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Perturbation Method for Linear and Non-Linear Fractional Order Systems and Integral Representation for Evaluation of Integrals

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ABSTRACT: In the present work, the authors used the Laplace transform - perturbation method to solve certain linear and non-linear systems of fractional differential and difference equations with constant coefficients with the fractional derivatives in the Caputo sense. We also considered the problems of string vibrations in different cases with fractional damping. Another purpose of this article is to evaluate certain integrals. Illustrative examples are also provided.

Key Words: Laplace transform, Integral representation, Caputo fractional derivative, Fractional differential equations, Perturbation method, Fractional difference equation.

Contents

1. Introduction and Definitions

In the present study, the fractional derivatives are understood in the Caputo sense. The reason for adopting the Caputo definition is as follows: There are several approaches to the generalization of the notion of differentiation to fractional orders e.g. Riemann-Liouville, Grnwald-Letnikov, Caputo and generalized functions approach [12]. Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since

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it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations [4].

By ${}_aI_t^\alpha f(t)$ we denote the fractional integral of f with order $\alpha > 0$ on $[0,t]$ defined as

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(x)}{(t-x)^{1-\alpha}} dx. $$

This integral is sometimes called the left-sided fractional integral. For the concept of fractional derivative we will adopt Caputo's definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which in the case in most physical processes. The Caputo fractional derivative is more suited than the usual Riemann-Liouville derivative for the applications in several engineering problems due to the fact that it has better relations with the Laplace transform and because the differentiation appears inside instead than outside, the integral, so to alleviate the effects of noise and numerical differentiation.

For an arbitrary real number $\alpha > 0\,(n-1 \leq \alpha < n, n \in N)$ Caputo fractional derivative is given as

$$ {}_a^CD_t^\alpha f(t) = {}_aI_t^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx. $$

The direct Laplace transform of a function $f(t)$ defined for $0 \leq t < \infty$ is the ordinary calculus integration problem

$$ L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt := F(s). $$

If $L\{f(t)\} = F(s)$, then $L^{-1}\{F(s)\}$ is given by

$$ f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} F(s) ds \ (t > 0), $$

where $F(s)$ is analytic in the region $Re(s) > c$ and $f(t) = 0$ for $t < 0$. This result is called complex inversion formula. It is also known as Bromwich's integral formula. The one-dimensional convolution theorem of $f(x)$ and $g(x)$ is given by

$$ f(x) * g(x) = \int_0^x f(x-w) g(w) dw. $$

Two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}. $$

Theorem 1.1. *When $n-1 \leq \alpha < n$, we have*

$$L\{{}_0^C D_t^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0).$$

Proof: See [11]. □

Theorem 1.2. *Let $f(t)$ be continues, positive, and increasing for $0 < t < +\infty$. Then the following complex integral relationships hold true,*

$$a)\ f(\frac{t}{a}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{ts}}{s}\left(\int_0^\infty \exp(-asf^{-1}(\varepsilon))d\varepsilon\right) ds,$$

$$b)\ \cosh^{-1}\frac{t}{a} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{ts}}{s}\left(\int_0^\infty \exp(-as\cosh t)d\varepsilon\right) ds,$$

$$c)\ f(\frac{1}{t}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{ts}}{s}\left(\int_0^\infty \exp(-\frac{s}{f^{-1}(\varepsilon)})d\varepsilon\right) ds,$$

$$d)\ g(f(t)) - g(0) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{ts}}{s}\left(\int_0^\infty g'(\varepsilon)\exp(-sf^{-1}(\varepsilon))d\varepsilon\right) ds.$$

Proof: *a*) Let us consider the following integral

$$\frac{1}{2\,\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{ts}}{s}\left(\int_0^\infty \exp(-asf^{-1}(\varepsilon))d\varepsilon\right) ds$$

changing the order of integration which is permissible yields

$$= \int_0^\infty \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{ts}}{s} e^{-asf^{-1}(\varepsilon)} ds\right) d\varepsilon = \int_0^\infty L^{-1}\left\{\frac{1}{s} e^{-asf^{-1}(\varepsilon)} ds\right\} d\varepsilon$$

$$= \int_0^\infty h(t - af^{-1}(\varepsilon))d\varepsilon = \int_0^{f(\frac{t}{a})} d\varepsilon = f(\frac{t}{a}).$$

where h is Heaviside's unit step function.

b, c) The proofs are straight forward. Note that (b) is a particular example of (a).

d) Let us consider the following integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{ts}}{s}\left(\int_0^\infty g'(\varepsilon)\exp(-sf^{-1}(\varepsilon))d\varepsilon\right) ds$$

changing the order of integration which is permissible yields

$$= \int_0^\infty \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{ts}}{s} g'(\varepsilon) \exp(-sf^{-1}(\varepsilon)) ds \right) d\varepsilon$$
$$= \int_0^\infty g'(\varepsilon) L^{-1} \left\{ \frac{1}{s} \exp(-sf^{-1}(\varepsilon)) ds \right\} d\varepsilon$$

at this point, by using the table of Laplace transform and elementary properties of Heaviside's unit step function, we get

$$= \int_0^\infty g'(\varepsilon) h(t - f^{-1}(\varepsilon)) d\varepsilon = \int_0^{f(t)} g'(\varepsilon) d\varepsilon = g(f(t)) - g(0).$$

□

2. Perturbation-Laplace Transform Method for Solving Fractional Order System

Most scientific problems and physical phenomena occur nonlinearly. Except in a limited number of these problems, finding the exact analytical solutions of such problems are rather difficult. Therefore, there have been attempts to develop new techniques for obtaining analytical solutions which reasonably approximate the exact solutions. In recent years, several such techniques have drawn special attention, such as Hirota's bilinear method, the homogeneous balance method, inverse scattering method, Adomian's decomposition method -ADM-, the variational iteration method -VIM-, and homotopy analysis method -HAM- as well as homotopy perturbation method -HPM-. The method has been used by many authors to handle a wide variety of scientific and engineering applications to solve various functional fractional equations. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions. Recently, considerable research work has been conducted in applying this method to the fractional linear and nonlinear equations. The concept of He's homotopy perturbation method is introduced briefly for applying this method for problem solving. The results of HPM as an analytical solution are then compared with those derived from Adomian's decomposition method -ADM- and the variational iteration method -VIM-. The results reveal that the HPM is very effective and convenient in predicting the solution of such problems, and it is predicted that HPM can find a wide application in new engineering problems [5]. Convergence of the homotopy perturbation method can be found in [20]. In following, we solve fractional Chen's system with perturbation-Laplace transform method. Li and Peng found that chaos does exist in ChenÆs system with a fractional order. Deng, Li and Lu [24] studied the stability of n-dimensional linear fractional differential equation with time delays. By using the Laplace transform, they introduced a characteristic equation for the above system with multiple time delays. they discovered that if all roots of the characteristic equation have negative parts, then the equilibrium of the above linear system with fractional order is Lyapunov globally asymptotical stable if the equilibrium exist that is almost the same as that of classical differential equations.

