u_m, u'_m)) dt &= \int_0^s (\langle \lambda_1 e_1, u'_m \rangle_{\Gamma_6} + \langle \lambda_2 e_1, u'_m \rangle_{\Gamma_4} \\ &\quad + \langle \lambda_3 e_2, u'_m \rangle_{\Gamma_3} + \langle \lambda_4 e_2, u'_m \rangle_{\Gamma_5}) dt, \ \forall s \in (0, T). \end{aligned}$$

If we put $A(m, s) = \int_0^s \|u'_m(t)\|_H^2 dt + \frac{1}{2}\|u_m(s)\|_V^2 - \frac{1}{2}\|u_m(0)\|_V^2$, then we have

$$\begin{aligned} A(m, s) &= \lambda_1 \int_0^s \int_0^1 u'_{m1}(5, y)(t) dy dt + \lambda_2 \int_0^s \int_1^3 u'_{m1}(1, y)(t) dy dt \\ &+ \lambda_3 \int_0^s \int_0^1 u'_{m2}(x, 3)(t) dx dt + \lambda_4 \int_0^s \int_0^5 u'_{m2}(x, 1)(t) dx dt. \end{aligned}$$

So

$$\begin{aligned} A(m, s) &\leq |\lambda_1| | \int_0^1 \int_0^s u'_{m1}(5, y)(t) dt dy| + |\lambda_2| | \int_0^3 \int_0^s u'_{m1}(1, y)(t) dt dy| \\ &+ |\lambda_3| | \int_0^1 \int_0^s u'_{m2}(x, 3)(t) dt dx| + |\lambda_4| | \int_0^5 \int_0^s u'_{m2}(x, 1)(t) dt dx|. \end{aligned}$$

Thus

$$\begin{aligned} A(m,s) &\leq |\lambda_1|| \int_0^1 u_{m1}(5,y)(s)dy - \int_0^1 u_{m1}(5,y)(0)dy| \\ &+ |\lambda_2|| \int_0^3 u_{m1}(1,y)(s)dy - \int_0^3 u_{m1}(1,y)(0)dy| \\ &+ |\lambda_3|| \int_0^1 u_{m2}(x,3)(s)dx - \int_0^1 u_{m2}(x,3)(0)dx| \\ &+ |\lambda_4|| \int_0^5 u_{m2}(x,1)(s)dx - \int_0^5 u_{m2}(x,1)(0)dx|. \end{aligned}$$

So

$$\begin{aligned} A(m,s) &\leq c_1[\int_{\Gamma_6} |u_{m1}(s)|d\sigma + \int_{\Gamma_6} |u_{m1}(0)|d\sigma + \int_{\Gamma_4} |u_{m1}(s)|d\sigma + \int_{\Gamma_4} |u_{m1}(0)|d\sigma \\ &+ \int_{\Gamma_3} |u_{m2}(s)|d\sigma + \int_{\Gamma_3} |u_{m2}(0)|d\sigma + \int_{\Gamma_5} |u_{m2}(s)|d\sigma + \int_{\Gamma_5} |u_{m2}(0)|d\sigma] \\ &\leq c_2(\int_{\partial\Omega} |u_m(s)|d\sigma + \int_{\partial\Omega} |u_m(0)|d\sigma) \\ &\leq c_3[(\int_{\partial\Omega} |u_m(s)|^2 d\sigma)^{\frac{1}{2}} + (\int_{\partial\Omega} |u_m(0)|^2 d\sigma)^{\frac{1}{2}}] \ \text{(Holder's inequality)} \\ &= c_3[\|u_m(s)\|_{L^2(\partial\Omega)} + \|u_m(0)\|_{L^2(\partial\Omega)}] \\ &\leq c_4[\|u_m(s)\|_V + \|u_m(0)\|_V] \ (V \hookrightarrow L^2(\partial\Omega) \text{ a continuous injection }), \end{aligned}$$

where $c_i, i = 1 \dots 4$, are positive reals.
On the one hand

$$\begin{aligned} \int_0^s \|u_m'(t)\|_H^2 dt &= A(m,s) - \frac{1}{2}\|u_m(s)\|_V^2 + \frac{1}{2}\|u_m(0)\|_V^2 \\ &\leq c_4[\|u_m(s)\|_V + \|u_m(0)\|_V] - \frac{1}{2}\|u_m(s)\|_V^2 + \frac{1}{2}\|u_m(0)\|_V^2. \end{aligned}$$

