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Existence and upper semicontinuity of global attractors for a p-Laplacian inclusion

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ABSTRACT: In this work we study the asymptotic behavior of a p-Laplacian inclusion of the form $\frac{\partial u_\lambda}{\partial t} - div(D^\lambda|\nabla u_\lambda|^{p-2}\nabla u_\lambda) + |u_\lambda|^{p-2}u_\lambda \in F(u_\lambda) + h$, where $p > 2$, $h \in L^2(\Omega)$, with $\Omega \subset \mathbb{R}^n$, $n \geq 1$, a bounded smooth domain, $D^\lambda \in L^\infty(\Omega)$, $\infty > M \geq D^\lambda(x) \geq \sigma > 0$ a.e. in Ω, $\lambda \in [0,\infty)$ and $D^\lambda \to D^{\lambda_1}$ in $L^\infty(\Omega)$ as $\lambda \to \lambda_1$, $F : \mathcal{D}(F) \subset L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$, given by $F(y(\cdot)) = \{\xi(\cdot) \in L^2(\Omega) : \xi(x) \in f(y(x))$ x-a.e. in $\Omega\}$ with $f : \mathbb{R} \to \mathcal{C}_v(\mathbb{R})$ a multivalued Lipschitz map, where $\mathcal{C}_v(\mathbb{R})$ is the set of all nonempty, bounded, closed, convex subsets of $\mathbb{R}$. We prove the existence of a global attractor in $L^2(\Omega)$ for each positive finite diffusion coefficient and we show that the family of attractors behaves upper semicontinuously on positive finite diffusion parameters.

Key Words: Partial differential inclusions, p-Laplacian, attractors, upper semicontinuity.

Contents

1. Introduction

Let us consider the problem

$$\begin{cases} \frac{\partial u_\lambda}{\partial t}(t) - div(D^\lambda|\nabla u_\lambda(t)|^{p-2}\nabla u_\lambda(t)) + |u_\lambda(t)|^{p-2}u_\lambda(t) \in F(u_\lambda(t)) + h, \; t > 0 \\ u_\lambda(0) = u_{0,\lambda}, \end{cases} \tag{1.1}$$

where $p > 2$, $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded smooth domain, $h, u_{0,\lambda} \in H := L^2(\Omega)$, $D^\lambda \in L^\infty(\Omega)$, $\infty > M \geq D^\lambda(x) \geq \sigma > 0$ a.e. in Ω, $\lambda \in [0,\infty)$ and $D^\lambda \to D^{\lambda_1}$ in $L^\infty(\Omega)$ as $\lambda \to \lambda_1$, $F : \mathcal{D}(F) \subset L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$, given by

$$F(y(\cdot)) = \{\xi(\cdot) \in L^2(\Omega) : \xi(x) \in f(y(x)) \; x\text{-a.e. in } \Omega\}$$

2000 *Mathematics Subject Classification:* 35A16, 35K10, 35K92, 35K55 with $f : \mathbb{R} \to \mathcal{C}_v(\mathbb{R})$ a multivalued map, where $\mathcal{C}_v(\mathbb{R})$ is the set of all nonempty, bounded, closed, convex subsets of $\mathbb{R}$. Assume that f is Lipschitz, i.e., there exists $C \geq 0$ such that

$$\text{dist}_{\mathcal{H}}(f(x), f(z)) \leq C\|x - z\| \text{ for all } x, z \in \mathbb{R}.$$

Consequently, the map $F(u) + h$ has values in $\mathcal{C}_v(L^2(\Omega))$ and is Lipschitz.