Problem 2.1. We consider generalized fractional Chen's system

$$\begin{cases} {}_0^C D_t^{\alpha_1} x(t) = ay(t) - bx(t) + f_1(t) \\ {}_0^C D_t^{\alpha_2} y(t) = cx(t) - x(t)z(t) + dy(t) + f_2(t) \\ {}_0^C D_t^{\alpha_3} z(t) = x(t)y(t) - ez(t) + f_3(t) \end{cases},$$

where $x(0) = y(0) = z(0) = 0$ and $0 < \alpha_i \leq 1$ for $i = 1, 2, 3$.

Solution. According to perturbation method, we replace the previous system by

$$\begin{cases} {}_0^C D_t^{\alpha_1} x_\varepsilon(t) = ay_\varepsilon(t) - bx_\varepsilon(t) + \varepsilon f_1(t) \\ {}_0^C D_t^{\alpha_2} y_\varepsilon(t) = c\,x_\varepsilon(t) - x_\varepsilon(t)z_\varepsilon(t) + dy_\varepsilon(t) + \varepsilon f_2(t) \\ {}_0^C D_t^{\alpha_3} z_\varepsilon(t) = x_\varepsilon(t)y_\varepsilon(t) - ez_\varepsilon(t) + \varepsilon f_3(t) \end{cases},$$

when

$$x_\varepsilon(t) = \sum_{n=1}^{\infty} \varepsilon^n x_n(t)\ ,\ y_\varepsilon(t) = \sum_{n=1}^{\infty} \varepsilon^n y_n(t)\ ,\ z_\varepsilon(t) = \sum_{n=1}^{\infty} \varepsilon^n z_n(t).$$

By setting the above representations of x_ε, y_ε and z_ε in the above system, we get

$$\begin{cases} \varepsilon {}_0^C D_t^{\alpha_1} x_1 + \varepsilon^2 {}_0^C D_t^{\alpha_1} x_2 + \cdots = \varepsilon\{f_1 + ay_1 - bx_1\} + \varepsilon^2\{ay_2 - bx_2\} + \cdots \\ \varepsilon {}_0^C D_t^{\alpha_2} y_1 + \varepsilon^2 {}_0^C D_t^{\alpha_2} y_2 + \cdots = \varepsilon\{f_2 + cx_1 + dy_1\} - \varepsilon^2 x_1 z_1 + \cdots \\ \varepsilon {}_0^C D_t^{\alpha_3} z_1 + \varepsilon^2 {}_0^C D_t^{\alpha_3} z_2 + \cdots = \varepsilon\{f_3 - ez_1\} + \varepsilon^2 x_1 y_1 + \cdots \end{cases}.$$

If we set equal all the coefficients of the same exponents of ε on both sides, we get

$$\begin{cases} {}_0^C D_t^{\alpha_1} x_1 = f_1 + ay_1 - bx_1 \\ {}_0^C D_t^{\alpha_2} y_1 = f_2 + cx_1 + dy_1 \\ {}_0^C D_t^{\alpha_3} z_1 = f_3 - ez_1 \end{cases},\ \begin{cases} {}_0^C D_t^{\alpha_1} x_2 = ay_2 - bx_2 \\ {}_0^C D_t^{\alpha_2} y_2 = -x_1 z_1 \\ {}_0^C D_t^{\alpha_3} z_2 = x_1 y_1 \end{cases},\ \ldots..$$

The Laplace transform of the first system gives

$$Z_1(s) = \frac{F_3(s)}{s^{\alpha_3} + e},$$

$$X_1(s) = \frac{aF_2(s) + (s^{\alpha_2} - d)F_1(s)}{s^{\alpha_1+\alpha_2} + bs^{\alpha_2} - ds^{\alpha_1} - (ac + bd)},$$

$$Y_1(s) = \frac{1}{s^{\alpha_2} - d}(F_2(s) + cX_1(s)).$$

where $F_i(s) = L^{-1}\{f_i(t)\}$ for $i = 1, 2, 3$. At first, we obtain

$$L^{-1}\left\{\frac{1}{s^{\alpha_1+\alpha_2} + bs^{\alpha_2} - ds^{\alpha_1} - (ac+bd)}\right\}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{d^k(ac+bd)^{n-k}}{n!}t^{\alpha_1 n+(n+1)\alpha_2-\alpha_1(k-1)-1}E^{(n)}_{\alpha_1,(n+1)\alpha_2-\alpha_1(k-1)}(-bt^{\alpha_1}).$$

Therefore

$$\begin{aligned} x_1(t) &= L^{-1}\{X_1(s)\} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{ad^k(ac+bd)^{n-k}}{n!} \\ &\times \Big\{t^{\alpha_1 n+(n+1)\alpha_2-\alpha_1(k-1)-1}E^{(n)}_{\alpha_1,(n+1)\alpha_2-\alpha_1(k-1)}(-bt^{\alpha_1}) * f_2(t) \\ &+ t^{\alpha_1 n+n\alpha_2-\alpha_1(k-1)-1}E^{(n)}_{\alpha_1,n\alpha_2-\alpha_1(k-1)}(-bt^{\alpha_1}) * f_1(t) \\ &- t^{\alpha_1 n+(n+1)\alpha_2-\alpha_1(k-1)-1}E^{(n)}_{\alpha_1,(n+1)\alpha_2-\alpha_1(k-1)}(-bt^{\alpha_1}) * f_1(t)\Big\}, \end{aligned}$$

and

$$\begin{aligned} y_1(t) &= L^{-1}\{Y_1(s)\} \\ &= \int_0^t (f_2(t-z) + cx_1(t-z))z^{\alpha_2-1}E_{\alpha_2,\alpha_2}(dz^{\alpha_2})dz, \end{aligned}$$

and lastly

$$z_1(t) = \int_0^t f_3(t-z)z^{\alpha_3-1}E_{\alpha_3,\alpha_3}(-ez^{\alpha_3})dz.$$

By application of the previous method successively, we get all x_2, y_2, z_2, ...

A special case: $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}$, $d = 0$, $a = b = c = e = 1$, $f_1 = f_2 = f_3 = 1$. Then we will have

$$Z_1(s) = \frac{1}{s}\frac{1}{\sqrt{s}+1}, X_1(s) = (\frac{1}{s} + \frac{1}{\sqrt{s}})\frac{1}{s+\sqrt{s}-1}, Y_1(s) = \frac{1}{\sqrt{s}}(\frac{1}{s} + X_1(s)).$$

Applying the Laplace transform inversion, we get

$$L^{-1}\{\frac{1}{\sqrt{s}+1}\} = \frac{1}{\sqrt{\pi t}} - e^t erfc(\sqrt{t}),$$

$$L^{-1}\{\frac{1}{s+\sqrt{s}-1}\} = \frac{1}{\sqrt{3}}\left(\alpha e^{t\alpha^2} erfc(-\alpha\sqrt{t}) + \beta e^{t\beta^2} erfc(\beta\sqrt{t})\right),$$

where $\alpha = \frac{\sqrt{3}-1}{2}$ and $\beta = \frac{\sqrt{3}+1}{2}$. Therefore

$$z_1(t) = \int_0^t \left(\frac{1}{\sqrt{\pi y}} - e^y erfc(\sqrt{y})\right) dy = 2\sqrt{\frac{t}{\pi}} - \int_0^t e^y erfc(\sqrt{y}) dy,$$

$$x_1(t) = \frac{1}{\sqrt{3}} \int_0^t \left(1 + \frac{1}{\sqrt{\pi(t-z)}}\right) \left(\alpha e^{z\alpha^2} erfc(-\alpha\sqrt{z}) + \beta e^{z\beta^2} erfc(\beta\sqrt{z})\right) dz,$$

$$y_1(t) = 2\sqrt{\frac{t}{\pi}} + \frac{1}{\sqrt{\pi}} \int_0^t \frac{x_1(z)}{\sqrt{t-z}} dz.$$

Fractional order systems, or systems containing fractional derivatives and integrals, have been studied by many authors in the engineering area. Additionally, very readable discussions, devoted specifically to the subject, are presented by Oldham and Spanier (1974) and Miller and Ross (1993) and Podlubny (1999). It should be noted that there are a growing number of physical systems whose behavior can be compactly described using fractional system theory.