Let us consider P_m the projection of V on the space $span(w_0, w_1, \dots, w_m)$, we have $u_0 \in V$ and $u_m(0) = \sum_{i=1}^{m} (u_0, w_i)w_i = P_m u_0$. Hence $\|u_m(0)\|_V \leq \|u_0\|_V$ because $\|P_m\| \leq 1$.
So

$$\int_0^s \|u_m'(t)\|_H^2 dt \leq c_4[\|u_m(s)\|_V + \|u_0\|_V] - \frac{1}{2}\|u_m(s)\|_V^2 + \frac{1}{2}\|u_0\|_V^2. \tag{3.6}$$

We put $\|u_m(s)\|_V = R_m(s)$, then we obtain

$$\int_0^s \|u_m'(t)\|_H^2 dt \leq c_4 R_m(s) - \frac{1}{2}(R_m(s))^2 + c_5, \quad \text{where } c_5 \geq 0. \tag{3.7}$$

Consequently there exists $c_6 > 0$ such that

$$\int_0^s \|u'_m(t)\|_H^2 dt \leq c_6 \ \forall\ m.$$

On the other hand, we have

$$\int_0^s \|u'_m(t)\|_H^2 dt + \frac{1}{2}\|u_m(s)\|_V^2 - \frac{1}{2}\|u_m(0)\|_V^2 \leq c_4[\|u_m(s)\|_V + \|u_m(0)\|_V]. \quad (3.8)$$

So

$$\frac{1}{2}\|u_m(s)\|_V^2 - \frac{1}{2}\|u_m(0)\|_V^2 \leq c_4[\|u_m(s)\|_V + \|u_m(0)\|_V]. \quad (3.9)$$

Thus

$$\begin{aligned}\frac{1}{2}\|u_m(s)\|_V^2 - c_4\|u_m(s)\|_V &\leq \frac{1}{2}\|u_m(0)\|_V^2 + c_4\|u_m(0)\|_V \\ &\leq \frac{1}{2}\|u_0\|_V^2 + c_4\|u_0\|_V.\end{aligned}$$

Finally there exists $c_7 > 0$ such that

$$\|u_m(s)\|_V \leq c_7 \ \forall\ m.$$

□

Theorem 3.3. *If $u_0 \in V$, then the solution of the problem (S_1) exists, unique and satisfies $u \in L^\infty(0,T;V)$, $\frac{\partial u}{\partial t} \in L^2(0,T;H)$.*

Proof: From Theorem 3.2, we have $\|u_m\|_{(C([0,T]);V)} \leq C$. Since $L^\infty(0,T;V) \sim (L^1(0,T;V'))'$ and V' is reflexive, separable, we deduce that there exists u_{m_k}; $u_{m_k} \rightharpoonup^* u$ when $m_k \to \infty$ in $L^\infty(0,T;V)$, we show also that $\|u'_m\|_{L^2(0,T.H)} \leq c'$. Now we can extract (u'_{m_k}) such that $u'_{m_k} \rightharpoonup^* \eta$ when $m_k \to \infty$ in $L^\infty(0,T;H)$. We have

$$(u'_{m_k}, v) + ((u_{m_k}, v)) = b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma),\ \forall\ v \in V. \quad (3.10)$$

Hence

$$(u_{m_k}, v') + ((u_{m_k}, v)) = b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma),\ \forall\ v \in V. \quad (3.11)$$

When $m_k \to \infty$, we deduce from (3.10) and (3.11) respectively:

$$(\eta, v') + ((u, v)) = b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma),\ \forall\ v \in V. \quad (3.12)$$

$$(u', v) + ((u, v)) = b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma),\ \forall\ v \in V. \quad (3.13)$$

Then, according to (3.12) and (3.13) we get $(\eta, v) = (u', v)\ \forall\ v \in V$, thus $\eta = u'$. When $m_k \to \infty$, we have

$$(\frac{\partial u}{\partial t}, v) + ((u, v)) = b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma),\ \forall\ v \in V. \quad (3.14)$$

So u is a solution of the problem (S_1) and it is unique according to Theorem 3.1.