The authors in [21] proved that the operator

$$A^{D^\lambda}(u) := -div(D^\lambda |\nabla u|^{p-2} \nabla u) + |u|^{p-2}u$$

is maximal monotone in H and is the subdifferential of a proper, convex and lower semicontinuos function $\varphi^{D^\lambda} : H \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi^{D^\lambda}(u) \doteq \begin{cases} \frac{1}{p}\Big[\int_{\mathbb{R}^n} D^\lambda(x)|\nabla u|^p dx + \int_{\mathbb{R}^n} |u|^p dx\Big], & u \in W^{1,p}(\Omega) \\ +\infty, & \text{otherwise} \end{cases}.$$

Moreover, it is not difficult to see that there are constants $w_1 = w_1(\sigma) > 0$, $w_2 = w_2(p, M) > 0$, $c_1 \doteq 0 \in \mathbb{R}$ and $p > 2$ such that for all $u \in E := W^{1,p}(\Omega)$ the following two conditions hold:

$$\langle A_1^{D^\lambda} u, u \rangle_{E^*, E} \geq w_1 \parallel u \parallel_E^p + c_1 \tag{1.2}$$

and

$$\parallel A_1^{D^\lambda} u \parallel_{E^*} \leq w_2 \parallel u \parallel_E^{p-1} < w_2(\parallel u \parallel_E^{p-1} + 1). \tag{1.3}$$

As a consequence we can conclude that $\overline{\mathcal{D}(A^{D^\lambda})} = H$ and the operator $A^{D^\lambda} : \mathcal{D}(A^{D^\lambda}) \subset H \to H$ generates a compact semigroup S^{D^λ}, [7].

During the last ten years, many researches have spent much effort in obtaining results on global attractors for p-Laplacian problems (see for example [1,4,8,6,10,11,12,13,16,18,19,21,22,23,24,25,26,27]). To prove existence of a global attractor for Partial Differential Inclusions it is necessary to use theory of multivalued semigroups or generalized semiflows (see [2,5,17,20]). In this work, in order to prove existence of a global attractor for problem (1.1) we use the theory developed in [17].

The paper is organized as follows. In Section 2 we present some preliminaries results. In Section 3 we prove the existence of the global attractor for the problem (1.1). Moreover, in Section 4 we obtain H and E estimates for the solutions u_λ of the problem (1.1), uniformly on $\lambda \in [0, \infty)$. Finally, in Section 5 we prove the upper semicontinuity of the global attractors.

2. A preliminary theory

This section is based on the paper [17], therefore we strongly suggest that the reader consult the original sources when using the results for further research. We include them to make this text self-contained.

Consider the following evolution inclusion

$$\frac{dy(t)}{dt} \in A(y(t)) + F(y(t)),\ t \in [0,T], \tag{2.1}$$

with the initial condition

$$y(0) = y_0 \in H. \tag{2.2}$$

Let us consider the next conditions:

(A) The operator A is maximal monotone in H.

(F_1) $F : H \to \mathcal{C}_v(H)$, where $\mathcal{C}_v(H)$ is the set of all nonempty, bounded, closed and convex subsets of H.

(F_2) The map F is Lipschitz on $\overline{\mathcal{D}(A)}$, i.e., there exists $c \geq 0$ such that

$$\mathrm{dist}_H(F(y_1), F(y_2)) \leq c \parallel y_1 - y_2 \parallel_H, \text{ for all } y_1, y_2 \in \overline{\mathcal{D}(A)},$$

where $\mathrm{dist}_H(\cdot,\cdot)$ denotes the Hausdorff metric of bounded sets.

Consider also the next inclusion

$$\frac{dy(t)}{dt} \in A(y(t)) + f(t),\ t \in [0,T], \tag{2.3}$$

with the initial condition

$$y(0) = y_0 \in H, \tag{2.4}$$

where $f(\cdot) \in L^1([0,T];H)$ and $L^1([0,T];H)$ is the space of Bochner integrable functions.

Definition 2.1. *[17] The continuous function $y : [0,T] \to H$ is called an integral solution of the problem (2.3), (2.4) if:*

i) $y(0) = y_0$;

ii) $\forall\, u \in \mathcal{D}(A)$, $\forall\, v \in A(u)$,

$$\|y(t) - u\|_H^2 \leq \|y(s) - u\|_H^2 + 2\int_s^t \langle f(\tau) + v, y(\tau) - u\rangle d\tau,\ t \geq s. \tag{2.5}$$

