



Pricing American bond options using a cubic spline collocation method

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ABSTRACT: In this paper, American options on a discount bond are priced under the Cox-Ingersoll-Ross (CIR) model. The linear complementarity problem of the option value is solved numerically by a penalty method. The problem is transformed into a nonlinear partial differential equation (PDE) by adding a power penalty term. The solution of the penalized problem converges to the one of the original problem. To numerically solve this nonlinear PDE, we use the horizontal method of lines to discretize the temporal variable and the spatial variable by means of trapezoidal method and a cubic spline collocation method, respectively. We show that this full discretization scheme is second order convergent, and hence the convergence of the numerical solution to the viscosity solution of the continuous problem is guaranteed. Numerical results are presented and compared with other collocation methods given in the literature.

Key Words: Cox Ingersoll Ross *CIR*; American put; Trapezoidal method; Spline collocation.

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1. Introduction

Although the concept of interest rates seems to be something natural that everybody knows to deal with, the management of interest rate risk, i.e. the control of changes in future cash flows due to fluctuations in interest rates is an issue of great complexity. In particular, the pricing and hedging of products depending in large part on interest rates create the necessity for mathematical models.

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Here we adopt the *CIR* model developed by Cox, Ingersoll and Ross in 1985 [8], is one of the most widely used term structure model and has several favorable features. Various mathematical results about *CIR* were known since the work presented in [13]. *CIR* model is defined as the unique strong solution of the equation

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dW,$$

where dW is the increment of a Wiener process, θ is the long-term level of the short rate, $\kappa > 0$ stands for the reversion speed, and $\sigma^2 r$ is the variance with constant $\sigma > 0$. In practice, r is positive, which enforces the constraint $0 < \sigma^2 < 2\kappa\theta$ (Fellers condition [8]). In [8], it has been shown that the price $P(r, t, s)$ of a pure discount bond with face value of one dollar at its maturity date s is given as follows

$$P(r, t, s) = A(t, s)e^{-B(t, s)r},$$

where

$$A(t, s) = \left[\frac{\phi_1 e^{\phi_2(s-t)}}{\phi_2 [e^{\phi_1(s-t)} - 1] + \phi_1} \right]^{\phi_3}, \quad B(t, s) = \frac{e^{\phi_1(s-t)} - 1}{\phi_2 [e^{\phi_1(s-t)} - 1] + \phi_1},$$

$$\phi_1 = \sqrt{\mu^2 + 2\sigma^2}, \quad \phi_2 = \frac{\mu + \phi_1}{2}, \quad \phi_3 = \frac{2\bar{\theta}}{\sigma^2}, \quad \bar{\theta} = \kappa\theta, \mu = \kappa + \zeta,$$

and ζ is the market risk premium.

We concentrate on the numerical solution of the *CIR* model which can be formulated as parabolic partial differential complementarity (PDC) problem with suitable boundary and terminal condition (see [1,17,22]). This complementarity problem is, in general, not analytically solvable. Hence, several approximation techniques have been developed for the solution of American bond option pricing problem. Numerical solution by penalty methods have been considered, e.g. by [1,2,9,12]. In this paper we develop a novel numerical method for solving a PDC problem by using the cubic spline collocation method and the generalized Newton method. First, the PDC problem is approximated by a sequence of nonlinear equation problems by using the penalty method given in [23,27]. It is shown that the penalized equation converges to the one of the PDC problem with an arbitrary order. This arbitrary order convergence rate allows one to achieve the required accuracy of the solution with a small penalty parameter. A numerical scheme for solving the penalized nonlinear PDE is also proposed. Then we apply the spline collocation method to approximate the solution of a boundary value problem of second order. The discret problem is formulated as to find the cubic spline coefficients of a nonsmooth system $\varphi(Y) = Y$, where $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m$. In order to solve the nonsmooth equation we apply the generalized Newton method (see [4,5,20], for instance). We prove that the cubic spline collocation method converges quadratically provided that a property coupling the penalty parameter λ and the discretization parameter h is satisfied.

Numerical methods to approximate the solution of boundary value problems have been considered by several authors. We only mention the paper [3] and references therein, which use the spline collocation method for solving the boundary value problems.

The organization of this paper is as follows. In Section 2, we present the penalty method to approximate the PDC problem by a sequence of second order boundary value problems. In Section 3 we construct a cubic spline to approximate the solution of the boundary problem. Section 4 is devoted to the presentation of the generalized Newton method. In Section 5 we show the convergence of the cubic spline to the solution of the boundary problem and provide an error estimate. Finally, some numerical results are given in Section 6 to validate our methodology and compare our method with [25].

2. Penalty problem

Now, let $u(r, t)$ be the value of an American put option on a zero-coupon bond with striking price K , where the holder can receive a given payoff $\Lambda(r, t)$ at the expiry date T . Introducing a time-reverse transformation $\tau = T - t$, the option pricing problem can be formulated as the following parabolic partial differential complementarity problem ([24]):

$$\begin{cases} Lu(r, \tau) \geq 0 \\ u(r, \tau) - \Lambda(r, \tau) \geq 0 \\ Lu(r, \tau) \cdot (u(r, \tau) - \Lambda(r, \tau)) = 0 \end{cases} \quad (2.1)$$

almost everywhere in $(0, +\infty) \times (0, T)$, where

$$L = \frac{\partial}{\partial \tau} - \left[\frac{1}{2} \sigma^2 r \frac{\partial^2}{\partial r^2} + (\bar{\theta} - \mu r) \frac{\partial}{\partial r} - rI \right]$$

is a degenerate parabolic partial differential operator. The initial condition is

$$u(r, 0) = \Lambda(r, 0) = \max[K - P(r, T, s), 0] \quad \text{for a put,}$$

and the boundary conditions are

$$\begin{aligned} u(0, \tau) &= \Lambda(0, \tau), \quad r \rightarrow 0, \\ u(r, \tau) &= \Lambda(r, \tau), \quad r \rightarrow \infty, \end{aligned} \quad (2.2)$$

For computational purposes, it is necessary to restrict r to a finite region $I = [0, R]$, where R denotes a sufficiently large number to ensure the accuracy of the solution, see [24]. Thus, relation (2.2) becomes

$$u(R, \tau) = \Lambda(R, \tau)$$

Remark 2.1. *It is worth noting that $T < s$ and $K < P(0, T, s) = A(T, s)$ for a call option or $K > A(T, s)$ for a put option, since otherwise the option would never be exercised and would be worthless.*

