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Generalized derivations in prime and semiprime rings

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ABSTRACT: Let R be a prime ring, I a nonzero ideal of R and m, n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F([x, y])^m = [x, y]_n$ for all $x, y \in I$, then R is commutative. Moreover we also examine the case when R is a semiprime ring.

Key Words: prime and semiprime rings, generalized derivations, GPIs.

Contents

1. Introduction

In all that follows, unless stated otherwise, R will be an associative ring, $Z(R)$ the center of R, Q its Martindale quotient ring and U its Utumi quotient ring. The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [3] for these objects). For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. For each $x, y \in R$ and each $n \geq 1$, define $[x, y]_1 = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k \geq 2$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In [4], Bresar introduced the definition of generalized derivation: an additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. Hence, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier. Basic examples are derivations and generalized inner derivations. We refer to call such mappings generalized inner derivations for the reason they present a generalization of the concept of inner derivations. In [9], Hvala studied generalized derivations in the context of algebras on certain norm spaces. In [13], Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \longrightarrow U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense left ideal of R and d is a derivation from I into U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a

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generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole of U. Lee obtained the following: every generalized derivation F on a dense left ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U.

This paper is included in a line of investigation concerning the relationship between the structure of a ring R and the behaviour of some additive mappings defined on R satisfy certain special identities. In [1], Ashraf and Rehman proved that if R is a prime ring, I a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In [2, Theorem 1], Argac and Inceboz generalized the above result as following: Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer, if R admits a derivation d with the property $(d(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative. In [7], Daif and Bell showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. At this point the natural question is what happens in case the derivation is replaced by a generalized derivation. In [18], Quadri et al., proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative. In [10], we studied a similar condition and proved that a prime ring R satisfying $(F(x \circ y))^n = x \circ y$ must be commutative. The present paper is motivated by the previous results and we here continue this line of investigation by examining what happens a ring R satisfying the identity $(F([x, y])^m = [x, y]_n$. Explicitly we shall prove the following:

Theorem 1.1. *Let R be a prime ring, I a nonzero ideal of R and m, n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F([x, y])^m = [x, y]_n$ for all $x, y \in I$, then R is commutative.*

Theorem 1.2. *Let R be a semiprime ring and m, n fixed positive integers. If R admits a generalized derivation F associated with a derivation d such that $(F([x, y])^m = [x, y]_n$ for all $x, y \in R$, then there exists a central idempotent element e in U such that on the direct sum decomposition $R = eU \oplus (1 - e)U$, d vanishes identically on eU and the ring $(1 - e)U$ is commutative.*

2. The case: R a prime ring

Theorem 2.1. *Let R be a prime ring, I a nonzero ideal of R and m, n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F([x, y])^m = [x, y]_n$ for all $x, y \in I$, then R is commutative.*

Proof: Since R is a prime ring and F is a generalized derivation of R, by Lee [13, Theorem 3], $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. By the given hypothesis we have now $[x, y]_n = (a[x, y] + d([x, y]))^m = (a[x, y] + [d(x), y] + [x, d(y)])^m$ for all $x, y \in I$. By Kharchenko [12], we divide the proof into two cases:

Case 1. Let d be an outer derivation of U, then I satisfies the polynomial identity $(a[x,y]+[s,y]+[x,t])^m=[x,y]_n$ for all $x,y,s,t\in I$. In particular, for $y=0$, I satisfies the blended component $([x,t])^m=0$ for all $x,t\in I$, by Herstein [11, Theorem 2], we have $I\subseteq Z(R)$, and so R is commutative by Mayne [17, Lemma 3].

Case 2. Let now d be the inner derivation induced by an element $q\in Q$, that is $d(x)=[q,x]$ for all $x,y\in U$. It follows that $(a[x,y]+[[q,x],y]+[x,[q,y]])^m=[x,y]_n$ for all $x,y\in I$. By Chuang [5, Theorem 2], I and Q satisfy the same generalized polynomial identities (GPIs), we have $(a[x,y]+[[q,x],y]+[x,[q,y]])^m=[x,y]_n$ for all $x,y\in Q$. In case center C of Q is infinite, we have $(a[x,y]+[[q,x],y]+[x,[q,y]])^m=[x,y]_n$ for all $x,y\in Q\bigotimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q\bigotimes_C\overline{C}$ are prime and centrally closed [8, Theorem 2.5 and Theorem 3.5], we may replace R by Q or $Q\bigotimes_C\overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e. $RC=C$) which is either finite or algebraically closed and $(a[x,y]+[[q,x],y]+[x,[q,y]])^m=[x,y]_n$ for all $x,y\in R$. By Martindale [16, Theorem 3], RC (and so R) is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D.
Assume that $dimV_D\geq 3$.