□

4. Classical problem

Theorem 4.1. *Let $u \in C([0,T];V)$. u is a solution of (S_1) if and only if there exists $p \in L^2(0,T;\mathbb{R})$ such that (u,p) is a solution of the problem:*

$$(S_2)\begin{cases} \frac{\partial u_1}{\partial t} - \triangle u_1 + \frac{\partial p}{\partial x} = 0 & in\ \Omega \times (0,T),\\ \frac{\partial u_2}{\partial t} - \triangle u_2 + \frac{\partial p}{\partial y} = 0 & in\ \Omega \times (0,T),\\ div\ u = 0 & in\ \Omega \times (0,T),\\ u_i|_{\Gamma_1} = u_i|_{\Gamma_6}, u_i|_{\Gamma_2} = u_i|_{\Gamma_4}, u_i|_{\Gamma_3} & \\ \quad = u_i|_{\Gamma_8}, u_i|_{\Gamma_5} = u_i|_{\Gamma_7} & for\ i=1,2,\\ \frac{\partial u_i}{\partial x}|_{\Gamma_1} = \frac{\partial u_i}{\partial x}|_{\Gamma_6}, \frac{\partial u_i}{\partial x}|_{\Gamma_2} = \frac{\partial u_i}{\partial x}|_{\Gamma_4} & for\ i=1,2,\\ \frac{\partial u_i}{\partial y}|_{\Gamma_3} = \frac{\partial u_i}{\partial y}|_{\Gamma_8}, \frac{\partial u_i}{\partial y}|_{\Gamma_5} = \frac{\partial u_i}{\partial y}|_{\Gamma_7} & for\ i=1,2,\\ p|_{\Gamma_6} - p|_{\Gamma_1} = -\lambda_1, p|_{\Gamma_4} - p|_{\Gamma_2} = -\lambda_2, p|_{\Gamma_3} - p|_{\Gamma_8} & \\ \quad = -\lambda_3, p|_{\Gamma_5} - p|_{\Gamma_7} = -\lambda_4, & \\ u_1(\mathbf{x},0) = a_{01}(\mathbf{x}) & a.e.\ \mathbf{x}\ in\ \Omega,\\ u_2(\mathbf{x},0) = a_{02}(\mathbf{x}) & a.e.\ \mathbf{x}\ in\ \Omega. \end{cases}$$

Proof: Let us assume that (S_2) have a solution $(u=(u_1,u_2),p)$.
Thus

$$\sum_{i=1}^{2}\int_\Omega \frac{\partial u_i}{\partial t} v_i d\mathbf{x} - \sum_{i=1}^{2}\int_\Omega \triangle u_i\ v_i d\mathbf{x} + \int_\Omega \frac{\partial p}{\partial x}\ v_1 d\mathbf{x} + \int_\Omega \frac{\partial p}{\partial y}\ v_2 d\mathbf{x} = 0, \forall v=(v_1,v_2)\in V_1,$$

so

$$\sum_{i=1}^{2}\int_\Omega \frac{\partial u_i}{\partial t} v_i d\mathbf{x} - \sum_{i=1}^{2}\int_\Omega \triangle u_i\ v_i d\mathbf{x} + \int_\Omega (\nabla p).v d\mathbf{x} = 0, \forall v=(v_1,v_2)\in V_1.$$