Let λ be a real positive number. The penalty problem is given by the following boundary value problem (see [26]):

$$\begin{cases} Lu_\lambda(r, t) = F(t, u_\lambda(r, t)), & \forall (r, t) \in \Omega, \\ u_\lambda(r, 0) = \psi_0(r), & \forall r \in \overline{\Omega_r}, \\ u_\lambda(0, t) = \psi_1(t), & \forall t \in \overline{\Omega_t}, \\ u_\lambda(R, t) = \psi_2(t), & \forall t \in \overline{\Omega_t}, \end{cases} \quad (2.3)$$

where $\Omega = \Omega_r \times \Omega_t = (0, R) \times (0, T)$ and $L : C(\overline{\Omega}) \cap C^{2,1}(\Omega) \rightarrow C(\Omega)$ is the Black-Scholes operator defined as

$$L = \frac{\partial}{\partial t} - \alpha(r) \frac{\partial^2}{\partial r^2} - \beta(r) \frac{\partial}{\partial r} - \gamma(r)I.$$

and

$$\begin{aligned} F(t, u_\lambda(r, t)) &= \lambda[\Lambda(r, t) - u_\lambda(r, t)]_+, \\ \alpha(r) &= \frac{1}{2}\sigma^2 r, \quad \beta(r) = \bar{\theta} - \mu r, \quad \gamma(r) = -r, \\ \psi_0(r) &= \Lambda(r, 0), \quad \psi_1(t) = \Lambda(0, t), \quad \psi_2(t) = \Lambda(R, t). \end{aligned}$$

with $\alpha(r) \geq \tilde{\alpha} > 0$, $\gamma(r) \leq \tilde{\gamma} < 0$ on $\overline{\Omega}$ and $\alpha, \beta, \gamma, \psi_0, \psi_1, \psi_2$, are sufficiently smooth functions.

Here we assume that the problem satisfies sufficient regularity and compatibility conditions which guarantee that the problem has a unique solution $u \in C(\overline{\Omega}) \cap C^{2,1}(\Omega)$ satisfying (see, [15,16,18]):

$$\left| \frac{\partial^{i+j} u_\lambda(r, t)}{\partial r^i \partial t^j} \right| \leq k \text{ on } \overline{\Omega}; \quad 0 \leq j \leq 3 \text{ and } 0 \leq i + j \leq 4, \quad (2.4)$$

where k is a constant. The following theorem proves the order of convergence of the solution u_λ to u .

Theorem 2.2. ([14]) *Let u and u_λ be solutions to problems (2.1) and (2.3) respectively. Then, there exists a constant $C > 0$, independent of u, u_λ and λ , such that*

$$\|u - u_\lambda\|_2 \leq \frac{C}{\lambda^{1/2}}$$

where λ is the penalty parameter used in (2.3)

Lemma 2.3. *The function F is a nonlinear continuous and λ -Lipschitz on u_λ .*

Proof: F is Lipschitz, indeed For any two function u_λ and v_λ , we have:

$$\begin{aligned}
 |F(t, u_\lambda) - F(t, v_\lambda)| &= \lambda |[\Lambda - u_\lambda]_+ - [\Lambda - v_\lambda]_+| \\
 &= \lambda \left| \frac{(\Lambda - u_\lambda) + |\Lambda - u_\lambda| - (\Lambda - v_\lambda) - |\Lambda - v_\lambda|}{2} \right| \\
 &\leq \lambda \left| \frac{v_\lambda - u_\lambda}{2} + \frac{|\Lambda - u_\lambda| - |\Lambda - v_\lambda|}{2} \right| \\
 &\leq \lambda \left(\left| \frac{v_\lambda - u_\lambda}{2} \right| + \left| \frac{|\Lambda - u_\lambda| - |\Lambda - v_\lambda|}{2} \right| \right) \\
 &\leq \lambda \left(\left| \frac{v_\lambda - u_\lambda}{2} \right| + \left| \frac{v_\lambda - u_\lambda}{2} \right| \right) \\
 &\leq \lambda |v_\lambda - u_\lambda|.
 \end{aligned}$$

Then F is a nonlinear continuous function on u_λ and satisfies the following Lipschitz condition:

$$|F(t, u_\lambda(r, t)) - F(t, v_\lambda(r, t))| \leq \lambda |u_\lambda(r, t) - v_\lambda(r, t)|, \quad \forall (r, t) \in \Omega.$$

□

In the sequel of this paper we will consider the model (2.3) for numerical treatment.

3. Time discretization and description of the Trapezoidal method

We denote by $\|\cdot\|_\infty$ the uniform norm.

The observation period has to be specified first and is set from today (time point $t = 0$) to time point T . This period will be divided into M equally spaced time intervals with length $\Delta t = \frac{T}{M}$, $t_m = m\Delta t$. The price of the underlying (either the project value or the value of a twin security with the same risk profile) is assumed to stay in a range Ω_r . We discretize the variable time in (2.3) by means of Trapezoidal method. Then the semi-discretization yields the following system of equations:

$$\frac{u_\lambda^{m+1} - u_\lambda^m}{\Delta t} - \frac{1}{2}L_r(u_\lambda^{m+1} + u_\lambda^m) = \frac{1}{2}[F(t_m, u_\lambda^m) + F(t_{m+1}, u_\lambda^{m+1})],$$

That is

$$(1 - \frac{\Delta t}{2}L_r)u_\lambda^{m+1} = (1 + \frac{\Delta t}{2}L_r)u_\lambda^m + \frac{\Delta t}{2}[F(t_m, u_\lambda^m) + F(t_{m+1}, u_\lambda^{m+1})], \quad (3.1)$$

where, $L_r : C(\overline{\Omega}_r) \cap C^2(\Omega_r) \rightarrow C(\Omega_r)$ is the differential operator defined by

$$L_r = \alpha(r)\frac{\partial^2}{\partial r^2} + \beta(r)\frac{\partial}{\partial r} + \gamma(r)I,$$

and u_λ^m approximate the exacte solution $u_\lambda(r, t)$ at the time level $t_m = m\Delta t$. Then, the approximate problem of (2.3) is

$$\left\{ \begin{array}{l} p(r) \frac{\partial^2 u_\lambda^{m+1}}{\partial r^2} + q(r) \frac{\partial u_\lambda^{m+1}}{\partial r} + l(r) u_\lambda^{m+1} = J_\lambda^m(\cdot, u_\lambda^{m+1}), \quad \forall r \in \Omega_r, \\ u_\lambda^0 = \psi_0(r), \quad \forall r \in \Omega_r, \\ u_\lambda^{m+1}(0) = \psi_1^{m+1}, \quad 0 \leq m < M, \\ u_\lambda^{m+1}(R) = \psi_2^{m+1}, \quad 0 \leq m < M. \end{array} \right. \quad (3.2)$$

where, for any $m \geq 0$ and for any $r \in \Omega_r$, we have

$$\begin{aligned} J_\lambda^m(\cdot, u_\lambda^{m+1}) &= -(1 + \frac{\Delta t}{2} L_r) u_\lambda^m - \frac{\Delta t}{2} [F(t_m, u_\lambda^m) + F(t_{m+1}, u_\lambda^{m+1})], \\ p(r) &= \frac{\Delta t}{2} \alpha(r), \\ q(r) &= \frac{\Delta t}{2} \beta(r), \\ l(r) &= \frac{\Delta t}{2} \gamma(r) - 1, \\ \psi_0(r) &= \Lambda^0(r) = \Lambda(r, 0), \\ \psi_1^{m+1} &= \Lambda^{m+1}(0) = \Lambda(0, t_{m+1}) \\ \psi_2^{m+1} &= \Lambda^{m+1}(R) = \Lambda(R, t_{m+1}), \end{aligned}$$

u_λ^{m+1} is solution of (3.2), at the $(m + 1)$ th-time level.