3. Computation of Certain Integrals and Inverse Laplace Transform of the Object Functions by Means of Integral Representation

An interesting application of Laplace transforms involves the evaluation of integrals. In this section, we have implemented integral representation method to evaluate certain integrals and inverse Laplace transform of the object functions.

Lemma 3.1. *By using the integral representation, we may show that*

$$\int_0^\infty J_v(\frac{a}{x}) J_v(bx) dx = \frac{1}{b} J_{2\nu}(2\sqrt{ab}),$$

where J_ν is Bessel function.

Proof: It is well-known that

$$(\frac{2}{z})^\nu J_\nu(z) = \frac{1}{2\pi i} \int_C \frac{e^\varepsilon e^{-\frac{z^2}{4\varepsilon}}}{\varepsilon^{v+1}} d\varepsilon.$$

Then, the left hand side of the above integral relation can be written as following

$$\begin{aligned}
\int_0^{\infty} J_v(\frac{a}{x})J_v(bx)dx &= \frac{(\frac{ab}{2})^{\nu}}{(2\pi i)^2}\int_{C_2}\int_{C_1}\frac{e^{\eta+\xi}}{(\eta\xi)^{\nu+1}}\left(\int_0^{\infty}e^{-\frac{b^2}{4\eta}x^2-\frac{a^2}{4\xi x^2}}dx\right)d\xi d\eta \\
&= \frac{\sqrt{\pi}a^{\nu}b^{\nu-1}}{2^{2\nu}}\frac{1}{(2\pi i)^2}\int_{C_2}\int_{C_1}\frac{e^{\eta+\xi}}{(\eta\xi)^{\nu+1}}e^{-\frac{ab}{2\sqrt{\xi\eta}}}d\xi d\eta \\
&= \frac{\sqrt{\pi}a^{\nu}b^{\nu-1}}{2^{2\nu}}\frac{1}{2\pi i}\int_{C_2}\left(\frac{1}{2\pi i}\int_{C_1}\frac{e^{\xi}}{\xi^{\nu+1}}e^{-\frac{ab}{2\sqrt{\eta}\sqrt{\xi}}}d\xi\right)\frac{e^{\eta}}{\eta^{\nu+\frac{1}{2}}}d\eta \\
&= \frac{\sqrt{\pi}a^{\nu}b^{\nu-1}}{2^{2\nu}}\frac{1}{2\pi i}\int_{C_2}L^{-1}\left\{\frac{e^{-\frac{ab}{2\sqrt{\eta}\sqrt{\xi}}}}{\xi^{\nu+1}},\xi\to x\right\}\Bigg| x=1\ \frac{e^{\eta}}{\eta^{\nu+\frac{1}{2}}}d\eta \\
&= \frac{\sqrt{\pi}a^{\nu}b^{\nu-1}}{2^{2\nu}}\frac{1}{2\pi i}\int_{C_2}\left(\sum_{n=0}^{\infty}\frac{(-\frac{ab}{2})^n}{n!\Gamma(\frac{n}{2}+\nu+1)\eta^{\frac{n}{2}}}\right)\frac{e^{\eta}}{\eta^{\nu+\frac{1}{2}}}d\eta \\
&= \frac{\sqrt{\pi}a^{\nu}b^{\nu-1}}{2^{2\nu}}\sum_{n=0}^{\infty}\frac{(-\frac{ab}{2})^n}{n!\Gamma(\frac{n}{2}+\nu+1)}\left(\frac{1}{2\pi i}\int_{C_2}\frac{e^{\eta}}{\eta^{\frac{n}{2}+\nu+\frac{1}{2}}}d\eta\right) \\
&= \frac{\sqrt{\pi}a^{\nu}b^{\nu-1}}{2^{2\nu}}\sum_{n=0}^{\infty}\frac{(-\frac{ab}{2})^n}{n!\Gamma(\frac{n}{2}+\nu+1)}\left(L^{-1}\left\{\frac{1}{\eta^{\frac{n}{2}+\nu+\frac{1}{2}}}\right\}\Bigg| y=1\right) \\
&= \frac{\sqrt{\pi}b}{2^{2\nu}}\sum_{n=0}^{\infty}\frac{(-1)^n(ab)^{n+\nu}}{2^n n!\Gamma(\frac{n}{2}+\nu+1)\Gamma(\frac{n}{2}+\nu+\frac{1}{2})}.
\end{aligned}$$

Note that $\int_{C_1}$ and $\int_{C_2}$ are contour integration. On the other hand

$$2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})=\sqrt{\pi}\Gamma(2x),$$

and by setting $x=\frac{n}{2}+\nu+\frac{1}{2}$ in the above relation, we get

$$\Gamma(\frac{n}{2}+\nu+1)\Gamma(\frac{n}{2}+\nu+\frac{1}{2})=\frac{\sqrt{\pi}}{2^{n+2\nu}}\Gamma(n+2\nu+1).$$

Finally, final solution is obtained

$$\int_0^{\infty}J_v(\frac{a}{x})J_v(bx)dx=\frac{1}{b}\sum_{n=0}^{\infty}\frac{(-1)^n(ab)^{n+\nu}}{n!\Gamma(2\nu+n+1)}=\frac{1}{b}J_{2\nu}(2\sqrt{ab}).$$

□

Lemma 3.2. *By applying the integral representation, show that*

$$\int_0^{\infty}\frac{\sin(t\sqrt{x^2-a^2})}{\sqrt{x^2-a^2}}\cos(bx)dx=\frac{\pi}{2}I_0\left(|a|\sqrt{t^2-b^2}\right)h(1-|\frac{b}{t}|),$$

where I_0 is modified Bessel function of zero order and h is Heaviside's unit step function.

