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The Almost Lacunary χ^2 sequence spaces defined by modulus

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abstract: In this paper we introduce a new concept for almost lacunary χ^2 sequence spaces strong $P-$ convergent to zero with respect to an modulus function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of almost lacunary χ^2 sequence spaces and also some inclusion theorems are discussed.

Key Words: analytic sequence, modulus function, double sequences, chi sequence.

Contents

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1\, for\, some\, l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

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$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n \to \infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gӧkhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all finite sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Orlicz [13] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \leq p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], Bektas and Altin [3], Tripathy et al. [18], Rao and Subramanian [15], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)\,(u \geq 0)$. The Δ_2- condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of u and for $\ell > 1$.

Remark 1.1. *An Orlicz function satisfies the inequality* $M(\lambda x) \leq \lambda M(x)$ *for all* λ *with* $0 < \lambda \leq 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\, for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \le p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.
If X is a sequence space, we give the following definitions:

(i)$X' =$ the continuous dual of X;

(ii)$X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty,\ for\, each\, x \in X\right\};$

(iii)$X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent,\ foreach\, x \in X\right\};$

(iv)$X^\gamma = \left\{a = (a_{mn}) : sup_{mn} \ge 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, foreachx \in X\right\};$

(v)$let\, X\, beanFK - space \supset \phi;\, then\, X^f = \left\{f(\Im_{mn}) : f \in X'\right\};$

(vi)$X^\delta = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty,\ foreach\, x \in X\right\};$

$X^\alpha.X^\beta, X^\gamma$ are called $\alpha - (orK\ddot{o}the - Toeplitz)$dual of $X, \beta - (or\, generalized - K\ddot{o}the - Toeplitz)\, dual\, of\, X, \gamma - \ dual\, of\, X,\ \delta\ -\ dual\, of\, X\, respectively. X^\alpha$ is defined by Gupta and Kamptan [20]. It is clear that $x^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\beta \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \le p \le \infty$ by BaŞar and Altay in [42] and in the case $0 < p < 1$ by Altay and BaŞar in [43]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\ge1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \le p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

2. Definitions and Preliminaries

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{mn})$ has Pringsheim limit 0 (denoted by $P - limx = 0$) (i.e) $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. We shall write more briefly as "$'P - convergent\, to\, 0$"$'$

Definition 2.1. *A modulus function was introduced by Nakano [12]. We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$, such that*
(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from here that f is continuous on $[0,\infty)$.

Definition 2.2. *Let $A = \left(a_{k,\ell}^{mn}\right)$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k,\ell-$ th term to Ax is as follows:*

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$$

such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [40] and Toeplitz [41]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is $P-$ convergent is not necessarily bounded.

Definition 2.3. *A double sequence $x = (x_{mn})$ of real numbers is called almost $P-$ convergent to a limit 0 if*

$$P - lim_{p,q\to\infty} sup_{r,s\geq 0} \frac{1}{pq} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \left((m+n)!\,|x_{mn}|\right)^{1/m+n} \to 0.$$

that is, the average value of (x_{mn}) taken over any rectangle
$\{(m,n) : r \leq m \leq r+p-1, s \leq n \leq s+q-1\}$ tends to 0 as both p and q to ∞, and this $P-$ convergence is uniform in r and s. Let denot the set of sequences with this property as $\left[\widehat{\chi^2}\right]$.

By a lacunary $\theta = (m_k)$; $k = 0, 1, 2, \cdots$ where $m_0 = 0$, we shall mean an incrasing sequence of non-negative integers with $m_k - m_{k-1}$ as $k \to \infty$. The intervals determined by θ will be denoted by $I_k = (m_{k-1}, m_r]$ and $h_k = m_r - m_{r-1}$. The ratio $\frac{m_k}{m_{k-1}}$ will be denoted by q_k.

Definition 2.4. *The double sequence $\theta_{k,\ell} = \{(m_k, n_\ell)\}$ is called double lacunary if there exist two increasing sequences of integers such that*

$$m_0 = 0, h_k = m_k - m_{r-1} \to \infty \text{ as } k \to \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$

Let $m_{k,\ell} = m_k n_\ell$, $h_{k,\ell} = h_k \overline{h_\ell}$, and $\theta_{k,\ell}$ is determine by
$I_{k,\ell} = \{(m,n) : m_{k-1} < m < m_k \,\text{and}\, n_{\ell-1} < n \leq n_\ell\}$, $q_k = \frac{m_k}{m_{k-1}}$, $\overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}$.