Definition 2.2. *[14] The continuous function $y : [0,T] \to H$ is called a strong solution of the problem (2.3), (2.4) if $y(0) = y_0$ and $y(\cdot)$ is absolutely continuous on every compact subsets of $(0,T)$ and satisfies (2.3) almost everywhere on $(0,T)$.*

Definition 2.3. *[17] The continuous function $y : [0,T] \to H$ is called an integral solution of the problem (2.1), (2.2) if:*

i) $y(0) = y_0$;

ii) *For some selection* $f \in L^1([0,T],H)$, $f(t) \in F(y(t))$ *a.e. on* $[0,T]$ *and the inequality* (2.5) *holds.*

Definition 2.4. *[14,21] The continuous function* $y : [0,T] \to H$ *is called a strong solution of the problem* (2.1), (2.2) *if there exists a selection* $f \in L^1([0,T],H)$, $f(t) \in F(y(t))$ *a.e. on* $[0,T]$ *such that* $y : [0,T] \to H$ *is a strong solution of the problem* (2.3), (2.4).

Remark 2.5. *[3,17] If the condition* (A) *holds and* $f \in L^1([0,T];H)$, *then for every* $y_0 \in \overline{\mathcal{D}(A)}$, *there exists a unique integral solution* $y(\cdot)$ *of the problem* (2.3), (2.4) *for each* $T > 0$. *We shall denote* $y(\cdot) = I(y_0)f(\cdot)$. *Moreover, for any integral solutions* $y_i(\cdot) = I(y_{i0})f_i(\cdot)$, $i = 1,2$, *the next inequality holds:*

$$\|y_1(t) - y_2(t)\| \leq \|y_1(s) - y_2(s)\| \ + \int_s^t \|f_1(\tau) - f_2(\tau)\| d\tau, \ t \geq s. \tag{2.6}$$

Let us denote by $D(y_0)$ the set of all integral solutions of (2.1) such that $y(0) = y_0$.

Lemma 2.6. *[17] The multivalued map* $G : \mathbb{R}_+ \times \overline{\mathcal{D}(A)} \to \mathcal{P}(\overline{\mathcal{D}(A)})$ *defined by* $G(t,y_0) := \{y(t) : y(\cdot) \in D(y_0)\}$ *is a multivalued semigroup.*

3. Existence of the global attractor

Using the properties on the external forcing term and on the operator we obtain from Lemma 2.6 the following

Proposition 3.1. *The inclusion (1.1) defines a multivalued semigroup (or m-semiflow)* $G_\lambda(t,\cdot) : H \to \mathcal{P}(H)$ *where* $G_\lambda(t,u_0)$ *is the set of all integral solutions of (1.1) beginning at* $u_0 \in H$ *valuated at time* t.

Let us consider the following condition:

$(\mathcal{H})$ The sets $M_K := \{u \in D(\varphi) : \|u\|_H \leq K, \ \varphi(u) \leq K\}$ are compact in H for any $K > 0$.

We intend to use the following

Theorem 3.2. *[17] Let* $(\mathcal{H})$ *be satisfied. Suppose that there exist* $\delta > 0$, $M > 0$ *such that for every* $u \in \mathcal{D}(\partial\varphi)$ *with* $\|u\| \geq M$ *and for each* $y \in -\partial\varphi(u) + F(u) + h$, *we have*

$$(y,u) \leq -\delta. \tag{3.1}$$

Then the multivalued semigroup G *has a global attractor* R. *It is the minimal closed set attracting each bounded set. It is compact, invariant and maximal among all negatively semi-invariant bounded subsets in* H.

Now, considering a growth condition on f, we establish the following

Theorem 3.3. *If there exist constants $M_0 > 0$ and $\epsilon_0 > \frac{1}{2} + \frac{1}{M_0|\Omega|+\frac{1}{2}\|h\|^2}$ such that for all $s \in \mathbb{R}$ and for every $z \in f(s)$,*

$$zs \leq \frac{w_1}{\gamma^p}|s|^p - \epsilon_0 |s|^2 + M_0, \tag{3.2}$$

where γ is the immersion constant of $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, then the multivalued semigroup associated with problem (1.1) has a global attractor $\mathcal{A}_\lambda$. It is the minimal closed set attracting each bounded set. It is compact, invariant and maximal among all negatively semi-invariant bounded subsets in H.