Proof: It is shown that

$$\begin{aligned}\frac{\sin(t\sqrt{x^2-a^2})}{\sqrt{x^2-a^2}} &= \frac{\sqrt{\pi}t}{2}L^{-1}\left\{\frac{e^{-\frac{x^2-a^2}{4z}t^2}}{z^{\frac{3}{2}}}, z\to\varepsilon\right\}\Bigg|\ \varepsilon=1 \\ &= \frac{\sqrt{\pi}t}{2}\frac{1}{2\pi i}\int_C e^z\frac{e^{-\frac{x^2-a^2}{4z}t^2}}{z^{\frac{3}{2}}}dz.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^\infty \frac{\sin(t\sqrt{x^2-a^2})}{\sqrt{x^2-a^2}}\cos(bx)dx &= \int_0^\infty\left(\frac{\sqrt{\pi}t}{2}\frac{1}{2\pi i}\int_C e^z\frac{e^{-\frac{x^2-a^2}{4z}t^2}}{z^{\frac{3}{2}}}dz\right)\cos(bx)dx \\ &= \frac{\sqrt{\pi}t}{2}\frac{1}{2\pi i}\int_C\frac{e^z e^{\frac{a^2}{4z}t^2}}{z^{\frac{3}{2}}}\left(\int_0^\infty e^{-\frac{t^2}{4z}x^2}\cos(bx)dx\right)dz \\ &= \frac{\sqrt{\pi}t}{2}\frac{1}{2\pi i}\int_C\frac{e^z e^{\frac{a^2}{4z}t^2}}{z^{\frac{3}{2}}}\left(\frac{\sqrt{\pi z}}{t}\right)e^{-\frac{b^2}{t^2}z}dz \\ &= \frac{\pi}{2}\frac{1}{2\pi i}\int_C\frac{e^{(1-\frac{b^2}{t^2})z}e^{\frac{a^2}{4z}t^2}}{z}dz \\ &= \frac{\pi}{2}L^{-1}\left\{\frac{e^{\frac{a^2}{4z}t^2}}{z}, z\to\varepsilon\right\}\Bigg|\ \varepsilon=1-\frac{b^2}{t^2}\quad .\end{aligned}$$

According to definition of Laplace transform, we must have $\varepsilon > 0$ or $|\frac{b}{t}| \leq 1$. Then

$$\int_0^\infty \frac{\sin(t\sqrt{x^2-a^2})}{\sqrt{x^2-a^2}}\cos(bx)dx = \frac{\pi}{2}I_0\left(|a|\sqrt{t^2-b^2}\right)h(1-|\frac{b}{t}|).$$

□

4. Evaluation of Integrals

In applied mathematics, the **Kelvin functions** $ber_\nu(x)$ and $bei_\nu(x)$ are the real and imaginary parts, respectively, of $J_\nu(xe^{3\pi i/4})$ where x is real, and $J_\nu(z)$, is the ν-th order Bessel function of the first kind. Similarly, the functions $Ker_\nu(x)$ and $Kei_\nu(x)$ are the real and imaginary parts, respectively, of $K_\nu(xe^{\pi i/4})$, where $K_\nu(z)$ is the ν-th order modified Bessel function of the second kind. The Kelvin functions were investigated because they are involved in solutions of various engineering problems occurring in the theory of electrical currents, elasticity and in fluid mechanics. One of the main applications of Laplace transform is evaluating the integrals as discussed in the following.

Lemma 4.1. *The following integral relationship holds true*

$$\frac{2}{\pi}\int_0^\infty \frac{\lambda bei(\sqrt{2\lambda})d\lambda}{\lambda^2-\xi^2} = ber(2\sqrt{\xi}).$$

Proof: Let us define the following function

$$I(t) = \frac{2}{\pi}\int_0^\infty \frac{\lambda bei(\sqrt{2t\lambda})d\lambda}{\lambda^2-\xi^2}.$$

Laplace transform of $I(t)$ yields

$$L\{I(t)\} = \int_0^\infty e^{-st}\left(\frac{2}{\pi}\int_0^\infty \frac{\lambda bei(\sqrt{2t\lambda})d\lambda}{\lambda^2-\xi^2}\right)dt.$$

Changing the order of integration, which is permissible, leads to

$$L\{I(t)\} = \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda^2-\xi^2}\left(\int_0^\infty e^{-st}bei(\sqrt{2\lambda t})dt\right)d\lambda,$$

or,

$$L\{I(t)\} = \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda^2-\xi^2}(\frac{1}{s}\sin\frac{\lambda}{2s})d\lambda.$$

After simplifying, we get

$$L\{I(t)\} = \frac{2}{\pi s}\int_0^\infty \frac{\sin\frac{\lambda}{2s}}{\lambda^2-\xi^2}d\lambda.$$

At this point, by using table of integrals or residue theorem, we have the following

$$L\{I(t)\} = \frac{2}{\pi s}\{\frac{\pi}{2}\cos\frac{\xi}{s}\} = \frac{1}{s}\cos\frac{\xi}{s}.$$

Taking inverse Laplace transform of the above relationship, we arrive at

$$I(t) = L^{-1}\{\frac{1}{s}\cos\frac{\xi}{s}\} = ber(2\sqrt{\xi t}).$$

Letting $t = 1$, we get

$$\frac{2}{\pi}\int_0^\infty \frac{\lambda bei(\sqrt{2\lambda})d\lambda}{\lambda^2-\xi^2} = ber(2\sqrt{\xi}).$$

□

Problem 4.2. With the aid of integral representation, we can find inverse Laplace transform of the following object functions

$$f_1(t) = L^{-1}\left\{\frac{e^{-\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}\right\},$$

$$f_2(t) = L^{-1}\left\{\frac{\sinh\sqrt{s^2+a^2}}{\sqrt{s^2+a^2}}\right\},\ (a>0)$$

$$f_3(t) = L^{-1}\left\{\frac{1}{\sqrt{s^2+a^2}(s+\sqrt{s^2+a^2})^{n+\frac{1}{2}}}\right\},\ (\, Re(s) > |Im(a)|)$$

Solution. Using the Sonine integral relation, we have

$$\int_0^\infty \frac{K_\mu(s\sqrt{x^2+z^2})}{(x^2+z^2)^{\frac{\mu}{2}}} x^{\nu+1} J_\nu(ax)dx = \frac{a^\nu}{s^\nu}\frac{(s^2+a^2)^{\frac{\mu-\nu-1}{2}}}{z^{\mu-\nu-1}} K_{\mu-\nu-1}(z\sqrt{s^2+a^2})$$

$$s>0, a>0, Re(\nu) > -1, |\arg(z)| < \frac{\pi}{2}$$

where K_μ is modified Bessel function.