The following theorem proves the order of convergence of the solution u_λ^m to $u_\lambda(r, t)$

Theorem 3.1. *problem (3.2) is second order convergent i.e.*

$$\|u_\lambda(r, t_m) - u_\lambda^m\|_\infty \leq C(\Delta t)^2.$$

Proof: We introduce the notation $e_m = u_\lambda(r, t_m) - u_\lambda^m$ the error at step m . By Taylor series expansion of u_λ , we have

$$u_\lambda(r, t_{m+1}) = u_\lambda(r, t_{m+\frac{1}{2}}) + \frac{\Delta t}{2} \frac{\partial u_\lambda}{\partial t}(r, t_{m+\frac{1}{2}}) + \frac{(\Delta t)^2}{8} \frac{\partial^2 u_\lambda}{\partial t^2}(r, t_{m+\frac{1}{2}}) + O((\Delta t)^3),$$

$$u_\lambda(r, t_m) = u_\lambda(r, t_{m+\frac{1}{2}}) - \frac{\Delta t}{2} \frac{\partial u_\lambda}{\partial t}(r, t_{m+\frac{1}{2}}) + \frac{(\Delta t)^2}{8} \frac{\partial^2 u_\lambda}{\partial t^2}(r, t_{m+\frac{1}{2}}) + O((\Delta t)^3).$$

By using these expansions, we get

$$\frac{u_\lambda(r, t_{m+1}) - u_\lambda(r, t_m)}{\Delta t} = \frac{\partial u_\lambda}{\partial t}(r, t_{m+\frac{1}{2}}) + O((\Delta t)^2), \quad (3.3)$$

and by Taylor series expansion of $\frac{\partial u_\lambda}{\partial t}$, we have

$$\frac{\partial u_\lambda}{\partial t}(r, t_{m+1}) = \frac{\partial u_\lambda}{\partial t}(r, t_{m+\frac{1}{2}}) + \frac{\Delta t}{2} \frac{\partial^2 u_\lambda}{\partial t^2}(r, t_{m+\frac{1}{2}}) + \frac{(\Delta t)^2}{8} \frac{\partial^3 u_\lambda}{\partial t^3}(r, t_{m+\frac{1}{2}}) + O((\Delta t)^3),$$

$$\frac{\partial u_\lambda}{\partial t}(r, t_m) = \frac{\partial u_\lambda}{\partial t}(r, t_{m+\frac{1}{2}}) - \frac{\Delta t}{2} \frac{\partial^2 u_\lambda}{\partial t^2}(r, t_{m+\frac{1}{2}}) + \frac{(\Delta t)^2}{8} \frac{\partial^3 u_\lambda}{\partial t^3}(r, t_{m+\frac{1}{2}}) + O((\Delta t)^3).$$

By using these expansions, and $\left| \frac{\partial^3 u_\lambda}{\partial t^3} \right| \leq c$ on $\bar{\Omega}$ (see relation (2.4)), we have

$$\frac{1}{2} \frac{\partial u_\lambda}{\partial t} [u_\lambda(r, t_{m+1}) + u_\lambda(r, t_m)] = \frac{\partial u_\lambda}{\partial t} u_\lambda(r, t_{m+\frac{1}{2}}) + O((\Delta t)^2).$$

This implies

$$\begin{aligned} \frac{\partial u_\lambda}{\partial t} u_\lambda(r, t_{m+\frac{1}{2}}) &= \frac{1}{2} \frac{\partial u_\lambda}{\partial t} [u_\lambda(r, t_{m+1}) + u_\lambda(r, t_m)] + O((\Delta t)^2) \\ &= \frac{1}{2} [L_r u_\lambda(r, t_{m+1}) + F(t_{m+1}, u_\lambda^{m+1}) + L_r u_\lambda(r, t_m) + F(t_m, u_\lambda^m)] \\ &\quad + O((\Delta t)^2). \end{aligned}$$

By using this relation in (3.3) we get

$$\begin{aligned} \left(1 - \frac{\Delta t}{2} L_r\right) u_\lambda(r, t_{m+1}) &= \left(1 + \frac{\Delta t}{2} L_r\right) u_\lambda(r, t_m) \\ &\quad + \frac{\Delta t}{2} [F(t_m, u_\lambda(r, t_m)) + F(t_{m+1}, u_\lambda(r, t_{m+1}))] \\ &\quad + O((\Delta t)^3), \end{aligned}$$

by (3.1). Then, we obtain

$$\begin{aligned} \left(1 - \frac{\Delta t}{2} L_r\right) e_{m+1} &= \left(1 + \frac{\Delta t}{2} L_r\right) e_m + \frac{\Delta t}{2} [F(t_m, u_\lambda(r, t_m)) - F(t_m, u_\lambda^m)] \\ &\quad + \frac{\Delta t}{2} [F(t_{m+1}, u_\lambda^{m+1}) - F(t_{m+1}, u_\lambda(r, t_{m+1}))] + O((\Delta t)^3). \end{aligned}$$

For an analytic function F we may bound the term $O((\Delta t)^3)$ by $c(\Delta t)^3$ for some $c > 0$, and this upper bound is valid uniformly throughout $[0, T]$. Therefore, it follows from the Lipschitz condition of Lemma 2.3 and the triangle inequality that

$$\left(1 - \frac{\Delta t}{2} L_r\right) \|e_{m+1}\|_\infty \leq \left(1 + \frac{\Delta t}{2} L_r\right) \|e_m\|_\infty + \frac{\Delta t}{2} \lambda (\|e_m\|_\infty + \|e_{m+1}\|_\infty) + c(\Delta t)^3.$$

Clearly, the operator $\left(1 \pm \frac{\Delta t}{2} L_r\right)$ satisfies a maximum principle (see, [6,7]) and consequently

$$\left\| \left(1 \pm \frac{\Delta t}{2} L_r\right)^{-1} \right\|_\infty \leq \left(\frac{1}{1 + \frac{\Delta t}{2} \gamma} \right).$$

Since we are ultimately interested in letting $\Delta t \rightarrow 0$, there is no harm in assuming that $\Delta t.\eta < 2$, with $\eta = \|L_r\|_\infty + \lambda$. We can thus deduce that

$$\|e_{m+1}\|_\infty \leq \left(\frac{1 + \frac{1}{2}\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right) \|e_m\|_\infty + \left(\frac{c}{1 - \frac{1}{2}\Delta t.\eta} \right) (\Delta t)^3. \tag{3.4}$$

We now claim that

$$\|e_m\|_\infty \leq \frac{c}{\eta} \left[\left(\frac{1 + \frac{1}{2}\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right)^m - 1 \right] (\Delta t)^2. \tag{3.5}$$

The proof is by induction on m . When $m = 0$ we need to prove that $\|e_0\| \leq 0$ and hence that $e_0 = 0$. This is certainly true, since at $t_0 = 0$ the numerical solution matches the initial condition and the error is zero.