Proof: First, we will to prove that the condition $(\mathcal{H})$ is satisfied. Indeed, since $E \subset\subset H$ and

$$M_K := \left\{u \in \mathcal{D}(\varphi^{D^\lambda}); \|u\|_H \leq K,\ \varphi^{D^\lambda}(u) \leq K\right\} = \overline{M_k},$$

it is sufficient to show that for each $K > 0$, M_K is a bounded set in E. Let be $K > 0$ and $u \in M_K$. Using (1.2), we have

$$w_1\|u\|_E^p \leq \frac{p}{p}\left[\int_\Omega D^\lambda(x)\,|\nabla u|^p\,dx + \int_\Omega |u|^p\,dx\right] = p\varphi^{D^\lambda}(u) \leq pK =: K_1.$$

So, $\|u\|_E \leq [\frac{K_1}{w_1}]^{\frac{1}{p}}$ and the the condition $(\mathcal{H})$ is satisfied.

Now, we intend to show that the condition (3.1) in Theorem 3.2 is satisfied. Let $u \in \mathcal{D}(A^{D^\lambda})$ and $\xi \in F(u)$. Then, using Cauchy Schwarz and the hypothesis (3.2) we get

$$\begin{aligned}
&\left\langle -A^{D^\lambda}(u) + \xi + h, u\right\rangle \leq \\
\leq\ & -w_1\|u\|_E^p + \int_\Omega \left(\frac{w_1}{\gamma^p}|u(x)|^p - \epsilon_0\|u(x)\|^2 + M_0\right)dx + \|h\|_H\|u\|_H \\
\leq\ & -\frac{w_1}{\gamma^p}\|u\|_{L^p}^p + \frac{w_1}{\gamma^p}\|u\|_{L^p}^p - \epsilon_0\|u\|_H^2 + \frac{1}{2}\|h\|_H^2 + \frac{1}{2}\|u\|_H^2 + M_0\,|\Omega| \\
=\ & \left(\frac{1}{2} - \epsilon_0\right)\|u\|_H^2 + \left(M_0\,|\Omega| + \frac{1}{2}\|h\|_H^2\right).
\end{aligned}$$

Considering $M := M_0\,|\Omega| + \frac{1}{2}\|h\|_H^2 > 0$ and $\delta := \left(\epsilon_0 - \frac{1}{2}\right)M^2 - M > 0$ we have that

$$\left\langle -A^{D^\lambda}(u) + \xi + h, u\right\rangle \leq -\delta, \text{ for all } u \in \mathcal{D}(A^{D^\lambda}) \text{ with } \|u\|_H > M.$$

So, condition (3.1) is satisfied and the result follows from Theorem 3.2. $\square$

4. Uniform Estimates

In this section we obtain H and E estimates for the solutions u_λ's of the problem (1.1), uniformly on $\lambda \in [0,\infty)$. Since the map f has values in $\mathcal{C}_v(\mathbb{R})$ it's easy to see that there exist $D_1,\ D_2 \geq 0$ such that

$$\sup_{y\in f(s)} |y| \leq D_1 + D_2|s|, \text{ for all } s\in\mathbb{R}.$$

Consequently, there exist $\tilde{D_1},\ \tilde{D_2} \geq 0$ such that

$$\sup_{v_\lambda\in F(u_\lambda)} \|v_\lambda\| \leq \tilde{D_1} + \tilde{D_2}\|u_\lambda\|, \text{ for all } \lambda\in[0,\infty). \tag{4.1}$$

By Lemma 1 in [14] each integral solution u_λ of problem (1.1) is a strong solution of this problem. Since $\infty > M \geq D^\lambda(x) \geq \sigma > 0$ a.e. in Ω, $\lambda \in [0,\infty)$, working with selections we can repeat the same arguments used in [21,22] to obtain the desired estimates. What essentially change is the control on the right hand side, i.e., being u_λ a solution of (1.1), then there exists $\xi_\lambda \in L^1(0,T;H)$, $\xi_\lambda(t) \in F(u_\lambda(t))$ $t-a.e.$ in $(0,T)$ such that

$$\frac{\partial u_\lambda}{\partial t}(t) - div(D^\lambda|\nabla u_\lambda(t)|^{p-2}\nabla u_\lambda(t)) + |u_\lambda(t)|^{p-2}u_\lambda(t) = \xi_\lambda(t) + h.$$

