Bol. Soc. Paran. Mat. (3s.) **v. 33** 1 (2015): 33–39.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v33i1.21712

A Characterization for Bishop Equations of Parallel Curves according to Bishop Frame in $\mathbb{E}^3$

Talat Körpinar, Vedat Asil, Muhammed T. Sariaydin, Muhsin İncesu

ABSTRACT: In this paper, we study Bishop equations of parallel curves according to Bishop frame in Euclidean 3-space. We obtain a new characterization of parallel curve by using Bishop frame in $\mathbb{E}^3$.

Key Words: Bishop frame, Curves, Euclidean 3-space, Parallel curves.

Contents

1. Introduction

A parallel of a curve is the envelope of a family of congruent circles centered on the curve. It generalises the concept of parallel lines. It can also be defined as a curve whose points are at a fixed normal distance of a given curve, [11]. Parallel curves is not a subject widely studied in research papers. The first study in this regard, Chrastinova developed a new construction. This construction is carried over the three-dimensional space and as a result, two parallel curves are obtained as well. Then, Chrastinova studyed parallel helices in three-dimensional space in [4]. In [9], Korpinar et al obtained some characterizations about parallel curves by using Bishop frame in $\mathbb{E}^3$.

On the other hand Bishop frame, which is also called alternetive or parallel frame of the curves, was introduced by L.R. Bishop in 1975 by means of parallel vector fields. Recently, many research papers related to this concept have been treated in Euclidean space. For example, in [13,14] the outhors introduced a new version of Bishop frame and an application to spherical images and they studied Minkowski space in $\mathbb{E}_1^3$, respectively. In [7,8], Korpinar and Turhan explored biharmonic $B-$slant helices and dual spacelike biharmonic curves with timelike principal normal for dual variable in dual Lorentzian space D_1^3 according to Bishop frame.

In this paper, we obtain a new characterization of parallel curve by using Bishop frame in $\mathbb{E}^3$. The firstly, we summarize properties Bishop frame and Frenet frame which are parameterized by arc-length parameter s and the basic concepts on curves. Finally, we give Frenet frame of parallel curves according to Bishop frame in $\mathbb{E}^3$.

2000 *Mathematics Subject Classification:* 53A04

2. Background on parallel curves

Let $\alpha : I \to \mathbb{E}^3$ be a regular curve with parametrized by arc-length. Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet–Serret frame along the curve α in the space $\mathbb{E}^3$. For an arbitrary curve α with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet–Serret formulae is given

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N}, \\ \mathbf{N}' &= -\kappa \mathbf{T} + \tau \mathbf{B}, \\ \mathbf{B}' &= -\tau \mathbf{N}. \end{aligned}$$

The Bishop frame is expressed as

$$\mathbf{T}' = \kappa_1 \mathbf{M}_1 + \kappa_2 \mathbf{M}_2, \tag{2.1}$$
$$\mathbf{M}_1' = -\kappa_1 \mathbf{T}, \tag{2.2}$$
$$\mathbf{M}_2' = -\kappa_2 \mathbf{T}, \tag{2.3}$$

where we shall call the set $\{\mathbf{T}, \mathbf{M_1}, \mathbf{M_2}\}$ as Bishop trihedra and κ_1 and κ_2 as Bishop curvatures. The relation matrix may be expressed as

$$\begin{aligned} \mathbf{T} &= \mathbf{T}, \\ \mathbf{N} &= \cos\theta(s)\mathbf{M_1} + \sin\theta(s)\mathbf{M_2}, \\ \mathbf{B} &= -\sin\theta(s)\mathbf{M_1} + \cos\theta(s)\mathbf{M_2}, \end{aligned}$$

where $\theta(s) = \arctan \frac{\kappa_2}{\kappa_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{\kappa_1^2 + \kappa_2^2}$. Here, Bishop curvatures are defined by

$$\begin{aligned} \kappa_1 &= \kappa(s)\cos\theta(s), \\ \kappa_2 &= \kappa(s)\sin\theta(s). \end{aligned}$$

For planar curve α its unit tangent and unit normal vectors are $\mathbf{T}(s)$ and $\mathbf{N}(s)$, respectively. Then, we get

$$\mathbf{P}_+(S_+) = \alpha(s) + t\mathbf{N}(s) \text{ and } \mathbf{P}_-(S_-) = \alpha(s) - t\mathbf{N}(s),$$

where $S_\pm = S_\pm(s)$ and $S_\pm$ denotes the length along $\mathbf{P}\pm$, at the distance t. Determining the length $S_\pm$, we can write

$$\frac{dS_\pm}{ds} = 1 \pm t\kappa,$$

where κ is the curvature of $\alpha(s)$, [4,11].