Definition 2.5. *Let f be an modulus function and $P = (p_{mn})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence space:* $\chi_f^2 \left[AC_{\theta_{k,\ell}}, P\right] =$
$\Big\{P - lim_{k,\ell} \frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell}} \left[f\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} = 0,$
$uniformly\ in\ r\ and\ s.\}$

We shall denote $\chi_f^2 \left[AC_{\theta_{k,\ell}}, P\right]$ as $\chi^2 \left[AC_{\theta_{k,\ell}}, P\right]$ respectively when $p_{mn} = 1$ for all m and n. If x is in $\chi^2 \left[AC_{\theta_{k,\ell}}, P\right]$, we shall say that x is almost lacunary χ^2 strongly $P-$convergent with respect to the modulus function f. Also note if $f(x) = x, p_{mn} = 1$ for all m and n, then $\chi_f^2 \left[AC_{\theta_{k,\ell}}, P\right] = \chi^2 \left[AC_{\theta_{k,\ell}}\right]$ which are defined as follows: $\chi^2 \left[AC_{\theta_{k,\ell}}\right] =$
$\Big\{P - lim_{k,\ell} \frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell}} \left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right] = 0,$
$uniformly\ in\ r\ and\ s.\}$
Again note if $p_{mn} = 1$ for all m and n, then $\chi_f^2 \left[AC_{\theta_{k,\ell}}, P\right] = \chi_f^2 \left[AC_{\theta_{k,\ell}}\right]$. we define $\chi_f^2 \left[AC_{\theta_{k,\ell}}, P\right] =$
$\Big\{P - lim_{k,\ell} \frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell}} \left[f\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} = 0,$
$uniformly\ in\ r\ and\ s.\}$

Definition 2.6. *Let f be an modulus function $P = (p_{mn})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence space:* $\chi_f^2 [P] =$
$\Big\{P - lim_{p,q \to \infty} \frac{1}{pq} \sum_{m=1}^{p} \sum_{n=1}^{q} \left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} = 0,$
$uniformly\ in\ r\ and\ s.\}$

If we take $f(x) = x, p_{mn} = 1$ for all m and n, then $\chi_f^2 [P] = \chi^2$.

Definition 2.7. *Let $\theta_{k,\ell}$ be a double lacunary sequence; the double number sequence x is $\widehat{S_{\theta k,\ell}} - P-$ convergent to 0 then*

$$P - lim_{k,\ell} \frac{1}{h_{k,\ell}} max_{r,s} \left|\left\{(m,n) \in I_{k,\ell} : \left((m+n)! \left|x_{m+r,n+s} - 0\right|\right)^{1/m+n}\right\}\right| = 0.$$

In this case we write $\widehat{S_{\theta k,\ell}} - lim \left((m+n)! \left|x_{m+r,n+s} - 0\right|\right)^{1/m+n} = 0.$

3. Main Results

Theorem 3.1. *If f be any modulus function and a bounded factorable positive double number sequence p_{mn} then $\chi_f^2 \left[AC_{\theta_{k,\ell}}, P\right]$ is linear space*

Proof: The proof is easy. Theorefore omit the proof. □

Lemma 3.2. *Let f be an modulus function which sastisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $f(x) < K\delta^{-1} f(2)$ for some constant $K > 0$.*

Theorem 3.3. *For any modulus function f which satisfies Δ_2- condition we have* $\chi^2 \left[AC_{\theta_{k,\ell}}\right] \subset \chi_f^2 \left[AC_{\theta_{k,\ell}}\right]$

Proof: Let $x \in \chi^2 \left[AC_{\theta_{k,\ell}}\right]$ so that for each r and s
$\chi^2 \left[AC_{\theta_{k,\ell}}\right] = \left\{P - lim_{k,\ell} \frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell}} \left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right] = 0\right\}.$
Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for every t with $0 \leq t \leq \delta$. We obtain the following,