Multiplying the equation by $u_\lambda(t)$ we control the right hand side using (4.1):

$$\langle \xi_\lambda(t) + h, u_\lambda(t)\rangle \leq \tilde{D_2}\|u_\lambda(t)\|_H^2 + (\tilde{D_1} + \|h\|_H)\|u_\lambda(t)\|_H,\ \forall\, \lambda\in[0,\infty).$$

Thus, we obtain

Lemma 4.1. *If u_λ is a solution of (1.1) in $(0,\infty)$, then there are positive constants $r_0,\ t_0$ such that $\|u_\lambda(t)\|_H \leq r_0$, for each $t \geq t_0$ and $\lambda\in[0,\infty)$.*

Remark 4.2. *We observe that the constants $r_0,\ t_0$ in Lemma 4.1 depend neither on the initial data nor on λ.*

Remark 4.3. *For each fixed $\lambda\in[0,\infty)$, there exists a positive constant $\widetilde{r_0}(u_{0,\lambda},t_0)$ such that $\|u_\lambda(t)\|_H < \widetilde{r_0}(u_{0,\lambda},t_0)$, for each $t\in[0,t_0]$ and, for initial conditions in bounded subsets of H, we have that $\|u_\lambda(t)\|_H < \widetilde{r_0}$, for each $\lambda\in[0,\infty)$ and $t\in[0,t_0]$.*

Corollary 4.4. *There is a bounded set B_0 in H such that $\mathcal{A}_\lambda \subset B_0$, $\forall\,\lambda\in[0,\infty)$.*

Lemma 4.5. *If u_λ is a solution of (1.1) in $(0,\infty)$, then there exist positive constants $r_1 > 0$ and $t_1 > t_0$ such that $\|u_\lambda(t)\|_E \leq r_1$, for each $t\geq t_1$ and $\lambda\in[0,\infty)$, with t_0 as in the Lemma 4.1.*

Remark 4.6. *If u_λ is a solution of (1.1) with initial conditions in bounded subsets of E, we have that there is a constant $\widetilde{r_3} > 0$ such that $\|u_\lambda(t)\|_E < \widetilde{r_3}$, for each $\lambda\in[0,\infty)$ and $t\in[0,t_1]$.*

As an important consequence of Lemma 4.5 it follows that $\bigcup_{\lambda\in[0,\infty)}\mathcal{A}_\lambda$ is a bounded subset of E and once $E \subset\subset H$, we can conclude:

Corollary 4.7. *$\mathcal{A} := \overline{\bigcup_{\lambda\in[0,\infty)}\mathcal{A}_\lambda}$ is a compact subset of H.*

5. Upper semicontinuity of the global attractors

In this section we guarantee that $\{\mathcal{A}_\lambda\}_{\lambda\in[0,\infty)}$ is upper semicontinuous at λ_1, i.e.,

$$dist(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_1}) = \sup_{a_\lambda\in\mathcal{A}_\lambda} dist(a_\lambda, \mathcal{A}_{\lambda_1}) \to 0 \ \ as\ \lambda \to \lambda_1.$$