For general $m \geq 0$, we assume that (3.5) is true up to m and use (3.4) to argue that

$$\begin{aligned} \|e_{m+1}\|_\infty &\leq \frac{c}{\eta} \left(\frac{1 + \frac{1}{2}\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right) \left[\left(\frac{1 + \frac{1}{2}\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right)^m - 1 \right] (\Delta t)^2 + \left(\frac{c}{1 - \frac{1}{2}\Delta t.\eta} \right) (\Delta t)^3 \\ &\leq \frac{c}{\eta} \left[\left(\frac{1 + \frac{1}{2}\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right)^{m+1} - 1 \right] (\Delta t)^2. \end{aligned}$$

This advances the inductive argument from m to $m + 1$ and proves that (3.5) is true. Since $0 < \Delta t.\eta < 2$, it is true that

$$\left(\frac{1 + \frac{1}{2}\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right) = 1 + \left(\frac{\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right) \leq \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right)^l = \exp \left(\frac{\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right).$$

Consequently, relation (3.5) yields

$$\|e_m\|_\infty \leq \frac{c(\Delta t)^2}{\eta} \left(\frac{1 + \frac{1}{2}\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right)^m \leq \frac{c(\Delta t)^2}{\eta} \exp \left(\frac{m\Delta t.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right).$$

This bound is true for every nonnegative integer m such that $m\Delta t < T$. Therefore

$$\|e_m\|_\infty \leq \frac{c(\Delta t)^2}{\eta} \exp \left(\frac{T.\eta}{1 - \frac{1}{2}\Delta t.\eta} \right).$$

We deduce that

$$\|u_\lambda(r, t_m) - u_\lambda^m\|_\infty \leq C(\Delta t)^2.$$

In other words, problem (3.2) is second order convergent. □

For any $m \geq 0$, problem (3.2) has a unique solution and can be written on the following form:

$$\begin{cases} p(r)u_\lambda''(r) + q(r)u_\lambda'(r) + l(r)u_\lambda(r) &= J_\lambda(\cdot, u_\lambda(r)), \quad \forall r \in \Omega_r, \\ u_\lambda(0) &= \psi_1, \\ u_\lambda(R) &= \psi_2, \end{cases} \quad (3.6)$$

where $J_\lambda(\cdot, u_\lambda(r)) = \Phi(r) - \frac{\Delta t}{2}\lambda[\psi(r) - u_\lambda(r)]_+$, with Φ is a known function. It is easy to see that J_λ is a nonlinear continuous function on u_λ and satisfies the following Lipschitz condition:

$$|J_\lambda(r, u_\lambda(r)) - J_\lambda(r, v_\lambda(r))| \leq L_\lambda |u_\lambda(r) - v_\lambda(r)|, \quad \forall r \in \Omega_r, \quad (3.7)$$

where $L_\lambda = \frac{\Delta t}{2}\lambda$.

In the sequel of this paper, we will focus on the solution of problem (3.6).

4. Spatial discretization and cubic spline collocation method

In this section we construct a cubic spline which approximates the solution u_λ of problem (3.6), in the interval $\Omega_r \subset \mathbb{R}$.

We denote by $\|\cdot\|$ the Euclidean norm on \mathbb{R}^{n+1} and $S^{(k)}$ the k^{th} derivative of a function S .

Let $\Theta = \{0 = r_{-3} = r_{-2} = r_{-1} = r_0 < r_1 < \dots < r_{n-1} < r_n = r_{n+1} = r_{n+2} = r_{n+3} = R\}$ be a subdivision of the interval Ω_r . Without loss of generality, we put $r_i = a + ih$, where $0 \leq i \leq n$ and $h = \frac{R}{n}$. Denote by $\mathbb{S}_4(\Omega_r, \Theta) := \mathbb{P}_3^2(\Omega_r, \Theta)$ the space of piecewise polynomials of degree less than or equal to 3 over the subdivision Θ and of class C^2 everywhere on Ω_r . Let B_i , $i = -3, \dots, n-1$, be the B-splines of degree 3 associated with Θ . These B-splines are positives and form a basis of the space $\mathbb{S}_4(\Omega_r, \Theta)$.

Consider the local linear operator Q_3 which maps the function u_λ onto a cubic spline space $\mathbb{S}_4(\Omega_r, \Theta)$ and which has an optimal approximation order. This operator is the discrete C^2 cubic quasi-interpolant (see [19]) defined by

$$Q_3 u_\lambda = \sum_{i=-3}^{n-1} \mu_i(u_\lambda) B_i,$$

where the coefficients $\mu_j(u_\lambda)$ are determined by solving a linear system of equations given by the exactness of Q_3 on the space of cubic polynomial functions $\mathbb{P}_3(\Omega_r)$. Precisely, these coefficients are defined as follows:

$$\begin{cases} \mu_{-3}(u_\lambda) = u_\lambda(r_0) = u_\lambda(0), \\ \mu_{-2}(u_\lambda) = \frac{1}{18}(7u_\lambda(r_0) + 18u_\lambda(r_1) - 9u_\lambda(r_2) + 2u_\lambda(r_3)), \\ \mu_j(u_\lambda) = \frac{1}{6}(-u_\lambda(r_{j+1}) + 8u_\lambda(r_{j+2}) - u_\lambda(r_{j+3})), \quad \text{for } j = -1, \dots, n-3, \\ \mu_{n-2}(u_\lambda) = \frac{1}{18}(2u_\lambda(r_{n-3}) - 9u_\lambda(r_{n-2}) + 18u_\lambda(r_{n-1}) + 7u_\lambda(r_n)), \\ \mu_{n-1}(u_\lambda) = u_\lambda(r_n) = u_\lambda(R). \end{cases}$$

It is well known (see e.g. [11], chapter 5) that there exists constants C_k , $k = 0, 1, 2, 3$, such that, for any function $u_\lambda \in C^4(\Omega_r)$,

$$\|u_\lambda^{(k)} - Q_3 u_\lambda^{(k)}\|_{\Omega_r} \leq C_k h^{4-k} \|u_\lambda^{(4-k)}\|_{\Omega_r}, \quad k = 0, 1, 2, 3, \quad (4.1)$$

where $\|u_\lambda\|_{\Omega_r} = \max_{r \in \Omega_r} |u_\lambda(r)|$.