Lemma 2.1. *Two curves $\alpha, \beta : I \to \mathbb{E}^3$ are parallel if their velocity vectors $\alpha'(s)$ and $\beta'(s)$ are parallel for each s. In this case, if $\alpha(s_0) = \beta(s_0)$ for some one s_0 in I then, $\alpha = \beta$, [11].*

Theorem 2.2. *If $\alpha, \beta : I \to \mathbb{E}^3$ are unit-speed curves such that $\kappa_\alpha = \kappa_\beta$ and $\tau_\alpha = \pm\tau_\beta$ then, α and β are congruent, [11].*

3. Bishop Frame of Parallel Curve in $\mathbb{E}^3$

Let $\alpha : I \to \mathbb{E}^3$ be a regular curve with parametrized by arc-length and $\mathbf{P}$ its a parallel curve. We obtained that for any parallel curve with Bishop frame, [9];

$$\mathbf{P} = \alpha + \mu \mathbf{M_1} + \eta \mathbf{M_2},$$

where

$$\mu = \frac{1}{\kappa_1} - \frac{2\kappa_2 \tan\theta - \kappa_1\kappa_2 C}{2\kappa_1^2 \sec^2\theta} \text{ and } \eta = \frac{2\tan\theta - \kappa_1 C}{2\kappa_1 \sec^2\theta}. \tag{3.1}$$

In the rest of the paper, assume that $S = s$ and Bishop Frame, Bishop curvatures, curvature and torsion of $\mathbf{P}$ with respect to arc-length parameter s denote $\{\tilde{\mathbf{T}},\ \tilde{\mathbf{M}}_1,\ \tilde{\mathbf{M}}_2\}$ and $\tilde{\kappa}_1,\ \tilde{\kappa}_2,\ \tilde{\kappa},\ \tilde{\tau}$ respectively.

Firstly, we need following lemma:

Lemma 3.1. *Let $\alpha : I \to \mathbb{E}^3$ be a regular curve with parametrized by arc-length and $\mathbf{P}$its a parallel curve $\mathbb{E}^3$. Then, curvature and torsion of $\mathbf{P}$ are given by*

$$\begin{aligned} \tilde{\kappa} &= ((-2\eta'\kappa_2 - 2\mu'\kappa_1 - \mu\kappa_1' - \eta\kappa_2')^2 + \left(\kappa_1 - \mu\kappa_1^2 - \eta\kappa_1\kappa_2 + \mu''\right)^2 \\ &\quad + \left(\kappa_2 - \mu\kappa_1\kappa_2 - \eta\kappa_2^2 + \eta''\right)^2)^{\frac{1}{2}}, \\ \tilde{\tau} &= -< \frac{d\mathbf{B^P}}{ds}, \mathbf{N^P} >, \end{aligned}$$

where

$$\mu = \frac{1}{\kappa_1} - \frac{2\kappa_2 \tan\theta - \kappa_1\kappa_2 C}{2\kappa_1^2 \sec^2\theta} \text{ and } \eta = \frac{2\tan\theta - \kappa_1 C}{2\kappa_1 \sec^2\theta}.$$

Proof: We take the derivative of the (3.1) formula

$$\mathbf{P}' = \mathbf{T^P} = \tilde{\mathbf{T}} = (1 - \kappa_1\mu - \kappa_2\eta)\mathbf{T} + \mu'\mathbf{M_1} + \eta'\mathbf{M_2}, \tag{3.2}$$

$$\begin{aligned} \mathbf{P}'' &= \tilde{\mathbf{T}}' = \left(\mathbf{T^P}\right)' = (-2\eta'\kappa_2 - 2\mu'\kappa_1 - \mu\kappa_1' - \eta\kappa_2')\mathbf{T} \\ &\quad + \left(\kappa_1 - \mu\kappa_1^2 - \eta\kappa_1\kappa_2 + \mu''\right)\mathbf{M_1} + \left(\kappa_2 - \mu\kappa_1\kappa_2 - \eta\kappa_2^2 + \eta''\right)\mathbf{M_2}. \end{aligned} \tag{3.3}$$