For the proof, we may use the following integral representation for modified Bessel function

$$K_\mu(w) = \frac{1}{2}(\frac{w}{2})^\mu \int_0^\infty e^{-\tau} e^{-\frac{w^2}{4\tau}} \frac{d\tau}{\tau^{\mu+1}}.$$

So that

$$\begin{aligned}
\int_0^\infty & \frac{K_\mu(s\sqrt{x^2+z^2})}{(x^2+z^2)^{\frac{\mu}{2}}}\, x^{\nu+1} J_\nu(ax)dx \\
&= \frac{s^\mu}{2^{\mu+1}} \int_0^\infty \left(\int_0^\infty e^{-\tau} e^{-\frac{s^2(x^2+z^2)}{4\tau}} \frac{d\tau}{\tau^{\mu+1}}\right) x^{\nu+1} J_\nu(ax)dx \\
&= \frac{s^\mu}{2^{\mu+1}} \int_0^\infty e^{-\tau} e^{-\frac{s^2z^2}{4\tau}} \left(\int_0^\infty e^{-\frac{s^2x^2}{4\tau}} x^{\nu+1} J_\nu(ax)dx\right) \frac{d\tau}{\tau^{\mu+1}} \\
&= \frac{s^\mu}{2^{\mu+1}} \int_0^\infty e^{-\tau} e^{-\frac{s^2z^2}{4\tau}} \left(2^{\nu+1} s^{-2\nu-2} \tau^{\nu+1} a^\nu e^{-\frac{a^2\tau}{s^2}}\right) \frac{d\tau}{\tau^{\mu+1}} \\
&= \frac{a^\nu}{2^{\mu-\nu} s^{2\nu+2-\mu}} \int_0^\infty e^{-(\frac{s^2+a^2}{s^2})\tau} e^{-\frac{s^2z^2}{4\tau}} \frac{d\tau}{\tau^{\mu-\nu}}.
\end{aligned}$$

With the change of variable $\frac{s^2+a^2}{s^2}\tau = t$, we obtain

$$\int_0^\infty \frac{K_\mu(s\sqrt{x^2+z^2})}{(x^2+z^2)^{\frac{\mu}{2}}} x^{\nu+1} J_\nu(ax)\, dx$$

$$= \frac{a^{\nu}(s^2+a^2)^{\frac{\mu-\nu-1}{2}}}{s^{\nu}z^{\mu-\nu-1}}\frac{1}{2}(\frac{z\sqrt{s^2+a^2}}{2})^{\mu-\nu-1}\int_0^{\infty} e^{-t}e^{-\frac{z^2(s^2+a^2)}{4t}}\frac{d\tau}{t^{(\mu-\nu-1)+1}}$$

$$= \frac{a^{\nu}(s^2+a^2)^{\frac{\mu-\nu-1}{2}}}{s^{\nu}z^{\mu-\nu-1}}K_{\mu-\nu-1}(z\sqrt{s^2+a^2}).$$

By replacing $\nu = 0$ and $\mu = 0.5$, we get

$$\int_0^{\infty} \frac{e^{-s\sqrt{x^2+z^2}}}{(x^2+z^2)^{\frac{1}{2}}}xJ_0(ax)dx = \frac{e^{-z\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}.$$

So that

$$\begin{aligned} L^{-1}\left\{\frac{e^{-z\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}\right\} &= \int_0^{\infty} \frac{xJ_0(ax)}{(x^2+z^2)^{\frac{1}{2}}}L^{-1}\left\{e^{-s\sqrt{x^2+z^2}}\right\}dx \\ &= \int_0^{\infty} \frac{\delta(t-\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}}xJ_0(ax)dx. \end{aligned}$$

By replacing $u = \sqrt{x^2+z^2}$, we obtain

$$L^{-1}\left\{\frac{e^{-z\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}\right\} = \int_z^{\infty} \delta(t-u)J_0(a\sqrt{u^2-z^2})du = J_0(a\sqrt{t^2-z^2}).$$

Also,

$$\begin{aligned} L^{-1}\left\{\frac{\sinh\sqrt{s^2+a^2}}{\sqrt{s^2+a^2}}\right\} &= \frac{1}{2}\int_{-1}^{1} L^{-1}\left\{e^{-xs}\right\}I_0(a\sqrt{1-x^2})dx \\ &= \frac{1}{2}\int_{-1}^{1} \delta(t-x)I_0(a\sqrt{1-x^2})dx = \frac{1}{2}I_0(a\sqrt{1-t^2}), \end{aligned}$$

$$\begin{aligned} L^{-1}\left\{\frac{1}{\sqrt{s^2+a^2}(s+\sqrt{s^2+a^2})^{n+\frac{1}{2}}}\right\} &= \frac{(-1)^n}{a^{n+\frac{1}{2}}}\int_0^{\infty} L^{-1}\{e^{-xs}\}H_{-n-\frac{1}{2}}(ax)dx \\ &= \frac{(-1)^n}{a^{n+\frac{1}{2}}}H_{-n-\frac{1}{2}}(at), \end{aligned}$$

where H is Struve's function [22].

5. Fractional Oscillations and Fractional Delay Systems

In this section, the authors considered certain time fractional differential equations which are a generalization to the problems of harmonic oscillators studied earlier by many researchers in the literature. In this work, only the Laplace transformation is considered as it is easily understood and being popular among engineers and scientists. The basic goal of this work has been to employ the Laplace transform method for studying the above mentioned problem. The goal has been achieved by formally deriving exact analytical solution. The transform method introduces a significant improvement in this field over existing techniques. The study on fractional calculus equations, i.e., fractional-order differential equation (FODE) and fractional-order integral equation (FOIE) which can describe more accurate behaviors of real physical phenomenon and systems have become a hot topic in the last decades. Fractional derivative provides a perfect tool when it is used to describe the memory and hereditary properties of various materials and processes, this is the main reason that fractional differential equations are being used in modeling mechanical and electrical properties of real materials, rheological properties of rocks, and many other fields. As an important application field of fractional calculus, the topic about fractional-order control and system has attracted many researchers to work on.

Problem 5.1. Solve the following Fractional differential equation

$$ {}_0^C D_t^{2\alpha} y(t) + ky(t) = -\lambda {}_0^C D_t^{\beta} y(t) \ : \ y(0) = y_0, y'(0) = v_0 $$

where $0.5 < \alpha \leq 1$ and $0 < \beta \leq 1$.

Solution. Applying the Laplace transform term wise to the above fractional differential equation, we get

$$ s^{2\alpha} Y - v_0 s^{2\alpha-2} - y_0 s^{2\alpha-1} + kY = -\lambda (s^{\beta} Y - s^{\beta-1} y_0). $$

Therefore

$$ Y = \lambda y_0 \frac{s^{\beta-1}}{s^{2\alpha} + \lambda s^{\beta} + k} + v_0 \frac{s^{2\alpha-2}}{s^{2\alpha} + \lambda s^{\beta} + k} + y_0 \frac{s^{2\alpha-1}}{s^{2\alpha} + \lambda s^{\beta} + k}. $$

The Laplace transform inversion yields

$$ \begin{aligned} y(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Big(\lambda y_0 t^{\beta n - 2\alpha n} E^{(n)}_{\beta, 1-2\alpha n}(-\frac{k}{\lambda} t^{\beta}) \\ &+ v_0 t^{\beta(n+1)+1-2\alpha(n+1)} E^{(n)}_{\beta, \beta+2-2\alpha(n+1)}(-\frac{k}{\lambda} t^{\beta}) \\ &+ y_0 t^{\beta(n+1)-2\alpha(n+1)} E^{(n)}_{\beta, \beta+1-2\alpha(n+1)}(-\frac{k}{\lambda} t^{\beta}) \Big) \end{aligned} $$