$\frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell}} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]$
$= \frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell} \, and \, |x_{m+r,n+s}-0| \leq \delta} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right] +$
$\frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell} \, and \, |x_{m+r,n+s}-0| > \delta} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right] \leq$
$\frac{1}{h_{k\ell}}\left(h_{k\ell}\epsilon\right) + \frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell} \, and \, |x_{m+r,n+s}-0| > \delta} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]$
$\frac{1}{h_{k\ell}}\left(h_{k\ell}\epsilon\right) + \frac{1}{h_{k\ell}} K \delta^{-1} f(2) h_{k\ell} \chi^2 \left[AC_{\theta_{k,\ell}}\right].$

Therefore by 3.2 as k and ℓ goes to infinity in the Pringsheim sense, for each r and s we are granted $x \in \chi^2_f \left[AC_{\theta_{k,\ell}}\right]$. □

Theorem 3.4. *Let* $\theta_{k,\ell} = \{m_k, n_\ell\}$ *be a double lacunary sequence with* $liminf_k q_k > 1$ *and* $liminf_\ell \overline{q_\ell} > 1$ *then for any modulus function* f, $\chi^2_f(P) \subset \chi^2_f\left(AC_{\theta_{k,\ell}}, P\right)$

Proof: Suppose $liminf_k q_k > 1$ and $liminf_\ell \overline{q_\ell} > 1$; then there exists $\delta > 0$ such that $q_k > 1 + \delta$ and $\overline{q_\ell} > 1 + \delta$. This implies $\frac{h_k}{m_k} \geq \frac{\delta}{1+\delta}$ and $\frac{h_\ell}{n_\ell} \geq \frac{\delta}{1+\delta}$. Then for $x \in \chi^2_f(P)$, we can write for each r and s.

$$B_{k,\ell} = \frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell}} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} =$$
$$\frac{1}{h_{k\ell}} \sum_{m=1}^{m_k} \sum_{n=1}^{n_\ell} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} -$$
$$\frac{1}{h_{k\ell}} \sum_{m=1}^{m_{k-1}} \sum_{n=1}^{n_{\ell-1}} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} -$$
$$\frac{1}{h_{k\ell}} \sum_{m=m_{k-1}+1}^{m_k} \sum_{n=1}^{n_{\ell-1}} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} -$$
$$\frac{1}{h_{k\ell}} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}$$
$$= \frac{m_k n_\ell}{h_{k\ell}}\left(\frac{1}{m_k n_\ell} \sum_{m=1}^{m_k} \sum_{n=1}^{n_\ell} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right) -$$
$$\frac{m_{k-1} n_{\ell-1}}{h_{k\ell}}\left(\frac{1}{m_{k-1} n_{\ell-1}} \sum_{m=1}^{m_{k-1}} \sum_{n=1}^{n_{\ell-1}} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right) -$$
$$\frac{n_{\ell-1}}{h_{k\ell}}\left(\frac{1}{n_{\ell-1}} \sum_{m=m_{k-1}+1}^{m_k} \sum_{n=1}^{n_{\ell-1}} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right) -$$
$$\frac{m_{k-1}}{h_{k\ell}}\left(\frac{1}{m_{k-1}} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right).$$

Since $x \in \chi^2_f(P)$ the last two terms tend to zero uniformly in m, n in the Pringsheim sense, thus, for each r and s

$$B_{k,\ell} = \frac{m_k n_\ell}{h_{k\ell}}\left(\frac{1}{m_k n_\ell} \sum_{m=1}^{m_k} \sum_{n=1}^{n_\ell} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right)$$
$$- \frac{m_{k-1} n_{\ell-1}}{h_{k\ell}}\left(\frac{1}{m_{k-1} n_{\ell-1}} \sum_{m=1}^{m_{k-1}} \sum_{n=1}^{n_{\ell-1}} f\left[\left((m+n)! \left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right)$$
$$+ o(1).$$

Since $h_{k\ell} = m_k n_\ell - m_{k-1} n_{\ell-1}$ we are granted for each r and s the following

$$\frac{m_k n_\ell}{h_{k\ell}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{m_{k-1} n_{\ell-1}}{h_{k\ell}} \leq \frac{1}{\delta}.$$