By using the boundary conditions of problem (3.6), we obtain $\mu_{-3}(u_\lambda) = Q_3 u_\lambda(0) = u_\lambda(0) = \psi_1$ and $\mu_{n-1}(u_\lambda) = Q_3 u_\lambda(R) = u_\lambda(R) = \psi_2$. Hence

$$Q_3 u_\lambda = z_1 + S_1,$$

where

$$z_1 = \psi_1 B_{-3} + \psi_2 B_{n-1} \quad \text{and} \quad S_1 = \sum_{i=-2}^{n-2} \mu_i(u_\lambda) B_i.$$

From equation: (4.1), we can easily see that the spline S_1 satisfies the following equation

$$p(r_i) S_1^{(2)}(r_i) + q(r_i) S_1^{(1)}(r_i) + l(r_i) S_1^{(0)}(r_i) = g(r_i) + O(h^2), \quad i = 0, \dots, n \quad (4.2)$$

with

$$g(r_i) = J_\lambda(r_i, u_\lambda(r_i)) - (p(r_i) z_1^{(2)}(r_i) + q(r_i) z_1^{(1)}(r_i) + l(r_i) z_1^{(0)}(r_i)), \quad i = 0, \dots, n.$$

The goal of this section is to compute a cubic spline collocation $\tilde{S}_\lambda = \sum_{i=-3}^{n-1} \tilde{c}_{i,\lambda} B_i$

which satisfies the equation (3.6) at the points τ_i , $i = 0, \dots, n+2$ with $\tau_0 = r_0$, $\tau_i = \frac{r_{i-1} + r_i}{2}$, $i = 1, \dots, n$, $\tau_{n+1} = r_{n-1}$ and $\tau_{n+2} = r_n$.

Then, it is easy to see that

$$\tilde{c}_{-3,\lambda} = \psi_1 \quad \text{and} \quad \tilde{c}_{n-1,\lambda} = \psi_2.$$

Hence

$$\tilde{S}_\lambda = z_1 + \tilde{S}_1, \quad \text{where} \quad \tilde{S}_1 = \sum_{i=-2}^{n-2} \tilde{c}_{i,\lambda} B_i,$$

and the coefficients $\tilde{c}_{i,\lambda}$, $i = -2, \dots, n-2$ satisfy the following collocation conditions:

$$p(\tau_i) \tilde{S}_1^{(2)}(\tau_i) + q(\tau_i) \tilde{S}_1^{(1)}(\tau_i) + l(\tau_i) \tilde{S}_1^{(0)}(\tau_i) = f(\tau_i), \quad i = 1, \dots, n+1, \quad (4.3)$$

where

$$f(\tau_i) = J_\lambda(\tau_i, \tilde{S}_1(\tau_i)) - (p(\tau_i) z_1^{(2)}(\tau_i) + q(\tau_i) z_1^{(1)}(\tau_i) + l(\tau_i) z_1^{(0)}(\tau_i)), \quad i = 1, \dots, n+1.$$

Taking $C_\lambda = [\mu_{-2}, \dots, \mu_{n-2}]^T$ and $\tilde{C}_\lambda = [\tilde{c}_{-2,\lambda}, \dots, \tilde{c}_{n-2,\lambda}]^T$, and using equations (4.2) and (4.3), we get:

$$(PA_h^{(2)} + QA_h^{(1)} + LA_h^{(0)})C_\lambda = F_\lambda + E \quad (4.4)$$

and

$$(PA_h^{(2)} + QA_h^{(1)} + LA_h^{(0)})\tilde{C}_\lambda = F_{\tilde{C}_\lambda}, \tag{4.5}$$

with

$$\begin{aligned} F_\lambda &= [g_1, \dots, g_{n+1}]^T \text{ and } g_i = \frac{1}{\Delta t}g(\tau_i), \\ F_{\tilde{C}_\lambda} &= [f_1, \dots, f_{n+1}]^T \text{ and } f_i = \frac{1}{\Delta t}f(\tau_i), \\ E &= [O(\frac{h^2}{\Delta t}), \dots, O(\frac{h^2}{\Delta t})]^T \in \mathbb{R}^{n+1}, \\ P &= (\text{diag}(\frac{1}{2}\alpha(\tau_i))_{1 \leq i \leq n+1}), \\ Q &= (\text{diag}(\frac{1}{2}\beta(\tau_i))_{1 \leq i \leq n+1}), \\ L &= (\text{diag}(\frac{1}{2}\gamma(\tau_i) - \frac{1}{\Delta t})_{1 \leq i \leq n+1}), \\ A_h^{(k)} &= (B_{-3+j}^{(k)}(\tau_i))_{1 \leq i, j \leq n+1}, \quad k = 0, 1, 2, \end{aligned}$$

It is well known that $A_h^{(k)} = \frac{1}{h^k}A_k$ for $k = 0, 1, 2$ where matrices A_0, A_1 and A_2 are independent of h , with expression A_2 is:

$$A_2 = \begin{bmatrix} \frac{-15}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \dots & & & 0 \\ \frac{3}{4} & \frac{-3}{4} & \frac{-1}{2} & \frac{1}{2} & 0 & \dots & & 0 \\ 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & \dots & & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-3}{4} & \frac{3}{4} \\ 0 & \dots & & & 0 & \frac{1}{2} & \frac{1}{4} & \frac{-15}{4} \\ 0 & \dots & & & & \frac{1}{2} & \frac{4}{4} & \frac{3}{4} \\ 0 & \dots & & & & 0 & 1 & \frac{-5}{2} \end{bmatrix}.$$

Results of this work are basically based on the invertibility of the matrix A_2 . Then, in order to prove that A_2 is invertible we give the flowing lemma.

Lemma 4.1. (de Boor [10]) Let $S \in \mathcal{S}_{k+1}$ such that $S = 0$ on $[r_{p-1}, r_p] \cup [r_q, r_{q+1}]$ where $p < q$. If S admits d zeros in $[r_p, r_q]$ then $d \leq p - q - (k + 1)$.

Lemma 4.2. The matrix A_2 is invertible.

Proof: Let $D = [d_1, \dots, d_{n+1}]^T$ be a vector of \mathbb{R}^{n+1} such that $A_2D = 0$. If we put $S(r) = \sum_{j=-2}^{n-2} d_j B_j$, then we have $S(0) = S(R) = 0$ and $S^{(2)}(\tau_i) = 0$ for any

$i = 1, \dots, n + 1$. Since $S \in \mathbb{S}_4(\Omega_r, \Theta)$ then $S^{(2)} \in \mathbb{S}_2(\Omega_r, \Theta)$. If we assume that $S^{(2)} \neq 0$ in $[r_0, r_n]$, then using the Lemma 4.1 and the fact that $S^{(2)}$ has $n + 1$ zeros in $[r_0, r_n]$, we conclude that $n + 1 \leq n - 2$, which is impossible. Therefore $S^{(2)} = 0$ for each $r \in \Omega_r$. This means that the function S is a piecewise linear polynomial on Ω_r . Since $S(0) = S(R) = 0$, then we obtain $S(r) = 0$ for any $r \in \Omega_r$. Consequently $D = 0$ and the matrix A_2 is invertible. \square

Then, relations (4.4) and (4.5) can be written in the following form

$$PA_2(I + U + V)C_\lambda = h^2 F_\lambda + h^2 E, \quad (4.6)$$

$$PA_2(I + U + V)\tilde{C}_\lambda = h^2 F_{\tilde{C}_\lambda}, \quad (4.7)$$

with

$$U = hA_2^{-1}P^{-1}QA_1, \quad (4.8)$$

$$V = h^2A_2^{-1}P^{-1}LA_0. \quad (4.9)$$

In order to determine the bounded of $\|C - \tilde{C}\|_\infty$, we need the following Lemma.