Hence $\forall v \in V_1$, we have

$$\sum_{i=1}^{2}\int_\Omega \frac{\partial u_i}{\partial t} v_i d\mathbf{x} + \sum_{i=1}^{2}\int_\Omega \nabla u_i\ \nabla v_i d\mathbf{x} - \sum_{i=1}^{2}\int_{\partial\Omega} \frac{\partial u_i}{\partial n}.v_i d\sigma + \int_{\partial\Omega} pv.\vec{\eta} d\sigma - \int_\Omega p\ div\ v d\mathbf{x} = 0,$$

where $\vec{\eta}$ is the unit outward normal to $\partial\Omega$. On the other hand, we have

$$\int_{\partial\Omega} \frac{\partial u_i}{\partial n}.v_i d\sigma = \sum_{j=1}^{8}\int_{\Gamma_j} \frac{\partial u_i}{\partial n}.v_i d\sigma = 0.$$

$$\begin{aligned} \int_{\partial\Omega} pv.\vec{\eta} d\sigma = &-\int_0^1 p|_{\Gamma_1}(x,y)v_1(5,y)dy - \int_1^3 p|_{\Gamma_2}(x,y)v_1(1,y)dy \\ &+ \int_0^1 p|_{\Gamma_3}(x,y)v_2(x,3)dx + \int_1^3 p|_{\Gamma_4}(x,y)v_1(1,y)dy \\ &+ \int_1^5 p|_{\Gamma_5}(x,y)v_2(x,1)dx + \int_0^1 p|_{\Gamma_6}(x,y)v_1(5,y)dy \\ &- \int_1^3 p|_{\Gamma_7}(x,y)v_2(x,1)dx - \int_0^1 p|_{\Gamma_8}(x,y)v_2(x,3)dx. \end{aligned}$$

Hence

$$\begin{aligned}\int_{\partial\Omega} pv.\vec{\eta}d\sigma &= \int_0^1 (-p|_{\Gamma_1}(x,y) + p|_{\Gamma_6}(x,y))v_1(5,y)dy + \int_1^3 (-p|_{\Gamma_2}(x,y) \\ &+ p|_{\Gamma_4}(x,y))v_1(1,y)dy + \int_0^1 (p|_{\Gamma_3}(x,y) - p|_{\Gamma_8}(x,y))v_2(x,3)dx \\ &+ \int_1^5 (p|_{\Gamma_5}(x,y) - p|_{\Gamma_7}(x,y))v_2(x,1)dx \\ &= -\lambda_1 \int_0^1 v_1(5,y)dy - \lambda_2 \int_1^3 v_1(1,y)dy - \lambda_3 \int_0^1 v_2(x,3)dx \\ &- \lambda_4 \int_1^5 v_2(x,1)dx.\end{aligned}$$

And $\int_\Omega p \; div \; vdx = 0$ because $v \in V_1$.
Now we assume that (S_1) have a solution u. $\forall v \in V_1$

$$\begin{aligned}\sum_{i=1}^2 \int_\Omega \frac{\partial u_i}{\partial t} v_i d\mathbf{x} + \sum_{i=1}^2 \int_\Omega \nabla u_i \nabla v_i d\mathbf{x} &= \lambda_1 \int_0^1 v_1(5,y)dy + \lambda_2 \int_1^3 v_1(1,y)dy \\ &+ \lambda_3 \int_0^1 v_2(x,3)dx + \lambda_4 \int_1^5 v_2(x,1)dx.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial}{\partial t}(u,v) + ((u,v)) &= \langle \lambda_1 e_1, v\rangle_{\Gamma_6} + \langle \lambda_2 e_1, v\rangle_{\Gamma_4} + \langle \lambda_3 e_2, v\rangle_{\Gamma_3} + \langle \lambda_4 e_2, v\rangle_{\Gamma_5} \\ &= b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma).\end{aligned}$$

Put $\forall v \in V_1, \; \forall s \in [0,T]$,

$$\alpha(s) := (u(s), v), \; \beta(s) := ((u(s), v)), \; \gamma(s) := b(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma).$$