The terms
$\left(\frac{1}{m_k n_\ell}\sum_{m=1}^{m_k}\sum_{n=1}^{n_\ell} f\left[\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right)$ and
$\left(\frac{1}{m_{k-1} n_{\ell-1}}\sum_{m=1}^{m_{k-1}}\sum_{n=1}^{n_{\ell-1}} f\left[\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right)$ are both Pringsheim gai sequences for all r and s. Thus $B_{k\ell}$ is a pringsheim gai sequence for each r and s. Hence $x \in \chi_f^2\left(AC_{\theta_{k,\ell}}, P\right)$. □

Theorem 3.5. *Let $\theta_{k,\ell} = \{m, n\}$ be a double lacunary sequence with $limsup_k q_k < \infty$ and $limsup_k \overline{q_k} < \infty$ then for any modulus function f, $\chi_f^2\left(AC_{\theta_{k,\ell}}, P\right) \subset \chi_f^2(p)$.*

Proof: Since $limsup_k q_k < \infty$ and $limsup_k \overline{q_k} < \infty$ there exists $H > 0$ such that $q_k < H$ and $\overline{q_\ell} < H$ for all k and ℓ. Let $x \in \chi_f^2\left(AC_{\theta_{k,\ell}}, P\right)$. Also there exist $k_0 > 0$ and $\ell_0 > 0$ such that for every $i \geq k_0$ and $j \geq \ell_0$ and r and s,

$$A_{ij}^{'} = \frac{1}{h_{ij}}\sum_{m\in I_{i,j}}\sum_{n\in I_{i,j}} f\left[\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} \to 0\, as\, m, n \to \infty.$$

Let $G^{'} = max\left\{A_{i,j}^{'} : 1 \leq i \leq k_0\, and\, 1 \leq j \leq \ell_0\right\}$, and p and q be such that $m_{k-1} < p \leq m_k$ and $n_{\ell-1} < q \leq n_\ell$. Thus we obtain the following:

$$\frac{1}{pq}\sum_{m=1}^{p}\sum_{n=1}^{q}\left[\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}$$
$$\leq \frac{1}{m_{k-1}n_{\ell-1}}\sum_{m=1}^{m_k}\sum_{n=1}^{n_\ell}\left[\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}$$
$$\leq \frac{1}{m_{k-1}n_{\ell-1}}\sum_{t=1}^{k}\sum_{u=1}^{\ell}\left(\sum_{m\in I_{t,u}}\sum_{n\in I_{t,u}}\left[\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}}\right)$$
$$= \frac{1}{m_{k-1}n_{\ell-1}}\sum_{t=1}^{k_0}\sum_{u=1}^{\ell_0} h_{t,u}A_{t,u}^{'} + \frac{1}{m_{k-1}n_{\ell-1}}\sum_{(k_0<t\leq k)\bigcup(\ell_0<u\leq\ell)} h_{t,u}A_{t,u}^{'}$$
$$\leq \frac{G^{'}}{m_{k-1}n_{\ell-1}}\sum_{t=1}^{k_0}\sum_{u=1}^{\ell_0} h_{t,u} + \frac{1}{m_{k-1}n_{\ell-1}}\sum_{(k_0<t\leq k)\bigcup(\ell_0<u\leq\ell)} h_{t,u}A_{t,u}^{'}$$
$$\leq \frac{G^{'}m_{k_0}n_{\ell_0}k_0\ell_0}{m_{k-1}n_{\ell-1}} + \frac{1}{m_{k-1}n_{\ell-1}}\sum_{(k_0<t\leq k)\bigcup(\ell_0<u\leq\ell)} h_{t,u}A_{t,u}^{'}$$
$$\leq \frac{G^{'}m_{k_0}n_{\ell_0}k_0\ell_0}{m_{k-1}n_{\ell-1}} + \left(sup_{t\geq k_0\bigcup u\geq\ell_0}A_{t,u}^{'}\right)\frac{1}{m_{k-1}n_{\ell-1}}\sum_{(k_0<t\leq k)\bigcup(\ell_0<u\leq\ell)} h_{t,u}$$
$$\leq \frac{G^{'}m_{k_0}n_{\ell_0}k_0\ell_0}{m_{k-1}n_{\ell-1}} + \frac{\epsilon}{m_{k-1}n_{\ell-1}}\sum_{(k_0<t\leq k)\bigcup(\ell_0<u\leq\ell)} h_{t,u}$$
$$\leq \frac{G^{'}m_{k_0}n_{\ell_0}k_0\ell_0}{m_{k-1}n_{\ell-1}} + \epsilon H^2.$$