If we take norm for (3.3), we can get

$$\begin{aligned} \tilde{\kappa} &= \| (-2\eta'\kappa_2 - 2\mu'\kappa_1 - \mu\kappa_1' - \eta\kappa_2')\mathbf{T} + \left(\kappa_1 - \mu\kappa_1^2 - \eta\kappa_1\kappa_2 + \mu''\right)\mathbf{M_1} \\ &\quad + \left(\kappa_2 - \mu\kappa_1\kappa_2 - \eta\kappa_2^2 + \eta''\right)\mathbf{M_2} \|, \end{aligned}$$

where κ_1 and κ_2 are Bishop curvatures of α curve. Then, we easy have

$$\begin{aligned} \tilde{\kappa}^2 &= (-2\eta'\kappa_2 - 2\mu'\kappa_1 - \mu\kappa_1' - \eta\kappa_2')^2 + \left(\kappa_1 - \mu\kappa_1^2 - \eta\kappa_1\kappa_2 + \mu''\right)^2 \\ &\quad + \left(\kappa_2 - \mu\kappa_1\kappa_2 - \eta\kappa_2^2 + \eta''\right)^2. \end{aligned}$$

Now, we can calculate $\mathbf{N^P}$ and $\mathbf{B^P}$ components of the Frenet-Serret formulas of $\mathbf{P}$ by

$$\begin{aligned}
\mathbf{N^P} &= \tilde{\mathbf{N}} = \frac{1}{\tilde{\kappa}}[(-2\eta'\kappa_2 - 2\mu'\kappa_1 - \mu\kappa_1' - \eta\kappa_2')\mathbf{T} + \left(\kappa_1 - \mu\kappa_1^2 - \eta\kappa_1\kappa_2 + \mu''\right)\mathbf{M_1} \\
&\quad + \left(\kappa_2 - \mu\kappa_1\kappa_2 - \eta\kappa_2^2 + \eta''\right)\mathbf{M_2}], \qquad (3.4)\\
\mathbf{B^P} &= \tilde{\mathbf{B}} = \frac{1}{\tilde{\kappa}}[(\mu'\kappa_2 - \mu\mu'\kappa_1\kappa_2 - \mu'\eta\kappa_2^2 \\
&\quad +\mu'\eta'' - \eta'\kappa_1 + \mu\eta'\kappa_1^2 + \eta\eta'\kappa_1\kappa_2 - \eta'\mu'')\mathbf{T} \\
&\quad +(-\kappa_2 + 2\mu\kappa_1\kappa_2 + 2\eta\kappa_2^2 - 2\mu\eta\kappa_1\kappa_2^2 - 2\left(\eta'\right)^2\kappa_2 - 2\mu'\eta'\kappa_1 \\
&\quad -\eta'' - \mu^2\kappa_1^2\kappa_2 + \mu\eta''\kappa_1 - \eta^2\kappa_2^3 + \eta\eta''\kappa_2 - \mu\eta'\kappa_1' - \eta\eta'\kappa_2')\mathbf{M_1} \qquad (3.5)\\
&\quad +(\kappa_1 - 2\mu\kappa_1^2 - 2\eta\kappa_1\kappa_2 + 2\mu'\eta'\kappa_2 + 2\left(\mu'\right)^2\kappa_1 + 2\mu\eta\kappa_1^2\kappa_2 \\
&\quad +\mu'' + \mu^2\kappa_1^3 - \mu\mu''\kappa_1 + \eta^2\kappa_1\kappa_2^2 - \mu''\eta\kappa_2 + \mu\mu'\kappa_1' + \mu'\eta\kappa_2')\mathbf{M_2}].
\end{aligned}$$