Thus

$$\alpha'(s) + \beta(s) - \gamma(s) = 0 \text{ in } \mathcal{D}'(0,T).$$

So

$$\int_0^t \alpha'(s)ds + \int_0^t \beta(s)ds - \int_0^t \gamma(s)ds = 0 \; \forall t \in (0,T).$$

As $\gamma(s)$ is independent of time t, we have

$$\alpha(t) - \alpha(0) + \int_0^t ((u(s),v))ds - tb(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma) = 0.$$

$$\alpha(t) - \alpha(0) = -\int_0^t ((u(s),v))ds + tb(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma).$$

Therefore

$$\alpha(t) - \alpha(0) = -((\int_0^t u(s)ds, v)) + tb(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma).$$

Hence $\forall t \in (0,T), \forall v \in V_1, (u(t)-u_0, v)+((U(t), v)) = tb(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v, \Gamma)$, where $U(t) := \int_0^t u(s)ds$. We have

$$\int_0^1 v_1(5,y)dy = \int_\Omega e_1 v(x,y)\chi_{\Gamma_6} dxdy,$$
$$\int_1^3 v_1(1,y)dy = \int_\Omega e_1 v(x,y)\chi_{\Gamma_4} dxdy,$$
$$\int_0^1 v_2(x,3)dx = \int_\Omega e_2 v(x,y)\chi_{\Gamma_3} dxdy,$$
$$\int_1^5 v_2(x,1)dx = \int_\Omega e_2 v(x,y)\chi_{\Gamma_5} dxdy,$$

where χ_{Γ_i} is the characteristic function of Γ_i, for $i = 3, \ldots, 6$. Thus

$$(u(t) - u_0, v) + (\nabla U(t), \nabla v) - t((\lambda_1\chi_{\Gamma_6} + \lambda_2\chi_{\Gamma_4})e_1 + (\lambda_3\chi_{\Gamma_3} + \lambda_4\chi_{\Gamma_5})e_2, v) = 0.$$

In particular for $v \in \boldsymbol{\vartheta}$, we have

$$(u(t) - u_0 - \triangle U(t) - t((\lambda_1\chi_{\Gamma_6} + \lambda_2\chi_{\Gamma_4})e_1 + (\lambda_3\chi_{\Gamma_3} + \lambda_4\chi_{\Gamma_5})e_2, v) = 0.$$

According to [3], there exists $Q(t) \in \mathcal{D}'(\Omega)$ such that

$$u(t) - u_0 - \triangle U(t) - t((\lambda_1\chi_{\Gamma_6} + \lambda_2\chi_{\Gamma_4})e_1 + (\lambda_3\chi_{\Gamma_3} + \lambda_4\chi_{\Gamma_5})e_2 = -\nabla Q(t) \in \mathcal{D}'(\Omega), \quad \forall t \in [0,T].$$

Thus

$$u(t) - u_0 - \triangle U(t) + \nabla Q(t) = t((\lambda_1\chi_{\Gamma_6} + \lambda_2\chi_{\Gamma_4})e_1 + (\lambda_3\chi_{\Gamma_3} + \lambda_4\chi_{\Gamma_5})e_2,$$

where $Q \in C([0,T]; L^2(\Omega))$.

Thus, $\forall \varphi \in \mathcal{D}(\Omega), \ \forall t \in [0,T]$

$$\begin{aligned}\int_\Omega (u(t) - u_0)\varphi(\mathbf{x})d\mathbf{x} \quad & - \int_\Omega U(t)\triangle\varphi(\mathbf{x})d\mathbf{x} - \int_\Omega Q(t) div\varphi(\mathbf{x})d\mathbf{x} \\ & = \int_\Omega t((\lambda_1\chi_{\Gamma_6} + \lambda_2\chi_{\Gamma_4})e_1 + (\lambda_3\chi_{\Gamma_3} + \lambda_4\chi_{\Gamma_5})e_2)\varphi(\mathbf{x})d\mathbf{x}.\end{aligned}$$