Since m_k and ℓ_ℓ both approaches infinity as both p and q approaches infinity, it follows that

$$\frac{1}{pq}\sum_{m=1}^{p}\sum_{n=1}^{q}\left[\left((m+n)!\left|x_{m+r,n+s}\right|\right)^{1/m+n}\right]^{p_{mn}} = 0,\, uniformly\, in\, r\, and\, s.$$

Hence $x \in \chi_f^2(P)$. □

Theorem 3.6. *Let $\theta_{k,\ell} = \{m, n\}$ be a double lacunary sequence with $1 < liminf_{k,\ell}q_{k,\ell} \leq limsup_k q_k < \infty$, then for any modulus function f, $\chi_f^2\left(AC_{\theta_{k,\ell}}, P\right) = \chi_f^2(p)$.*

Theorem 3.7. *Let $\theta_{k,\ell}$ be a double lacunary sequence then*
(i) $(x_{mn}) \stackrel{P}{\to} \chi^2\left(\widehat{S_{\theta_{k,\ell}}}\right)$
(ii)$\left(AC_{\theta_{k,\ell}}\right)$ *is a proper subset of* $\left(\widehat{S_{\theta_{k,\ell}}}\right)$
(iii) If $x \in \Lambda^2$ *and* $(x_{mn}) \stackrel{P}{\to} \chi^2\left(\widehat{S_{\theta_{k,\ell}}}\right)$ *then* $(x_{mn}) \stackrel{P}{\to} \chi^2\left(AC_{\theta_{k,\ell}}\right)$
(iv) $\chi^2\left(\widehat{S_{\theta_{k,\ell}}}\right) \bigcap \Lambda^2 = \chi^2\left[AC_{\theta_{k,\ell}}\right] \bigcap \Lambda^2$.

Proof: (i) Since for all r and s
$\left|\left\{(m,n) \in I_{k,\ell} : ((m+n)!\,|x_{m+r,n+s} - 0|)^{1/m+n}\right\} = 0\right| \leq$
$\sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell}\, and\, |x_{m+r,n+s}|=0} ((m+n)!\,|x_{m+r,n+s} - 0|)^{1/m+n} \leq$
$\sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell}} ((m+n)!\,|x_{m+r,n+s} - 0|)^{1/m+n}$, for all r and s
$P - lim_{k,\ell} \frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell}} ((m+n)!\,|x_{m+r,n+s} - 0|)^{1/m+n} = 0$
This implies that for all r and s

$$P - lim_{k,\ell} \frac{1}{h_{k,\ell}} \left|\left\{(m,n) \in I_{k,\ell} : ((m+n)!\,|x_{m+r,n+s} - 0|)^{1/m+n} = 0\right\}\right| = 0.$$

(ii)let $x = (x_{mn})$ be defined as follows:

$$(x_{mn}) = \begin{pmatrix} 1 & 2 & 3 & \ldots\frac{\left[\sqrt[3]{h_{k,\ell}}\right]^{m+n}}{(m+n)!} & 0 & \ldots \\ 1 & 2 & 3 & \ldots\frac{\left[\sqrt[3]{h_{k,\ell}}\right]^{m+n}}{(m+n)!} & 0 & \ldots \\ . \\ . \\ . \\ 1 & 2 & 3 & \ldots\frac{\left[\sqrt[3]{h_{k,\ell}}\right]^{m+n}}{(m+n)!} & 0 & \ldots \\ . \\ . \\ . \\ 0 & 0 & 0 & \ldots 0 & 0 & \ldots \\ . \\ . \\ . \end{pmatrix};$$