To accomplish that we appeal to

Theorem 5.1. *[15] Let Λ be a metric space, λ_1 be a non-isolated point and let $\mathbb{G}_\lambda : \mathbb{R}_+ \times X \to P(X)$, $\lambda \in \Lambda$, a family of m-semiflows in the Banach space X satisfying:*
(i) For each $\lambda \in \Lambda$, $\mathbb{G}_\lambda$ has a compact and invariant global B-attractor $\mathcal{A}_\lambda$ and $\bigcup_{\lambda\in\Lambda}\mathcal{A}_\lambda \in B(X)$;
(ii) The multivalued map $\lambda \mapsto \mathbb{G}_\lambda(t,\mathcal{A})$, $\mathcal{A} \doteq \overline{\bigcup_{\lambda\in\Lambda}\mathcal{A}_\lambda}$, is w-upper semicontinuous at λ_1 for large t, i.e., there exists $t_0 > 0$ such that for each $t \geq t_0$ fixed, given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $\mathbb{G}_\lambda(t,\mathcal{A}) \subset O_\varepsilon(\mathbb{G}_{\lambda_1}(t,\mathcal{A}))$, $\forall\, \lambda \in O_\delta(\lambda_1)$.
Then $dist(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_1}) \to 0$, as $\lambda \to \lambda_1$.

To prove the next theorem we intend to use Theorem 3.3 in [9]. So, we need impose one more hypothesis on F. We suppose that F is upper w-semicontinuous on H, i.e., for any $\varepsilon > 0$ and $x_0 \in H$, there exists $\delta > 0$ such that, for any $x \in B_\delta(x_0)$, we have $F(x) \subset B_\varepsilon(F(x_0))$.

Theorem 5.2. *The map $\lambda \longmapsto G_\lambda(t,\mathcal{A})$ is w-upper semicontinuous at λ_1 for each $t > 0$.*

Proof: For simplicity, we consider $\lambda_1 \doteq 0$. Suppose, on contrary, that there exists a number $t_0 > 0$ such that the map $\lambda \mapsto G_\lambda(t_0,\mathcal{A})$ is not w-upper semicontinuous at λ_1. So, there exists a γ-neighborhood $O_\gamma(G_0(t_0,\mathcal{A}))$ such that for each $n \in \mathbb{N}$ there exists $0 \leq \lambda_n < \frac{1}{n}$ and $\xi_{\lambda_n} \in G_{\lambda_n}(t_0,\mathcal{A})$ with $\xi_{\lambda_n} \notin O_\gamma(G_0(t_0,\mathcal{A}))$. (Note that $\lambda_n \to \lambda_1 = 0$ as $n \to +\infty$). Then, $\xi_{\lambda_n} = u_{\lambda_n}(t_0)$, $u_{\lambda_n}(0) \in \mathcal{A}$. It is enough to show that there is a subsequence $\{\xi_{\lambda_{n_k}}\}$ of $\{\xi_{\lambda_n}\}$ with $\xi_{\lambda_{n_k}} \to \xi_0 \in G_0(t_0,\mathcal{A})$, and so we obtain a contradiction. Indeed, we have that u_{λ_n} is a solution of (1.1) with $u_{\lambda_n}(0) \in \mathcal{A}$. So, there exists $f_{\lambda_n} \in L^1(0,T;H)$, with $f_{\lambda_n}(t) \in F(u_{\lambda_n}(t)) + h$, *a.e.* in $(0,T)$, and such that u_{λ_n} is an integral solution over $(0,T)$ of the problem $(P^1_{\lambda_n})$ below:

$$(P^1_{\lambda_n})\ \ \frac{\partial u_{\lambda_n}}{\partial t} - div(D^{\lambda_n}|\nabla u_{\lambda_n}|^{p-2}\nabla u_{\lambda_n}) + |u_{\lambda_n}|^{p-2}u_{\lambda_n} = f_{\lambda_n} \qquad \text{in}\ \ (0,T).$$

We can suppose $t_0 \in (0,T)$. As $\mathcal{A}$ is compact $u_{\lambda_n}(0) \to u_0 \in \mathcal{A}$. Let $u_{\lambda_n}(\cdot) \doteq I(u_{0,\lambda_n})f_{\lambda_n}(\cdot)$ and $z_{\lambda_n}(\cdot) \doteq I(u_0)f_{\lambda_n}(\cdot)$ be the solution of the problem

$$(P_{f_{\lambda_n},u_0})\begin{cases} \dfrac{\partial z_{\lambda_n}}{\partial t} - div(D^{\lambda_n}|\nabla z_{\lambda_n}|^{p-2}\nabla z_{\lambda_n}) + |z_{\lambda_n}|^{p-2}z_{\lambda_n} = f_{\lambda_n} \\ z_{\lambda_n}(0) = u_0. \end{cases}$$

By (4.1) and Remark 4.3, there exists $L > 0$ such that $\| f_{\lambda_n}(t) \|_H \leq L$ for all $t \in [0,T]$, and for all $n \in \mathbb{N}$. Let $K \doteq \{f_{\lambda_n}; n \in \mathbb{N}\}$ and $M(K) \doteq \{z_{\lambda_n}; n \in \mathbb{N}\}$.