Consequently, $\forall \psi \in \mathcal{D}(0,T)$

$$\begin{aligned}& - \int_0^T \int_\Omega t((\lambda_1\chi_{\Gamma_6} + \lambda_2\chi_{\Gamma_4})e_1 + (\lambda_3\chi_{\Gamma_3} + \lambda_4\chi_{\Gamma_5})e_2)\varphi(\mathbf{x})\psi'(t)d\mathbf{x}dt \\ & = - \int_0^T \int_\Omega u(t)\varphi(\mathbf{x})\psi'(t)d\mathbf{x}dt + \int_0^T \int_\Omega u_0(\mathbf{x})\varphi(\mathbf{x})\psi'(t)d\mathbf{x}dt \\ & + \int_0^T \int_\Omega U(t)\triangle\varphi(\mathbf{x})\psi'(t)d\mathbf{x}dt + \int_0^T \int_\Omega Q(t)div\varphi(\mathbf{x})\psi'(t)d\mathbf{x}dt,\end{aligned}$$

and

$$\int_0^T \int_\Omega u_0(\mathbf{x})\varphi(\mathbf{x})\psi'(t)d\mathbf{x}dt = 0.$$

Let $\Omega_T = \Omega \times (0,T)$, we get

$$\begin{aligned}\int_0^T \int_\Omega U(t)\triangle\varphi(\mathbf{x})\psi'(t)d\mathbf{x}dt &= \int_\Omega \triangle\varphi(\mathbf{x})d\mathbf{x}\int_0^T U(t)\psi'(t)dt \\ &= -\int_\Omega \triangle\varphi(\mathbf{x})d\mathbf{x}\int_0^T u(t)\psi(t)dt \\ &= \int_{\Omega_T} u(t)\triangle\varphi(\mathbf{x})\psi(t)d\mathbf{x}dt.\end{aligned}$$

Thus

$$\begin{aligned}&-\int_0^T \int_\Omega t((\lambda_1\chi_{\Gamma_6}+\lambda_2\chi_{\Gamma_4})e_1+(\lambda_3\chi_{\Gamma_3}+\lambda_4\chi_{\Gamma_5})e_2)\varphi(\mathbf{x})\psi'(t)d\mathbf{x}dt \\ &= \int_{\Omega_T} ((\lambda_1\chi_{\Gamma_6}+\lambda_2\chi_{\Gamma_4})e_1+(\lambda_3\chi_{\Gamma_3}+\lambda_4\chi_{\Gamma_5})e_2)\varphi(\mathbf{x})\psi(t)d\mathbf{x}dt \\ &= 0,\end{aligned}$$

because $\varphi \in \mathcal{D}(\Omega)$.

$$\int_0^T \int_\Omega Q(t)div\varphi(\mathbf{x})\psi'(t)d\mathbf{x}dt = -\int_{\Omega_T} Q'(t)div\varphi(\mathbf{x})\psi(t)d\mathbf{x}dt.$$

Put $P_1(t) = -Q'(t) \in \mathcal{D}'(0,T;L^2(\Omega))$ and $\phi(t,\mathbf{x}) = \varphi(\mathbf{x})\psi(t)$, we conclude that

$$-\int_{\Omega_T} u(t,\mathbf{x})\frac{\partial\phi}{\partial t}(t,\mathbf{x})d\mathbf{x}dt - \int_{\Omega_T} u(t,\mathbf{x})\triangle_{\mathbf{x}}\phi(t,\mathbf{x})d\mathbf{x}dt + \int_{\Omega_T} P_1(t,\mathbf{x})div_{\mathbf{x}}\phi(t,\mathbf{x})d\mathbf{x}dt = 0.$$

Since

$$\int_{\Omega_T} P_1(t,\mathbf{x})div_{\mathbf{x}}\phi(t,\mathbf{x})dxdt = \int_{\partial\Omega\times(0,T)} P_1\phi(t,\mathbf{x})d\sigma - \int_{\Omega_T} \nabla P_1\phi(t,\mathbf{x})d\mathbf{x}dt$$

and

$$\int_{\partial\Omega\times(0,T)} P_1\phi(t,\mathbf{x})d\sigma = 0,$$

we obtain

$$-\int_{\Omega_T} u(t,\mathbf{x})\frac{\partial\phi}{\partial t}(t,\mathbf{x})d\mathbf{x}dt - \int_{\Omega_T} u(t,\mathbf{x})\triangle_{\mathbf{x}}\phi(t,\mathbf{x})d\mathbf{x}dt - \int_{\Omega_T} \nabla P_1\phi(t,\mathbf{x})d\mathbf{x}dt = 0.$$