Here x is an double sequence and for all r and s
$P - lim_{k,\ell} \frac{1}{h_{k,\ell}} \left|\left\{(m,n) \in I_{k,\ell} : ((m+n)!\,|x_{m+r,n+s} - 0|)^{1/m+n} = 0\right\}\right| =$
$P - lim_{k,\ell} \frac{1}{h_{k,\ell}} \left(\frac{(m+n)!\left[\sqrt[3]{h_{k,\ell}}\right]^{m+n}}{(m+n)!}\right)^{1/m+n} = 0.$
Therefore $(x_{mn}) \stackrel{P}{\to} \chi^2\left(\widehat{S_{\theta_{k,\ell}}}\right)$. Also
$P - lim_{k,\ell} \frac{1}{h_{k\ell}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{k,\ell}} ((m+n)!\,|x_{m+r,n+s}|)^{1/m+n} =$
$P - \frac{1}{2}\left(lim_{k,\ell} \frac{1}{h_{k,\ell}} \left(\frac{(m+n)!\left[\sqrt[3]{h_{k,\ell}}\right]^{m+n}\left[\sqrt[3]{h_{k,\ell}}\right]^{m+n}\left[\sqrt[3]{h_{k,\ell}}\right]^{m+n}}{(m+n)!}\right)^{1/m+n} + 1\right) = \frac{1}{2}.$

Therefore $(x_{mn}) \overset{P}{\nrightarrow} \chi^2\left(AC_{\theta_{k,\ell}}\right)$.
(iii) If $x \in \Lambda^2$ and $(x_{mn}) \overset{P}{\rightarrow} \chi^2\left(\widehat{S_{\theta_{k,\ell}}}\right)$ then $(x_{mn}) \overset{P}{\rightarrow} \chi^2\left(AC_{\theta_{k,\ell}}\right)$.
Suppose $x \in \Lambda^2$ then for all r and s, $((m+n)!\,|x_{m+r,n+s}-0|)^{1/m+n} \leq M$ for all m,n. Also for given $\epsilon > 0$ and k and ℓ large for all r and s we obtain the following:

$$\frac{1}{h_{k\ell}}\sum_{m\in I_{k,\ell}}\sum_{n\in I_{k,\ell}}((m+n)!\,|x_{m+r,n+s}-0|)^{1/m+n} =$$
$$\frac{1}{h_{k\ell}}\sum_{m\in I_{k,\ell}}\sum_{n\in I_{k,\ell}\ and\ |x_{m+r,n+s}|\geq 0}((m+n)!\,|x_{m+r,n+s}-0|)^{1/m+n} +$$
$$\frac{1}{h_{k\ell}}\sum_{m\in I_{k,\ell}}\sum_{n\in I_{k,\ell}\ and\ |x_{m+r,n+s}|\leq 0}((m+n)!\,|x_{m+r,n+s}-0|)^{1/m+n}$$
$$\leq \frac{M}{h_{k\ell}}\left|\left\{(m,n)\in I_{k,\ell} : ((m+n)!\,|x_{m+r,n+s}-0|)^{1/m+n}\right\} = 0\right| + \epsilon.$$

Therefore $x \in \Lambda^2$ and $(x_{mn}) \overset{P}{\rightarrow} \chi^2\left(\widehat{S_{\theta_{k,\ell}}}\right)$ then $(x_{mn}) \overset{P}{\rightarrow} \chi^2\left(AC_{\theta_{k,\ell}}\right)$.
(iv)$\chi^2\left(\widehat{S_{\theta_{k,\ell}}}\right)\bigcap\Lambda^2 = \chi^2\left[AC_{\theta_{k,\ell}}\right]\bigcap\Lambda^2$. follows from (i),(ii) and (iii). □

Theorem 3.8. *If f be any modulus function then* $\chi^2_f\left[AC_{\theta_{k,\ell}}\right] \subset \chi^2\left(\widehat{S_{\theta_{k,\ell}}}\right)$

Proof: Let $x \in \chi^2_f\left[AC_{\theta_{k,\ell}}\right]$, for all r and s.
Therefore we have

$$\frac{1}{h_{k\ell}}\sum_{m\in I_{k,\ell}}\sum_{n\in I_{k,\ell}} f\left[((m+n)!\,|x_{m+r,n+s}-0|)^{1/m+n}\right] \geq$$
$$\frac{1}{h_{k\ell}}\sum_{m\in I_{k,\ell}}\sum_{n\in I_{k,\ell}\ and\ |x_{m+r,n+s}|=0} f\left[((m+n)!\,|x_{m+r,n+s}-0|)^{1/m+n}\right] >$$
$$\frac{1}{h_{k\ell}} f(0)\left|\left\{(m,n)\in I_{k,\ell} : ((m+n)!\,|x_{m+r,n+s}-0|)^{1/m+n}\right\} = 0\right|.$$

Hence $x \in \chi^2\left(\widehat{S_{\theta_{k,\ell}}}\right)$. □

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