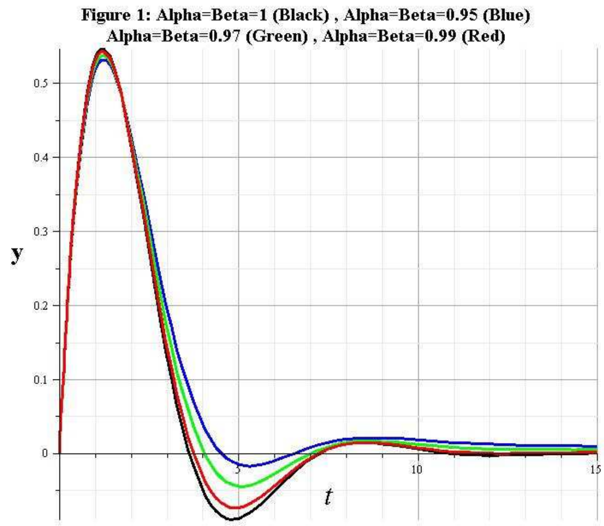
In special case, when $\alpha = \beta$, $v_0 = 1$ and $y_0 = 0$, we get

$$\begin{aligned}
Y(s) &= \frac{s^{2\alpha-2}}{s^{2\alpha}+s^{\alpha}+1} = \frac{s^{2\alpha-2}}{(s^{\alpha}-e^{\frac{2\pi}{3}i})(s^{\alpha}-e^{-\frac{2\pi}{3}i})} \\
&= \frac{1}{\sqrt{3}i}\left(\frac{s^{2\alpha-2}}{s^{\alpha}-e^{\frac{2\pi}{3}i}} - \frac{s^{2\alpha-2}}{s^{\alpha}-e^{-\frac{2\pi}{3}i}}\right) \\
&= L\left(\frac{t^{1-\alpha}}{\sqrt{3}i}\sum_{n=0}^{\infty}\frac{e^{\frac{2n\pi}{3}i}-e^{-\frac{2n\pi}{3}i}}{\Gamma(\alpha n+2-\alpha)}t^{\alpha n}\right) \\
&= L\left(\frac{2}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{\sin(\frac{2n\pi}{3})}{\Gamma(\alpha n+2-\alpha)}t^{\alpha n+1-\alpha}\right)
\end{aligned}$$

that leads to

$$y(t) = \frac{2}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{\sin(\frac{2n\pi}{3})}{\Gamma(\alpha n+2-\alpha)}t^{\alpha n+1-\alpha}.$$

We plot $y(t)$ for $v_0 = k = \lambda = 1$, $y_0 = 0$ and different values of α and β.



Problem 5.2. The vibrations of the mechanical system of two masses attached to three springs with fixed ends are governed by the following fractional differential equations ***with different orders***

$$\begin{cases} {}_0^C D_t^{2\alpha} x + k_1 x = -k(x-y) \\ {}_0^C D_t^{2\beta} y + k_2 y = k(x-y) \end{cases},$$

$$0.5 < \alpha, \beta \leq 1, y'(0) = -x'(0) = -\sqrt{3k}$$

where k, k_1 and k_2 are the springs modulus of each of the three springs and $x(t)$, $y(t)$ are the displacements of the masses from their position of static equilibrium. The masses of springs and the damping are neglected.

Let us assume that $L\{x(t)\} = X(s)$, $L\{y(t)\} = Y(s)$. Using the Laplace transform,

$$(s^{2\alpha} + k_1 + k)X - s^{2\alpha-2}x'(0) - s^{2\alpha-1}x(0) = kY,$$
$$(s^{2\beta} + k_2 + k)Y - s^{2\beta-2}y'(0) - s^{2\beta-1}y(0) = kX.$$

Then

$$\begin{aligned} X &= \frac{s^{2\alpha+2\beta-2} + (k+k_2)s^{2\alpha-2}}{s^{2\alpha+2\beta} + (k+k_2)s^{2\alpha} + (k+k_1)s^{2\beta} + k(k_1+k_2) + k_1k_2}x'(0) \\ &+ \frac{s^{2\alpha+2\beta-1} + (k+k_2)s^{2\alpha-1}}{s^{2\alpha+2\beta} + (k+k_2)s^{2\alpha} + (k+k_1)s^{2\beta} + k(k_1+k_2) + k_1k_2}x(0) \\ &- \frac{ks^{2\beta-2}}{s^{2\alpha+2\beta} + (k+k_2)s^{2\alpha} + (k+k_1)s^{2\beta} + k(k_1+k_2) + k_1k_2}y'(0) \\ &- \frac{ks^{2\beta-1}}{s^{2\alpha+2\beta} + (k+k_2)s^{2\alpha} + (k+k_1)s^{2\beta} + k(k_1+k_2) + k_1k_2}y(0), \end{aligned}$$

$$\begin{aligned} Y &= \frac{s^{2\alpha+2\beta-2} + (k+k_1)s^{2\beta-2}}{s^{2\alpha+2\beta} + (k+k_2)s^{2\alpha} + (k+k_1)s^{2\beta} + k(k_1+k_2) + k_1k_2}y'(0) \\ &+ \frac{s^{2\alpha+2\beta-1} + (k+k_1)s^{2\beta-1}}{s^{2\alpha+2\beta} + (k+k_2)s^{2\alpha} + (k+k_1)s^{2\beta} + k(k_1+k_2) + k_1k_2}y(0) \\ &- \frac{ks^{2\alpha-2}}{s^{2\alpha+2\beta} + (k+k_2)s^{2\alpha} + (k+k_1)s^{2\beta} + k(k_1+k_2) + k_1k_2}x'(0) \\ &- \frac{ks^{2\alpha-1}}{s^{2\alpha+2\beta} + (k+k_2)s^{2\alpha} + (k+k_1)s^{2\beta} + k(k_1+k_2) + k_1k_2}x(0). \end{aligned}$$

We may obtain

$$L^{-1}\left\{\frac{1}{s^{2\alpha+2\beta} + (k+k_2)s^{2\alpha} + (k+k_1)s^{2\beta} + k(k_1+k_2) + k_1k_2}\right\} =$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^n(k+k_1)^m(k(k_1+k_2)+k_1k_2)^{n-m}}{n!}t^{2\beta(1-m)+2\alpha(n+1)}$$

$$\times E^{(n)}_{2\beta,2\alpha(n+1)+2\beta(1-m)}(-(k+k_2)t^{2\beta}).$$

Therefore, applying the Laplace transform inversion, we get

$$\begin{aligned}
x(t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^n(k+k_1)^m(k(k_1+k_2)+k_1k_2)^{n-m}}{n!} \\
&\times \Big\{\Big(t^{2\beta(n-m)+2\alpha n+1}E^{(n)}_{2\beta,2\alpha n-2\beta m+2}(-(k+k_2)t^{2\beta}) \\
&+ (k+k_2)t^{2\beta(n-m+1)+2\alpha n+1}E^{(n)}_{2\beta,2\alpha n+2\beta(1-m)+2}(-(k+k_2)t^{2\beta})\Big)\,x'(0) \\
&+ \Big(t^{2\beta(n-m)+2\alpha m}E^{(n)}_{2\alpha,2\alpha n-2\beta m+1}(-(k+k_2)t^{2\beta}) \\
&+ (k+k_2)t^{2\beta(n-m+1)+2\alpha n}E^{(n)}_{2\beta,2\alpha n+2\beta(1-m)+1}(-(k+k_2)t^{2\beta})\Big)\,x(0) \\
&- \Big(kt^{2\beta(n-m)+2\alpha(n+1)+1}E^{(n)}_{2\beta,2\alpha(n+1)-2\beta m+2}(-(k+k_2)t^{2\beta})\Big)\,y'(0) \\
&- \Big(kt^{2\beta(n-m)+2\alpha(n+1)}E^{(n)}_{2\beta,2\alpha(n+1)-2\beta m+1}(-(k+k_2)t^{2\beta})\Big)\,y(0)\Big\},
\end{aligned}$$