From definition of torsion of $\mathbf{P}$, we have

$$\begin{aligned}
\tilde{\tau} &= -< \frac{d\mathbf{B^P}}{ds}, \mathbf{N^P} > \\
&= -\frac{1}{\tilde{\kappa}}[\frac{d}{ds}(\frac{1}{\tilde{\kappa}}(\mu'\kappa_2 - \mu\mu'\kappa_1\kappa_2 - \mu'\eta\kappa_2^2 + \mu'\eta'' - \eta'\kappa_1 + \mu\eta'\kappa_1^2 + \eta\eta'\kappa_1\kappa_2 \\
&\quad -\eta'\mu'')(-2\eta'\kappa_2 - 2\mu'\kappa_1 - \mu\kappa_1' - \eta\kappa_2')) + \frac{d}{ds}(\frac{1}{\tilde{\kappa}}(-\kappa_2 + 2\mu\kappa_1\kappa_2 + 2\eta\kappa_2^2 \\
&\quad -2\mu\eta\kappa_1\kappa_2^2 - 2\left(\eta'\right)^2\kappa_2 - 2\mu'\eta'\kappa_1 - \eta'' - \mu^2\kappa_1^2\kappa_2 + \mu\eta''\kappa_1 - \eta^2\kappa_2^3 + \eta\eta''\kappa_2 \\
&\quad -\mu\eta'\kappa_1' - \eta\eta'\kappa_2')(\kappa_1 - \mu\kappa_1^2 - \eta\kappa_1\kappa_2 + \mu'')) + \frac{d}{ds}(\frac{1}{\tilde{\kappa}}(\kappa_1 - 2\mu\kappa_1^2 - 2\eta\kappa_1\kappa_2 \\
&\quad +2\mu'\eta'\kappa_2 + 2\left(\mu'\right)^2\kappa_1 + 2\mu\eta\kappa_1^2\kappa_2 + \mu'' + \mu^2\kappa_1^3 - \mu\mu''\kappa_1 + \eta^2\kappa_1\kappa_2^2 - \mu''\eta\kappa_2 \\
&\quad +\mu\mu'\kappa_1' + \mu'\eta\kappa_2')(\kappa_2 - \mu\kappa_1\kappa_2 - \eta\kappa_2^2 + \eta''))].
\end{aligned}$$

□

Corollary 3.2. *Let $\alpha : I \to \mathbb{E}^3$ be a regular curve with parametrized by arc-length in $\mathbb{E}^3$. If $\mathbf{P}$ is a parallel curve of α, then*

$$\begin{aligned}
\tilde{\kappa}_1 &= \tilde{\kappa}\cos(\tilde{\theta}), \qquad (3.6)\\
\tilde{\kappa}_2 &= \tilde{\kappa}\sin(\tilde{\theta}), \qquad (3.7)
\end{aligned}$$

where

$$\tilde{\theta} = \int_0^s \tilde{\tau}(s)\, ds.$$

Proof: Using Lemma 3.1, we easily have (3.6) and (3.7). □

Finally, we give our main theorem.

Theorem 3.3. *Let $\alpha : I \to \mathbb{E}^3$ be a regular curve with parametrized by arc-length and* $\mathbf{P}$ *its a parallel curve in $\mathbb{E}^3$. Then, the Bishop equations of* $\mathbf{P}$ *are given by*

$$\begin{aligned}
\tilde{\mathbf{T}}' &= (-2\eta'\kappa_2 - 2\mu'\kappa_1 - \mu\kappa_1' - \eta\kappa_2')\mathbf{T} + \left(\kappa_1 - \mu\kappa_1^2 - \eta\kappa_1\kappa_2 + \mu''\right)\mathbf{M_1} \\
&\quad + \left(\kappa_2 - \mu\kappa_1\kappa_2 - \eta\kappa_2^2 + \eta''\right)\mathbf{M_2}, \\
\tilde{\mathbf{M}}_1' &= -\tilde{\kappa}_1\tilde{\mathbf{T}}, \\
\tilde{\mathbf{M}}_2' &= -\tilde{\kappa}_2\tilde{\mathbf{T}}.
\end{aligned}$$