Lemma 4.3. *If $h^2 < \frac{\Delta t}{4\rho}$, then $I + U + V$ is invertible, where $\rho = \frac{\|A_2^{-1}\|_\infty}{\tilde{\alpha}}$.*

Proof: From the relation (4.8), We have

$$\|U\|_\infty \leq h\|A_2^{-1}\|_\infty\|P^{-1}\|_\infty\|Q\|_\infty\|A_1\|_\infty \leq h\rho\|A_1\|_\infty \max_{i=0\dots n} |\beta(\tau_i)|.$$

For h sufficiently small, we conclude

$$\|U\|_\infty < \frac{1}{4}. \quad (4.10)$$

From the relation (4.9) and $\|A_0\|_\infty \leq 1$, we have

$$\begin{aligned} \|V\|_\infty &\leq h^2\|A_2^{-1}\|_\infty\|P^{-1}\|_\infty\|L\|_\infty\|A_0\|_\infty \\ &\leq h^2\frac{\|A_2^{-1}\|_\infty}{\tilde{\alpha}} \max_{i=0\dots n} |\gamma(\tau_i) - \frac{2}{\Delta t}| \\ &\leq h^2\rho \max_{i=0\dots n} |\gamma(\tau_i) - \frac{2}{\Delta t}| \\ &\leq h^2\rho \max_{i=0\dots n} (\tau_i) + \frac{2h^2\rho}{\Delta t}. \end{aligned}$$

For h sufficiently small, we conclude that $h^2\rho \max_{i=0\dots n} (\tau_i) < \frac{1}{4}$. Then

$$\|V\|_\infty < \frac{1}{4} + \frac{2h^2\rho}{\Delta t}. \quad (4.11)$$

As $\frac{2h^2\rho}{\Delta t} < \frac{1}{2}$. So, $\|U + V\| \leq \|U\| + \|V\| < 1$, and therefore $I + U + V$ is invertible. \square

Proposition 4.4. *Assume that the penalty parameter λ and the discretization parameter h satisfy the following relation:*

$$\lambda h^2 \|(I + U + V)^{-1}(PA_2)^{-1}\|_\infty < 2. \quad (4.12)$$

Then there exists a unique cubic spline which approximates the exact solution u_λ of problem (3.6).

Proof: From (4.7), we have $\tilde{C}_\lambda = h^2(I + U + V)^{-1}(PA_2)^{-1}F_{\tilde{C}_\lambda}$. Let $\varphi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be a function defined by

$$\varphi(Y) = h^2(I + U + V)^{-1}(PA_2)^{-1}F_{\tilde{Y}}. \quad (4.13)$$

To prove the existence of cubic spline collocation it suffices to prove that φ admits a unique fixed point. Indeed, let Y_1 and Y_2 be two vectors of \mathbb{R}^{n+1} . Then we have

$$\|\varphi(Y_1) - \varphi(Y_2)\| \leq h^2 \|(I + U + V)^{-1}(PA_2)^{-1}\|_\infty \|F_{Y_1} - F_{Y_2}\|_\infty. \quad (4.14)$$

Using relation (3.7) and the fact that $\sum_{j=-2}^{n-2} B_j \leq 1$, we get

$$|J_\lambda(\tau_i, S_{Y_1}(\tau_i)) - J_\lambda(\tau_i, S_{Y_2}(\tau_i))| \leq \frac{\lambda}{2} \Delta t |S_{Y_1}(\tau_i) - S_{Y_2}(\tau_i)| \leq L_\lambda \|Y_1 - Y_2\|_\infty,$$

Then we obtain

$$\|F_{Y_1} - F_{Y_2}\|_\infty \leq \frac{\lambda}{2} \|Y_1 - Y_2\|_\infty.$$

From relation (4.14), we conclude that

$$\|\varphi(Y_1) - \varphi(Y_2)\| \leq \frac{\lambda}{2} h^2 \|(I + U + V)^{-1}(PA_2)^{-1}\|_\infty \|Y_1 - Y_2\|_\infty.$$

Then we have

$$\|\varphi(Y_1) - \varphi(Y_2)\| \leq k \|Y_1 - Y_2\|_\infty,$$

where $k = \frac{\lambda}{2} h^2 \|(I + U + V)^{-1}(PA_2)^{-1}\|_\infty$. Hence the function φ admits a unique fixed point. \square

In order to calculate the coefficients of the cubic spline collocation given by the nonsmooth system

$$\tilde{C}_\lambda = \varphi(\tilde{C}_\lambda), \quad (4.15)$$

we propose the generalized Newton method defined by

$$\tilde{C}_\lambda^{(k+1)} = \tilde{C}_\lambda^{(k)} - (I_{n+1} - V_k)^{-1}(\tilde{C}_\lambda^{(k)} - \varphi(\tilde{C}_\lambda^{(k)})), \quad (4.16)$$

where I_{n+1} is the unit matrix of order $n + 1$ and V_k is the generalized Jacobian of the function $\tilde{C}_\lambda \mapsto \varphi(\tilde{C}_\lambda)$, (see [4,5,20], for instance).

5. Convergence of the method

Theorem 5.1. *Assume that the penalty parameter λ and the discretization parameter h satisfy the following relation:*

$$\lambda h^2 \|(I + U + V)^{-1}(PA_2)^{-1}\|_\infty < 1. \quad (5.1)$$

Then the cubic spline \tilde{S}_λ converges to the solution u_λ . Moreover the error estimate $\|u_\lambda - \tilde{S}_\lambda\|_\infty$ is of order $O(h^2)$.

Proof: We pose $\nu = \|(I + U + V)^{-1}(PA_2)^{-1}\|_\infty$. From (4.8), (4.9) and Lemma 4.2, we have

$$C_\lambda - \tilde{C}_\lambda = h^2(I + U + V)^{-1}(PA_2)^{-1}[(F_\lambda - F_{\tilde{C}_\lambda}) + E].$$

Since E is of order $O(\frac{h^2}{\Delta t})$, then there exists a constant K_1 such that

$$\|E\|_\infty \leq K_1 \frac{h^2}{\Delta t}.$$

Hence, we have

$$\|C_\lambda - \tilde{C}_\lambda\|_\infty \leq h^2 \nu \left[\|F_\lambda - F_{\tilde{C}_\lambda}\|_\infty + K_1 h^2 \right]. \quad (5.2)$$

On the other hand we have

$$\begin{aligned} |J_\lambda(\tau_i, u_\lambda(\tau_i)) - J_\lambda(\tau_i, \tilde{S}(\tau_i))| &\leq \frac{\lambda}{2} \Delta t |u_\lambda(\tau_i) - \tilde{S}_\lambda(\tau_i)| \\ &\leq \frac{\lambda}{2} \Delta t |u_\lambda(\tau_i) - S_\lambda(\tau_i)| + \frac{\lambda}{2} \Delta t |S_\lambda(\tau_i) - \tilde{S}_\lambda(\tau_i)|. \end{aligned}$$

From relation (4.1), there exists a constant K_2 such that

$$\|u_\lambda - Q_3 u_\lambda\|_\infty \leq K_2 h^4 \|u_\lambda^4\|_\infty. \quad (5.3)$$