We know that the set of all finite linear combinations of the functions $\phi(t,\mathbf{x}) = \varphi(\mathbf{x})\psi(t)$ ($\varphi \in \mathcal{D}(\Omega), \psi \in \mathcal{D}(0,T)$) is dense in $\mathcal{D}(\Omega_T)$. Thus

$$\frac{\partial u}{\partial t}(t,\mathbf{x}) - \triangle u(t,\mathbf{x}) + \nabla P = 0,$$

where $P = -P_1$.
From the first two equations of the problem (S_1) and (S_2), we find

$$\begin{aligned}\sum_{i=1}^{2}\int_{\partial\Omega}\frac{\partial u_i}{\partial x}.v_i d\sigma - \int_{\partial\Omega} Pv.\vec{\eta}d\sigma &= \lambda_1\int_0^1 v_1(5,y)dy + \lambda_2\int_1^3 v_1(1,y)dy \\ &+ \lambda_3\int_0^1 v_2(x,3)dx + \lambda_4\int_1^5 v_2(x,1)dx.\end{aligned}$$

Thus

$$\langle\frac{\partial u}{\partial x} - P\vec{\eta}, v\rangle = \langle\lambda_1 e_1, v\rangle_{\Gamma_1} + \langle\lambda_2 e_1, v\rangle_{\Gamma_2} + \langle\lambda_3 e_2, v\rangle_{\Gamma_3} + \langle\lambda_4 e_2, v\rangle_{\Gamma_4},$$

and

$$\begin{aligned}&\langle\frac{-\partial u}{\partial x} + Pe_1, v\rangle_{\Gamma_1} + \langle\frac{-\partial u}{\partial x} + Pe_1, v\rangle_{\Gamma_2} + \langle\frac{\partial u}{\partial y} - Pe_2, v\rangle_{\Gamma_3} + \langle\frac{\partial u}{\partial x} - Pe_1, v\rangle_{\Gamma_4} \\ &+\langle\frac{\partial u}{\partial y} - Pe_2, v\rangle_{\Gamma_5} + \langle\frac{\partial u}{\partial x} - Pe_1, v\rangle_{\Gamma_6} + \langle\frac{-\partial u}{\partial y} + Pe_2, v\rangle_{\Gamma_7} + \langle\frac{-\partial u}{\partial y} + Pe_2, v\rangle_{\Gamma_8} \\ &= \langle\lambda_1 e_1, v\rangle_{\Gamma_1} + \langle\lambda_2 e_1, v\rangle_{\Gamma_2} + \langle\lambda_3 e_2, v\rangle_{\Gamma_3} + \langle\lambda_4 e_2, v\rangle_{\Gamma_4}.\end{aligned} \tag{4.1}$$

We have $u|_{\Gamma_1} = u|_{\Gamma_6}$, thus $u_1|_{\Gamma_1} = u_1|_{\Gamma_6}$ and $u_2|_{\Gamma_1} = u_2|_{\Gamma_6}$, as $u_i(0,y) = u_i(5,y)$ for all $y \in [0,1]$, we have $\frac{\partial u_i}{\partial y}(0,y) = \frac{\partial u_i}{\partial y}(5,y)$. Moreover we know that $div\, u = 0$ in Ω and $u \in C^1(\bar{\Omega})^2$, we conclude that $\frac{\partial u_1}{\partial x}(0,y) = -\frac{\partial u_2}{\partial y}(0,y) = -\frac{\partial u_2}{\partial y}(5,y)$ for all $y \in [0,1]$. Thus $\frac{\partial u_1}{\partial x}(0,y) = \frac{\partial u_1}{\partial x}(5,y)$. We consider the space

$$H_{11}^{\frac{1}{2}}(\Gamma_1) = \{\varphi \in L^2(\Gamma_1); \exists v \in H^1(\Omega),\ v|_{\Gamma_3} = v|_{\Gamma_8},\ v|_{\Gamma_5} = v|_{\Gamma_7},\ v|_{\Gamma_2} = v|_{\Gamma_4},\ v|_{\Gamma_1\cup\Gamma_6} = \varphi\}$$