Proof: By using Bishop frame of parallel curve, we have

$$\tilde{\mathbf{M}}_1' = -\tilde{\kappa}_1\tilde{\mathbf{T}}.$$

If we write equation (3.6) in Corallary 3.2 and using (2.2), we get

$$\begin{aligned}
\tilde{\mathbf{M}}_1' &= -\cos(\tilde{\theta})[(-2\eta'\kappa_2 - 2\mu'\kappa_1 - \mu\kappa_1' - \eta\kappa_2')^2 + \left(\kappa_1 - \mu\kappa_1^2 - \eta\kappa_1\kappa_2 + \mu''\right)^2 \\
&\quad + \left(\kappa_2 - \mu\kappa_1\kappa_2 - \eta\kappa_2^2 + \eta''\right)^2]^{\frac{1}{2}}[(1 - \kappa_1\mu - \kappa_2\eta)\mathbf{T} + \mu'\mathbf{M_1} + \eta'\mathbf{M_2}].
\end{aligned}$$

Similarly, by using Bishop frame of parallel curve, we have

$$\tilde{\mathbf{M}}_2' = -\tilde{\kappa}_2\tilde{\mathbf{T}}.$$

If we write equation (3.7) in Corallary 3.2 and using (2.3), we get

$$\begin{aligned}
\tilde{\mathbf{M}}_2' &= -\sin(\tilde{\theta})[(-2\eta'\kappa_2 - 2\mu'\kappa_1 - \mu\kappa_1' - \eta\kappa_2')^2 + \left(\kappa_1 - \mu\kappa_1^2 - \eta\kappa_1\kappa_2 + \mu''\right)^2 \\
&\quad + \left(\kappa_2 - \mu\kappa_1\kappa_2 - \eta\kappa_2^2 + \eta''\right)^2]^{\frac{1}{2}}[(1 - \kappa_1\mu - \kappa_2\eta)\mathbf{T} + \mu'\mathbf{M_1} + \eta'\mathbf{M_2}].
\end{aligned}$$

□

Corollary 3.4. *Let $\alpha : I \to \mathbb{E}^3$ be a regular curve with parametrized by arc-length and* $\mathbf{P}$ *its a parallel curve in $\mathbb{E}^3$. Then, frenet frame of* $\mathbf{P}$ *are given by*

$$\begin{aligned}
\tilde{\mathbf{T}} &= (1 - \kappa_1\mu - \kappa_2\eta)\mathbf{T} + \mu'\mathbf{M_1} + \eta'\mathbf{M_2}, \\
\tilde{\mathbf{N}} &= \frac{1}{\tilde{\kappa}}[(-2\eta'\kappa_2 - 2\mu'\kappa_1 - \mu\kappa_1' - \eta\kappa_2')\mathbf{T} + \left(\kappa_1 - \mu\kappa_1^2 - \eta\kappa_1\kappa_2 + \mu''\right)\mathbf{M_1} \\
&\quad + \left(\kappa_2 - \mu\kappa_1\kappa_2 - \eta\kappa_2^2 + \eta''\right)\mathbf{M_2}], \\
\tilde{\mathbf{B}} &= \frac{1}{\tilde{\kappa}}[(\mu'\kappa_2 - \mu\mu'\kappa_1\kappa_2 - \mu'\eta\kappa_2^2 + \mu'\eta'' - \eta'\kappa_1 + \mu\eta'\kappa_1^2 + \eta\eta'\kappa_1\kappa_2 - \eta'\mu'')\mathbf{T} \\
&\quad +(-\kappa_2 + 2\mu\kappa_1\kappa_2 + 2\eta\kappa_2^2 - 2\mu\eta\kappa_1\kappa_2^2 - 2\left(\eta'\right)^2\kappa_2 - 2\mu'\eta'\kappa_1 \\
&\quad -\eta'' - \mu^2\kappa_1^2\kappa_2 + \mu\eta''\kappa_1 - \eta^2\kappa_2^3 + \eta\eta''\kappa_2 - \mu\eta'\kappa_1' - \eta\eta'\kappa_2')\mathbf{M_1} \\
&\quad +(\kappa_1 - 2\mu\kappa_1^2 - 2\eta\kappa_1\kappa_2 + 2\mu'\eta'\kappa_2 + 2\left(\mu'\right)^2\kappa_1 + 2\mu\eta\kappa_1^2\kappa_2 \\
&\quad +\mu'' + \mu^2\kappa_1^3 - \mu\mu''\kappa_1 + \eta^2\kappa_1\kappa_2^2 - \mu''\eta\kappa_2 + \mu\mu'\kappa_1' + \mu'\eta\kappa_2')\mathbf{M_2}].
\end{aligned}$$