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^n(k+k_1)^m(k(k_1+k_2)+k_1k_2)^{n-m}}{n!} \\
&\times \Big\{\Big(t^{2\beta(n-m)+2\alpha n+1}E^{(n)}_{2\beta,2\alpha n-2\beta m+2}(-(k+k_2)t^{2\beta}) \\
&+ (k+k_1)t^{2\beta(n-m)+2\alpha(n+1)+1}E^{(n)}_{2\beta,2\alpha(n+1)-2\beta m+2}(-(k+k_2)t^{2\beta})\Big)\,y'(0) \\
&+ \Big(t^{2\beta(n-m)+2\alpha m}E^{(n)}_{2\alpha,2\alpha n-2\beta m+1}(-(k+k_2)t^{2\beta}) \\
&+ (k+k_1)t^{2\beta(n-m)+2\alpha(n+1)}E^{(n)}_{2\beta,2\alpha(n+1)-2\beta m+1}(-(k+k_2)t^{2\beta})\Big)\,y(0) \\
&- \Big(kt^{2\beta(n-m+1)+2\alpha n+1}E^{(n)}_{2\beta,2\alpha n+2\beta(1-m)+2}(-(k+k_2)t^{2\beta})\Big)\,y'(0) \\
&- \Big(kt^{2\beta(n-m+1)+2\alpha n}E^{(n)}_{2\beta,2\alpha n+2\beta(1-m)+1}(-(k+k_2)t^{2\beta})\Big)\,y(0)\Big\}.
\end{aligned}$$

Problem 5.3.A) Consider the following time fractional delay differential equation

$$ {}_0^C D_t^{\alpha} y + ky(t-\lambda) = 1 \ : \ 0<\alpha\le 1, \lambda\ge 0$$

with the initial condition

$$y(t) = 0 \ : \ -\lambda \leq t \leq 0.$$

Solution. We apply Laplace transform to obtain

$$s^{\alpha} Y(s\ ) + k\, e^{-\lambda s} Y(s) = \frac{1}{s},$$

that leads to

$$Y(s) = \frac{1}{s} \frac{1}{s^{\alpha} + k\, e^{-\lambda s}}.$$

Then

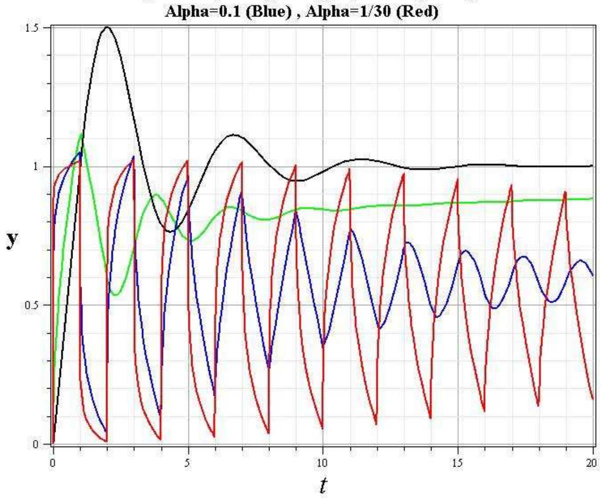
$$\begin{aligned} y(t) &= L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s}\frac{1}{s^{\alpha}+ke^{-\lambda s}}\right\} = L^{-1}\left\{\frac{1}{s^{\alpha+1}}\frac{1}{1+k\frac{e^{-\lambda s}}{s^{\alpha}}}\right\} \\ &= \sum_{n=0}^{\infty}(-k)^n L^{-1}\left\{\frac{e^{-\lambda n s}}{s^{n\alpha+\alpha+1}}\right\} = \sum_{n=0}^{\left[\frac{t}{\lambda}\right]}(-k)^n \frac{(t-\lambda n)^{n\alpha+\alpha}}{\Gamma(n\alpha+\alpha+1)}. \end{aligned}$$

Note. We use the fact that

$$L^{-1}\{\frac{e^{-as}}{s^{\nu}}\} = \begin{cases} \frac{(t-a)^{\nu-1}}{\Gamma(\nu)} & : t \geq a \\ 0 & : t < a \end{cases}.$$

In case of $\lambda = k = 1$, we have the following figure for different values of α.

Figure 2: Alpha=1 (Black) , Alpha=0.5 (Green)
Alpha=0.1 (Blue) , Alpha=1/30 (Red)



5.3.B) Let us consider the following fractional difference equation

$$ {}_0^C D_t^\alpha y + k_1 y(t-\lambda_1) + k_2 y(t-\lambda_2) = 1 \; : \; 0<\alpha\le 1, \lambda_i \ge 0 $$

with the initial condition

$$ y(t) = 0 \; : \; -\max(\lambda_i) \le t \le 0. $$

Solution. By using Laplace transform, we have

$$ Y(s) = \frac{1}{s}\frac{1}{s^\alpha + k_1 e^{-\lambda_1 s} + k_2 e^{-\lambda_2 s}}. $$

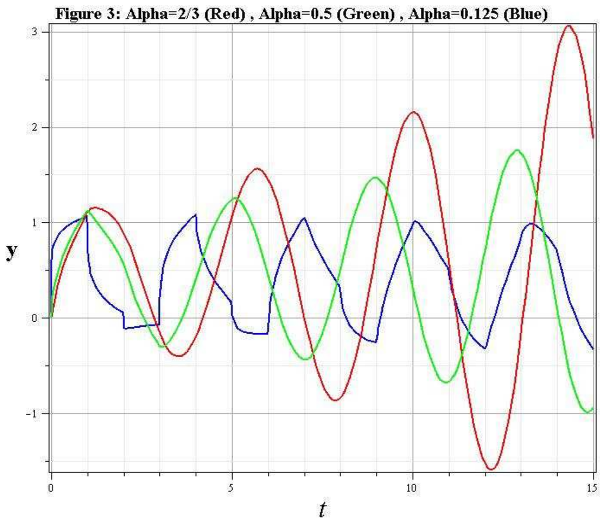
Hence

$$ \begin{aligned} y(t) &= L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s}\frac{1}{s^\alpha + k_1 e^{-\lambda_1 s} + k_2 e^{-\lambda_2 s}}\right\} \\ &= L^{-1}\left\{\frac{1}{s^{\alpha+1}}\frac{1}{1+\frac{k_1 e^{-\lambda_1 s}+k_2 e^{-\lambda_2 s}}{s^\alpha}}\right\} \\ &= \sum_{n=0}^{\infty}(-1)^n L^{-1}\left\{\frac{(k_1 e^{-\lambda_1 s}+k_2 e^{-\lambda_2 s})^n}{s^{n\alpha+\alpha+1}}\right\}. \end{aligned} $$