Let $\mu \in H_{11}^{\frac{1}{2}}(\Gamma_1)$, we put $\nu = (0,\mu_1)^t$ where $\mu_1 = \begin{cases}\mu \text{ on } \Gamma_1\cup\Gamma_6 \\ 0 \text{ on } \Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5\cup\Gamma_7\cup\Gamma_8.\end{cases}$
It is clear that $\nu \in (H^{\frac{1}{2}}(\Gamma))^2$ and $\int_{\partial\Omega}\nu.\vec{\eta}d\sigma = 0$, so there exists $v \in (H^1(\Omega))^2$ such that $div\, v = 0$ in Ω and $v = \nu$ on Γ (see [1,5]), therefore $v \in V_1$. According to (4.1), we have for all $\mu \in H_{11}^{\frac{1}{2}}(\Gamma_1)$

$$\int_0^1 \frac{\partial u_2}{\partial x}(0,y)\mu dy = \int_0^1 \frac{\partial u_2}{\partial x}(5,y)\mu dy,$$

thus

$$\frac{\partial u_2}{\partial x}(0,y) = \frac{\partial u_2}{\partial x}(5,y).$$

Similarly, we have

$$\frac{\partial u}{\partial x}|_{\Gamma_2} = \frac{\partial u}{\partial x}|_{\Gamma_4},\ \frac{\partial u}{\partial y}|_{\Gamma_3} = \frac{\partial u}{\partial y}|_{\Gamma_8},\ \frac{\partial u}{\partial y}|_{\Gamma_5} = \frac{\partial u}{\partial y}|_{\Gamma_7}.$$

According to (4.1), we have

$$\langle Pe_1, v\rangle_{\Gamma_1} + \langle Pe_1, v\rangle_{\Gamma_2} + \langle -Pe_2, v\rangle_{\Gamma_3} + \langle -Pe_1, v\rangle_{\Gamma_4} + \langle -Pe_2, v\rangle_{\Gamma_5} + \langle -Pe_1, v\rangle_{\Gamma_6}$$

$$+\langle Pe_2, v\rangle_{\Gamma_7} + \langle Pe_2, v\rangle_{\Gamma_8} = \langle \lambda_1 e_1, v\rangle_{\Gamma_1} + \langle \lambda_2 e_1, v\rangle_{\Gamma_2} + \langle \lambda_3 e_2, v\rangle_{\Gamma_3} + \langle \lambda_4 e_2, v\rangle_{\Gamma_4}. \quad (4.2)$$

On the other hand, let $\varepsilon_1 = \begin{cases} \varepsilon & \text{on } \Gamma_1 \cup \Gamma_6, \\ 0 & \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_7 \cup \Gamma_8, \end{cases}$ where $\varepsilon \in H_{11}^{\frac{1}{2}}(\Gamma_1)$ and $\tau = (\varepsilon_1, 0)^t$. We have $\tau \in (H^{\frac{1}{2}}(\Gamma))^2$ and $\int_\Gamma \tau.\vec{\eta} d\sigma = 0$. So, there exists $v \in (H^1(\Omega))^2$ such that $div\, v = 0$ in Ω and $v = \tau$ on Γ (see [1,5]). In particular $v \in V_1$. According to (4.2),

$$\int_0^1 (p(0,y) - p(5,y))\varepsilon = \int_0^1 \lambda_1 \varepsilon$$

we conclude that $P|_{\Gamma_1} - P|_{\Gamma_6} = \lambda_1$. In the same way we show the other relations of pressures. □

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