Proof: The proof of Corollary is obvious from Theorem 3.3. □

References

1. V. Asil: *Velocities of dual homothetic exponential motions in D^3*, Iranian Journal of Science & Tecnology Transaction A: Science 31 (4) (2007), 265-271.

2. V. Asil, T. Körpınar, E. Turhan: *On Inextensible Flows of Tangent Developable of Biharmonic $\mathcal{B}$-Slant Helices According to Bishop Frames in the Special 3-Dimensional Kenmotsu Manifold,* Bol. Soc. Paran. Mat., 1 (31) (2013), 89–97.

3. L. R. Bishop: *There is More Than One Way to Frame a Curve*, Amer. Math. Monthly, 82 (3) (1975), 246-251.

4. V. Chrastinova: *Parallel Curves in Three-Dimensional Space,* Sbornik 5. Konference o matematice a fyzice, 2007, UNOB.

5. T. Körpınar, E. Turhan, V. Asil: *Tangent Bishop Spherical Images of A Biharmonic $\mathcal{B}$-slant Helix in The Heisenberg Group Heis^3*, Iranian Journal of Science and Technology Transaction A: Science A4 (35) (2011), 265-271.

6. T. Körpınar, E. Turhan: *On Characterization of $\mathcal{B}$-Canal Surfaces in Terms of Biharmonic B-Slant Helices According to Bishop Frame in Heisenberg Group Heis^3*, Journal of Mathematical Analysis and Applications, 382 (1) (2011), 57-65.

7. T. Körpınar, E. Turhan: *New Solution of Differential Equation for Dual Curvatures of Dual Spacelike Biharmonic Curves with Timelike Principal Normal According to Dual Bishop Frames in The Dual Lorentzian Space,* Acta Universitatis Apulensis, 30 (2012), 77-86.

8. T. Körpınar, E. Turhan: *Biharmonic $\mathcal{B}$-Slant Helices According to Bishop Frame in The* $\widetilde{\mathcal{SL}_2}(\mathbb{R})$, Bol. Soc. Paran. Mat., 2 (31) (2013), 39–45.

9. T. Körpınar, M. T. Sarıaydın, V. Asil: *On Characterization of Parallel Curves According to Bishop Frame in $\mathbb{E}^3$*, (submitted).

10. W. F. Newton: *Theoretical and Practical Graphics,* Biblio Bazaar, LLC, 2009.

11. B. O'Neil: *Elementary Differential Geometry,* Academic Press, New York, 1967.

12. I. M. Yaglom, A. Shenitzer: *A Simple Non-Euclidean Geometry and Its Pysical Basis*, Springer-Verlag, New York, 1979.

13. S. Yılmaz, M. Turgut: *A New Version of Bishop Frame and An Aplication to Spherical Images* , J. Math. Anal. Appl., 371 (2010), 764-776.

14. S. Yılmaz: *Poition vectors of some special space-like curves according to Bishop frame in Minkowski space $\mathbb{E}_1^3$*, Sci Magna, 5 (1) (2010), 48-50.

Talat Körpinar
Muş Alparslan University, Department of Mathematics
49250, Muş, Turkey
E-mail address: talatkorpinar@gmail.com

and

Vedat Asil
Fırat University, Department of Mathematics
23119, Elazığ, Turkey
E-mail address: vasil@firat.edu.tr

and

Muhammed T. Sariaydin
Muş Alparslan University, Department of Mathematics
49250, Muş, Turkey
E-mail address: talatsariaydin@gmail.com

and

Muhsin İncesu
Muş Alparslan University, Department of Mathematics
49250, Muş, Turkey
E-mail address: m.incesu@alparslan.edu.tr