Without loss of generality, assume that $\lambda_1 < \lambda_2$. Then

$$ \begin{aligned} y(t) &= \sum_{n=0}^{\infty}(-1)^n k_1^n L^{-1}\left\{\frac{e^{-n\lambda_1 s}(1+\frac{k_2}{k_1}e^{-(\lambda_2-\lambda_1)s})^n}{s^{n\alpha+\alpha+1}}\right\} \\ &= \sum_{n=0}^{\infty}(-1)^n k_1^n L^{-1}\left\{\frac{e^{-n\lambda_1 s}}{s^{n\alpha+\alpha+1}}\sum_{m=0}^{n}\binom{n}{m}(\frac{k_2}{k_1})^m e^{-m(\lambda_2-\lambda_1)s}\right\} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^n\binom{n}{m}k_1^{n-m}k_2^m L^{-1}\left\{\frac{e^{-((n-m)\lambda_1+m\lambda_2)s}}{s^{n\alpha+\alpha+1}}\right\} \\ &= \sum_{n=0}^{\left[\frac{t}{\lambda_1}\right]}\sum_{m=0}^{\left[\frac{t-n\lambda_1}{\lambda_2-\lambda_1}\right]}(-1)^n\binom{n}{m}k_1^{n-m}k_2^m\frac{(t-n\lambda_1-m(\lambda_2-\lambda_1))^{n\alpha+\alpha}}{\Gamma(n\alpha+\alpha+1)}. \end{aligned} $$

For the special case $\lambda_1 = k_1 = k_2 = 1$ and $\lambda_2 = 2$, we plot $y(t)$.



Problem 5.4. Let us solve the following system of fractional delay differential equation

$$\begin{cases} {}_0^C D_t^\alpha y - z(t-T) = 1 \\ {}_0^C D_t^\alpha z + y(t-T) = 1 \end{cases},$$

where $t > 0$, $0 < \alpha \le 1$, $T > 0$ and $y = z \equiv 0$ $(-T \le t \le 0)$.
Solution. Setting $w = y + iz$, one has

$${}_0^C D_t^\alpha w(t) + iw(t-T) = 1 + i.$$

Applying the Laplace transform term wise on both sides of the above relation and using boundary conditions, we obtain

$$s^\alpha W(s) + e^{-Ts} iW(s) = \frac{1+i}{s},$$

which gives

$$W(s) = \frac{1}{s}\frac{1+i}{s^\alpha + ie^{-Ts}} = (1+i)\sum_{n=0}^{\infty} \frac{(-1)^n i^n e^{-nTs}}{s^{n\alpha+\alpha+1}}.$$

By the convolution theorem, we have

$$\begin{aligned} w(t) &= L^{-1}\{W(s)\} = (1+i)\sum_{n=0}^{\infty}(-1)^n i^n L^{-1}\left(\frac{e^{-nTs}}{s^{n\alpha+\alpha+1}}\right) \\ &= (1+i)\sum_{n=0}^{\infty}\frac{(-1)^n i^n}{\Gamma(n\alpha+\alpha+1)}(t-nT)^{n\alpha+\alpha}. \end{aligned}$$

Thus

$$w(t) = \sqrt{2}\sum_{n=0}^{\left[\frac{t}{T}\right]} \frac{(-1)^n e^{(\frac{n\pi}{2}+\frac{\pi}{4})i}}{\Gamma(n\alpha+\alpha+1)}(t-nT)^{n\alpha+\alpha},$$

that leads to

$$y(t) = \sqrt{2}\sum_{n=0}^{\left[\frac{t}{T}\right]} \frac{(-1)^n \cos(\frac{n\pi}{2}+\frac{\pi}{4})}{\Gamma(n\alpha+\alpha+1)}(t-nT)^{n\alpha+\alpha},$$

$$z(t) = \sqrt{2}\sum_{n=0}^{\left[\frac{t}{T}\right]} \frac{(-1)^n \sin(\frac{n\pi}{2}+\frac{\pi}{4})}{\Gamma(n\alpha+\alpha+1)}(t-nT)^{n\alpha+\alpha}.$$

We have shown $y(t)$ and $z(t)$ when $T = 1$.

Figure 4: Alpha=0.05

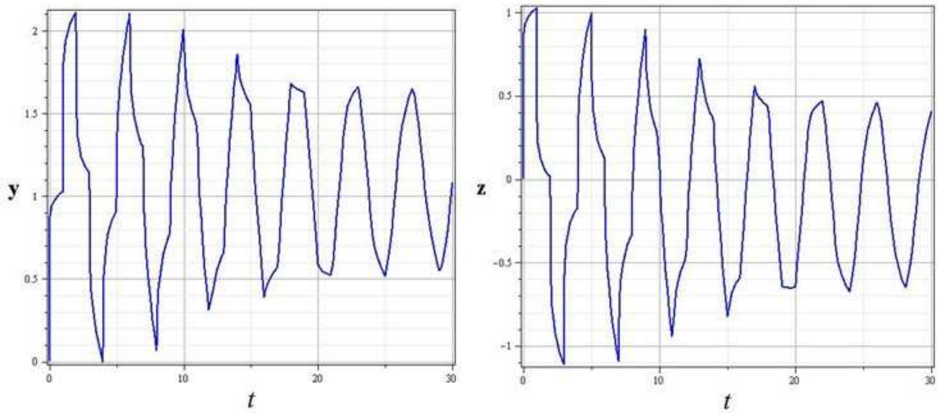
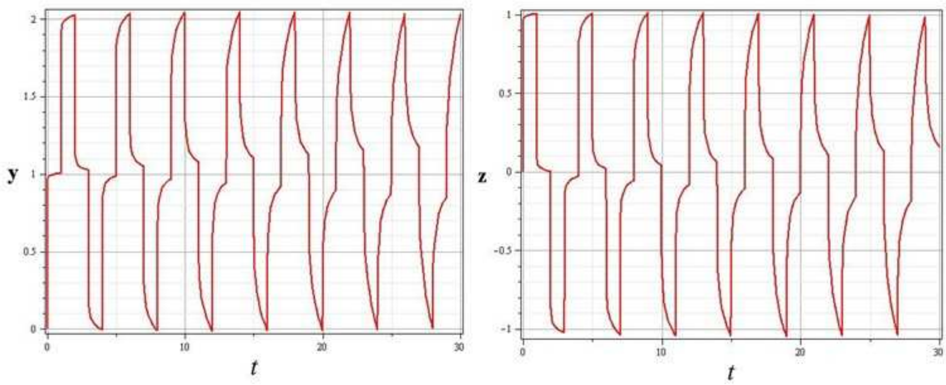


Figure 5: Alpha=0.01



6. Conclusion

In recent years, integral transforms have become essential working tools of every engineer and applied scientists. The Laplace transform, which undoubtedly is the most familiar example, is basic to the solution of initial value problems. In this article, the authors used the Hybrid perturbation-Laplace transform method to solve certain linear and non-linear systems of fractional differential and difference equations with constant coefficients. We also considered the problems of string vibrations in different cases with fractional damping. Constructive examples are also provided.

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