Using the fact that

$$|Q_3 u_\lambda - \tilde{S}_\lambda| \leq \|C_\lambda - \tilde{C}_\lambda\|_\infty \sum_{j=-2}^{n-2} B_j \leq \|C_\lambda - \tilde{C}_\lambda\|_\infty, \quad (5.4)$$

then, we obtain

$$\begin{aligned} |F_\lambda - F_{\tilde{C}_\lambda}| &\leq \frac{1}{\Delta t} |J_\lambda(\tau_i, u_\lambda(\tau_i)) - J_\lambda(\tau_i, \tilde{S}(\tau_i))| \\ &\leq \frac{\lambda}{2} \|C_\lambda - \tilde{C}_\lambda\|_\infty + \frac{\lambda}{2} K_2 h^4 \|u_\lambda^4\|_\infty. \end{aligned}$$

By using relation (5.2) and assumption (5.1) it is easy to see that

$$\begin{aligned} \|C_\lambda - \tilde{C}_\lambda\|_\infty &\leq \frac{h^2\nu}{1 - \frac{\lambda}{2}h^2\nu} (K_2h^4\frac{\lambda}{2}\|u_\lambda^{(4)}\|_\infty + K_1h^2) \\ &\leq \frac{K_2h^2\frac{\lambda}{2}\|u_\lambda^{(4)}\|_\infty + K_1}{\frac{\lambda}{2}}h^2. \end{aligned} \tag{5.5}$$

We have

$$\|u_\lambda - \tilde{S}_\lambda\|_\infty \leq \|u_\lambda - Q_3u_\lambda\|_\infty + \|Q_3u_\lambda - \tilde{S}_\lambda\|_\infty,$$

then from relations (5.3), (5.4) and (5.5), we deduce that $\|u_\lambda - \tilde{S}_\lambda\|_\infty$ is $O(h^2)$. Hence the proof is complete. \square

Remark 5.2. *Theorem 5.1 provides a relation coupling the penalty parameter λ and the discretization parameter h , which guarantees the quadratic convergence of the cubic spline collocation \tilde{S}_λ to the solution u_λ of the penalty problem.*

6. Numerical examples

In this section we verify numerically the obtained theoretical results in the previous section. If the exact solution is known, then at time $t \leq T$ the maximum error E^{max} can be calculated as:

$$E^{max} = \max_{r \in [0, R], t \in [0, T]} |S^{M, N}(r, t) - u(r, t)|.$$

Otherwise it can be estimated by the following double mesh principle:

$$E_{M, N}^{max} = \max_{r \in [0, R], t \in [0, T]} |S^{M, N}(r, t) - S^{2M, 2N}(r, t)|,$$

where $S^{M, N}(r, t)$ is the numerical solution on the $M + 1$ grids in space and $N + 1$ grids in time, and $S^{2M, 2N}(r, t)$ is the numerical solution on the $2M + 1$ grids in space and $2N + 1$ grids in time. Denote

$$B = (B_{-2}, \dots, B_{n-2})$$

and

$$d(\tilde{C}_\lambda) = [\psi - \sum_{i=-2}^{n-2} \tilde{c}_{i, \lambda} B_i]_+^{\frac{1}{k}} = [\psi - B\tilde{C}_\lambda]_+^{\frac{1}{k}}. \tag{6.1}$$

This last equation is nonlinear in \tilde{C}_λ . We now apply the generalized Newton method to this equation. Note that when $k > 1$, we see that vector derivative

of d at \tilde{C}_λ satisfies $d'(\tilde{C}_\lambda) \rightarrow \infty$ as $\psi - B\tilde{C}_\lambda \rightarrow 0^+$. To overcome this difficulty, we use the technique proposed in [21] to smooth (6.1), yielding the following approximation to $d(\tilde{C}_\lambda)$:

$$d(\tilde{C}_\lambda) = \begin{cases} (\psi - B\tilde{C}_\lambda)^{\frac{1}{k}}, & \psi - B\tilde{C}_\lambda \geq \epsilon, \\ W([\psi - B\tilde{C}_\lambda]_+), & \psi - B\tilde{C}_\lambda < \epsilon, \end{cases} \tag{6.2}$$

for $k > 0$, where $0 < \epsilon \ll 1$ is a transition parameter and $W(z)$ is a function which smooths out the original $d(z)$ around $z = 0$. We choose $W(z) = a_1 + a_2z + \dots + a_nz^{n-1} + a_{n+1}z^n$ for $n \geq 3$ and impose that $W(z)$ is such that $d(\cdot)$ is smooth. This requires that $W(z)$ satisfies

$$W(0) = W'(0) = 0, \quad W(\epsilon) = \epsilon^{\frac{1}{k}}, \quad W'(\epsilon) = \frac{1}{k}\epsilon^{\frac{1}{k}-1}. \tag{6.3}$$

In this case, the function defined in (6.2) is globally smooth. Using the four conditions given in relation (6.1) and setting $a_3 = \dots = a_{n-1} = 0$, we can easily find that

$$a_1 = a_2 = 0, \quad a_n = \epsilon^{\frac{1}{k-n+1}} \left(n - \frac{1}{k}\right), \quad a_{n+1} = \epsilon^{\frac{1}{k-n}} \left(\frac{1}{k} - n + 1\right).$$

Taking $X = \psi - B\tilde{C}_\lambda$, then relation (6.2) can be expressed as

$$d(\tilde{C}_\lambda) = \begin{cases} X^{\frac{1}{k}}, & X \geq \epsilon, \\ W(0), & X \leq 0, \\ a_n X^{n-1} + a_{n+1} X^n, & 0 < X < \epsilon. \end{cases} \tag{6.4}$$

For $z \in (0, \epsilon)$, differentiating $W(z)$ gives

$$\begin{aligned} W'(z) &= (n-1)\epsilon^{\frac{1}{k-n+1}} \left(n - \frac{1}{k}\right) z^{n-2} + n\epsilon^{\frac{1}{k-n}} \left(\frac{1}{k} - n + 1\right) z^{n-1} \\ &= z^{n-2}\epsilon^{\frac{1}{k-n+1}} \left[(n-1) \left(n - \frac{1}{k}\right) + n \left(\frac{1}{k} - n + 1\right) \frac{z}{\epsilon} \right]. \end{aligned}$$

Hence

$$\frac{\partial d}{\partial \tilde{c}_{i,\lambda}} = \begin{cases} -\frac{1}{k} B_i X^{\frac{1}{k}-1}, & X \geq \epsilon, \\ 0, & X \leq 0, \\ -B_i X^{n-2} \epsilon^{\frac{1}{k-n+1}} \left[(n-1) \left(n - \frac{1}{k}\right) + \frac{n}{\epsilon} \left(\frac{1}{k} - n + 1\right) X \right], & 0 < X < \epsilon. \end{cases}$$

In this section we apply the numerical method developed in this paper to an example presented in the paper of the authors Kai Zhang et al. [25]. Then, we compare the obtained results to the ones given in [25].