Once K is a bounded set, it is easy to see it is a uniformly integrable subset. Given $t \in (0,T]$ and $h > 0$ such that $t-h \in (0,T]$, consider the operator $T_h : M(K)(t) \to H$ defined by $T_h z_{\lambda_n}(t) = S^{\lambda_n}(h) z_{\lambda_n}(t-h)$. By Statement 1 in [21], the operator $T_h : M(K)(t) \to H$ is compact. Then, from Theorem 3.2 in [21], the set $M(K)$ is relatively compact in $C([0,T];H)$ and so there exists $z \in C([0,T];H)$ and there exists a subsequence $\{z_{\lambda_n}(\cdot)\}$ such that $z_{\lambda_n} \to z$ in $C([0,T];H)$. As each z_{λ_n} is a solution of $(P_{f_{\lambda_n}, u_0})$, then z_{λ_n} verify

$$\frac{1}{2} \parallel z_{\lambda_n}(t) - \theta \parallel^2 \leq \frac{1}{2} \parallel z_{\lambda_n}(s) - \theta \parallel^2 + \int_s^t \langle f_{\lambda_n}(\tau) - y_{\lambda_n}, z_{\lambda_n}(\tau) - \theta \rangle d\tau \qquad (5.1)$$

for all $\theta \in \mathcal{D}(A^{D^{\lambda_n}}) \subseteq W^{1,p}(\Omega) \subset H$ and $y_{\lambda_n} = A^{D^{\lambda_n}}(\theta)$ and for all $0 \leq s \leq t \leq T$. As $\parallel f_{\lambda_n}(\tau) \parallel_H \leq L$, for all $0 \leq \tau \leq T$ and for all $n \in \mathbb{N}$, we conclude that there exists a positive constant $\widetilde{L}$ such that $\parallel f_{\lambda_n} \parallel_{L^2(0,T;H)} \leq \widetilde{L}$ for all $n \in \mathbb{N}$. As $L^2(0,T;H)$ is a reflexive Banach space there is $f \in L^2(0,T;H)$ and subsequence, which we do not relabel, $\{f_{\lambda_n}\}$ such that $f_{\lambda_n} \rightharpoonup f$ in $L^2(0,T;H)$. Consequently $f_{\lambda_n} \rightharpoonup f$ in $L^1(0,T;H)$. Moreover,

$$\begin{aligned}
\sup_{t\in[0,T]} \parallel u_{\lambda_n}(t) - z(t) \parallel_H \quad &\leq \quad \sup_{t\in[0,T]} \parallel I(u_{0,\lambda_n}) f_{\lambda_n}(t) - I(u_0) f_{\lambda_n}(t) \parallel_H \\
&\quad + \sup_{t\in[0,T]} \parallel z_{\lambda_n}(t) - z(t) \parallel_H \\
&\leq \quad \parallel u_{0,\lambda_n} - u_0 \parallel_H \\
&\quad + \sup_{t\in[0,T]} \parallel z_{\lambda_n}(t) - z(t) \parallel_H \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