A vanilla put option on a zero-coupon bond has the payoff

$$\Lambda = \max[K - P(r, T, s), 0], \quad \text{with } P(r, T, s) = EA(T, s)e^{-B(T,s)r}.$$

The parameters used for this put option on a bond under the CIR model are listed in Table 1. For the put option on a bond with the parameters in Table 1, we choose $R = 2$. The coarsest grid is defined as a uniform partition of the solution domain $(0, 2) \times (0, 1)$.

The comparison of the maximum error values between the method developed in this paper with the one developed in [25] will be taken at five different values of the number of space steps $N = 201, 401, 801, 1601, 3201$ and time steps $M = 100, 200, 400, 800, 1600$, for $\sigma = 0.1$ and $\sigma = 0.5$.

We conduct experiments on different values of N , M and σ . Table 2 and Table 3 show values of the maximum error (`max_error`) obtained in our numerical experiments and the one obtained in [25]. We see that the values of maximum error obtained by our method improve the ones in [25].

Table 1: Data used to value American put options on a zero-coupon bond under the CIR model.

<i>Parameter</i>	<i>values</i>
κ	0.10
θ	0.08
σ	0.10 and 0.50
ζ	0
E	100
K	60
T	1
s	5

Table 2: Computed results with $\sigma = 0.1$

N	M	<i>our max_error</i>	max_error in [25]
201	100	0.00573	0.0721
401	200	0.00161	0.0366
801	400	0.00050	0.0144
1601	800	0.00014	0.0048
3201	1600	0.00004	0.0022

7. Conclusion

In this paper we have presented the American options on a bond under the Cox-Ingersoll-Ross model, this problem is approximated by a sequence of nonlinear equation problems by using the penalty method given in [23,27], and its time

Table 3: Computed results with $\sigma = 0.5$

N	M	<i>our max_error</i>	max_error in [25]
201	100	0.00252	0.0283
401	200	0.00073	0.0164
801	400	0.00021	0.0086
1601	800	0.00005	0.0046
3201	1600	0.00002	0.0024

discretization scheme. Then, we have developed and analyzed a cubic spline collocation method and the generalized Newton method for approximating solutions of the semi-discrete problem. We have shown the convergence of the method provided that the penalty and discrete parameters satisfy the relation (5.1). Moreover we have provided an error estimate of order $O(h^2)$ with respect to the maximum norm $\| \cdot \|_\infty$. Numerical experiment was performed on one known model to validate the convergence and efficiency of the method. The computational results show that the proposed numerical method is an efficient alternative method to the one proposed in [25].

References

1. Allegretto, W., Lin, Y., and Yang, H, Numerical pricing of American put options on zero-coupon bonds. *Applied Numerical Mathematics*, 46 (2003), 113-134.
2. Bensoussan, A., and Lions, J. L., *Applications of variational inequalities in stochastic control*. Amsterdam, New York, Oxford, North-Holland, (1982).
3. Çağlar, H. N., Çağlar, S. N., Twizell, E. H., The numerical solution of fifth-order boundary value problems with sixth-degree B-spline function, *Applied Mathematics Letter*, 12 (1999), 25-30.
4. Chen, X., A verification method for solutions of nonsmooth equations, *Computing* 58 (1997), 281-294.
5. Chen, X., Nashed, Z., Qi, L., Smoothing methods and semismooth methods for nondifferentiable operator equations, *SIAM J. Numer. Anal.* 38 (2000), 1200-1216.
6. Clavero, C., Jorge, J.C., Lisbona, F., Uniformly convergent schemes for singular perturbation problems combining alternating directions and exponential fitting techniques, in: J.J.H. Miller (Ed.), *Applications of Advanced Computational Methods for Boundary and Interior Layers*, Boole, Dublin (1993), 33-52.
7. Clavero, C., Jorge, J.C., Lisbona, F., A uniformly convergent scheme on a nonuniform mesh for convectiondiffusion parabolic problems, *Journal of Computational and Applied Mathematics* 154 (2003), 415-429.
8. Cox, J., Ingersoll, J., Ross, S., A theory of the term structure of interest rates, *Econometrica*, 53 (1985), 385-407.
9. Dauvaut, G., and Lions, J. L, *Inequalities in mechanics and physics*. Berlin, Heidelberg, New York, Springer-Verlag, (1976).
10. de Boor, C., *A Practical guide to Splines*. Springer Verlag, NewYork, (1978).
11. DeVore, R.A., Lorentz, G.G., *Constructive approximation*, Springer-Verlag, Berlin, (1993).

12. Elliot, E. M., and Ockendon, J. R., Weak and variational methods for moving boundary problems. Boston: Pitman, (1982).
13. Feller, W., Two singular diffusion problems, *Annals of Mathematics*, 54 (1951), 173-182.
14. Forsyth, P. A., Vetzal, K. R., Quadratic convergence for valuing American options using a penalty method. *SIAM J. on Sci. Comput.*, 23 (2002), 2095-2122.
15. Friedman, A., *Partial Differential Equation of Parabolic Type*, Robert E. Krieger Publishing Co., Huntington, NY, (1983).
16. Kadalbajoo, M. K., Tripathi, L. P., Kumar, A., A cubic B-spline collocation method for a numerical solution of the generalized BlackScholes equation, *Mathematical and Computer Modelling* 55 (2012), 1483-1505.
17. Kwok, Y. K, *Mathematical models of financial derivatives*. Berlin, Springer, (1998).
18. Ladyzenskaja, O. A., Solonnikov, V. A, Ural'ceva, N. N., *Linear and Quasilinear Equations of Parabolic Type*, In: Amer. Math. Soc. Transl., Vol. 23, Providence, RI, (1968).
19. Sablonnière, P., Univariate spline quasi-interpolants and applications to numerical analysis, *Rend. Sem. Mat. Univ. Pol. Torino* 63 (2005), 211-222.
20. Śmietański, M.J., A generalized Jacobian based Newton method for semismooth block-triangular system of equations, *Journal of Computational and Applied Mathematics*, 205 (2007), 305-313.
21. Wang, S., Yang, X. Q., Teo, K. L., Power penalty method for a linear complementarity problem arising from American option valuation. *J. Optim. Theory Appl.*, 129 (2006), 227-254.
22. Wang, S., A novel fitted finite volume method for the BlackScholes equation governing option pricing. *IMA Journal of Numerical Analysis*, 24 (2004), 699-720.
23. Wang, S., Yang, X. Q., Teo, K. L., Power penalty method for a linear complementarity problem arising from American option valuation. *Journal of Optimization Theory and Applications*, 129 (2006), 227-254.
24. Wilmott, P., *On quantitative finance*. New York, Wiley, (2000).
25. Zhang, K., Wang, S., Pricing American bond options using a penalty method. *Automatica* 48 (2012), 472-479.
26. Zhang, K., Wang, S., Yang, X. Q., Teo, K. L., A power penalty approach to numerical solutions of two-asset American options. *Numerical Mathematics. Theory, Methods and Applications*, 2 (2009), 202-223.
27. Zhang, K., Yang, X. Q., Teo, K. L., Convergence analysis of a monotonic penalty method for American option pricing. *Journal of Mathematical Analysis and Applications*, 348 (2008), 915-926.

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