Therefore $u_{\lambda_n} \to z$ in $C([0,T];H)$. So, from Theorem 3.3 in [9], $f(t) \in F(z(t))$ t-a.e. in $[0,T]$. Since $f_{\lambda_n} \rightharpoonup f$ in $L^2(0,T;H)$ implies that $f_{\lambda_n} \rightharpoonup f$ in $L^2(s,t;H)$, $\forall\, 0 \leq s \leq t \leq T$; and $z_{\lambda_n} \to z$ in $C([0,T];H)$ implies that $z_{\lambda_n} \to z$ in $C([s,t];H)$ and consequently $z_{\lambda_n} \to z$ in $L^2(s,t;H)$, $\forall\, 0 \leq s \leq t \leq T$; then

$$\langle f_{\lambda_n} - h, z_{\lambda_n} - \theta \rangle_{L^2(s,t;H)} \to \langle f - h, z - \theta \rangle_{L^2(s,t;H)}$$

for all $\theta, h \in H$. Now, consider $\overline{\theta} \in D(A^{D^0}) \subset W^{1,p}(\Omega) \subset H$ and let be $\overline{h} := A^{D^0}(\overline{\theta}) \in H$. We consider

$$y_{\lambda_n} := A^{D^{\lambda_n}}(\overline{\theta}) = -div(D^{\lambda_n} |\nabla\overline{\theta}|^{p-2} \nabla\overline{\theta}) + |\overline{\theta}|^{p-2}\overline{\theta}.$$

Note that $\mathcal{D}(A^{D^{\lambda_n}}) = D(A^{D^0})$, $\forall\, n \in \mathbb{N}$. We already knows by (5.1) that holds

$$\begin{aligned}
\frac{1}{2} \parallel z_{\lambda_n}(t) - \overline{\theta} \parallel^2 \quad &\leq \quad \frac{1}{2} \parallel z_{\lambda_n}(s) - \overline{\theta} \parallel^2 + \int_s^t \langle f_{\lambda_n}(\tau) - \overline{h}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle d\tau \\
&\quad + \int_s^t \langle \overline{h} - y_{\lambda_n}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle d\tau. \qquad (5.2)
\end{aligned}$$

Repeating the arguments as in [21], we have

$$\int_s^t \langle \overline{h} - y_{\lambda_n}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle d\tau \to 0$$

as $n \to +\infty$. So, taking the limit in inequality (5.2) as $n \to +\infty$, we obtain

$$\frac{1}{2} \parallel z(t) - \overline{\theta} \parallel^2 \leq \frac{1}{2} \parallel z(s) - \overline{\theta} \parallel^2 + \int_s^t \langle f(\tau) - \overline{h}, z(\tau) - \overline{\theta} \rangle d\tau$$

for all $\overline{\theta} \in D(A^{D^0})$ and $\overline{h} \doteq A^{D^0}(\overline{\theta})$ and for all $0 \leq s \leq t \leq T$. So $z \in G_0$ with $z(0) = u_0 \in \mathcal{A}$. Then, $z(t) \in G_0(t, \mathcal{A})$, $\forall\, t \geq 0$. Thus, defining $\xi_0 \doteq z(t_0) \in G_0(t_0, \mathcal{A})$, we obtain

$$\parallel \xi_{\lambda_n} - \xi_0 \parallel_H = \parallel u_{\lambda_n}(t_0) - z(t_0) \parallel_H \leq \sup_{\tau \in [0,T]} \parallel u_{\lambda_n}(\tau) - z(\tau) \parallel_H \to 0 \text{ as } n \to +\infty,$$

which is a contradiction, and so we conclude that the map

$$[0, \infty) \ni \lambda \mapsto G_\lambda(t, \mathcal{A})$$

is w-upper semicontinuous on λ_1 for each $t > 0$. □

Therefore, the family $\{G_\lambda\}_{\lambda \in [0,\infty)}$ satisfies the condition (ii) of the Theorem 5.1. Therefore, using Corollary 4.7, we obtain immediately by Theorem 5.1 the following result:

Theorem 5.3. *The family of global attractors $\{\mathcal{A}_\lambda; \lambda \in [0, \infty)\}$ of the problem (1.1) is upper semicontinuous at λ_1.*

Remark 5.4. *All the results in this work can be reproduced in a similar way for the problem*

$$\frac{\partial u_\lambda}{\partial t}(t) - div(D^\lambda |\nabla u_\lambda(t)|^{p-2} \nabla u_\lambda(t)) \in F(u_\lambda(t)) + h.$$

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