First of all, we want to show that v and qv are linearly D-dependent for all $v\in V$. Since if $qv=0$ then v,qv is D-dependent, suppose that $qv\neq 0$. If v and qv are D-independent, since $dimV_D\geq 3$, then there exists $w\in V$ such that v,qv,w are also D-independent. By the density of R, there exists $x,y\in R$ such that: $xv=0,xqv=w,xw=v;yv=0,yqv=0,yw=v$. These imply that $v=(a[x,y]+[[q,x],y]+[x,[q,y]])^m v=[x,y]_n v=0$, which is a contradiction. So we conclude that v and qv are linearly D-dependent for all $v\in V$.

Our next goal is to show that there exists $b\in D$ such that $qv=vb$ for all $v\in V$. In fact, choose $v,w\in V$ linearly independent. Since $dimV_D\geq 3$, then there exists $u\in V$ such that u,v,w are linearly independent, and so $b_u,b_v,b_w\in D$ such that $qu=ub_u$, $qv=vb_v$, $qw=wb_w$, that is $q(u+v+w)=ub_u+vb_v+wb_w$. Moreover $q(u+v+w)=(u+v+w)b_{u+v+w}$ for a suitable $b_{u+v+w}\in D$. Then $0=u(b_{u+v+w}-b_u)+v(b_{u+v+w}-b_v)+w(b_{u+v+w}-b_w)$ and because u,v,w are linearly independent, $b_u=b_v=b_w=b_{u+v+w}$, that is b does not depend on the choice of v. Hence now we have $qv=vb$ for all $v\in V$.

Now for $r\in R$, $v\in V$, we have $(rq)v=r(qv)=r(vb)=(rv)b=q(rv)$, that is $[q,R]V=0$. Since V is a left faithful irreducible R-module, hence $[q,R]=0$, i.e. $q\in Z(R)$ and so $d=0$, a contradiction.
Suppose now that $dimV_D\leq 2$.

In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [14, Lemma 2], it follows that there exists a suitable filed F such that $R\subseteq M_k(F)$, the ring of all $k\times k$ matrices over F, and moreover $M_k(F)$ satisfies the same GPI as R.

Assume $k\geq 3$, by the same argument as in the above, we can get a contradiction.

Obviously if $k=1$, then R is commutative.

Thus we may assume that $k = 2$ i.e., $R \subseteq M_2(F)$, where $M_2(F)$ satisfies $(a[x,y] + [[q,x],y] + [x,[q,y]])^m = [x,y]_n$.

Denote e_{ij} the usual matrix unit with 1 in (i,j)-entry and zero elsewhere. Let $[x,y] = [e_{21}, e_{11}] = e_{21}$. Then $[x,y]_n = e_{21}$. In this case we have $(ae_{21} + qe_{21} - e_{21}q)^m = e_{21}$. Right multiplying by e_{21}, we get $(-1)^m(e_{21}q)^m e_{21} = (ae_{21} + qe_{21} - e_{21}q)^m e_{21} = e_{21}e_{21} = 0$. Set $q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. By calculation we find that $(-1)^m \begin{pmatrix} 0 & 0 \\ q_{12}^m & 0 \end{pmatrix} = 0$, which implies that $q_{12} = 0$. Similarly we can see that $q_{21} = 0$. Therefore q is diagonal in $M_2(F)$. Let $f \in Aut(M_2(F))$. Since $(f(a)[f(x),f(y)] + [[f(q),f(x)],f(y)] + [f(x),[f(q),f(y)]])^m = [f(x),f(y)]_n$ so $f(q)$ must be a diagonal matrix in $M_2(F)$. In particular, let $f(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$, then $f(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $M_2(F)$, which leads to $d = 0$, a contradiction. This completes the proof of the theorem. □

The following example demonstrates that R to be prime is essential in the hypothesis.

Example 2.2. *Consider S be any ring and let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$ and let $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in S \right\}$ be a nonzero ideal of R. We define a map $F : R \to R$ by $F(x) = 2e_{11}x - xe_{11}$. Then it is easy to see that F is a generalized derivation associated with a nonzero derivation $d(x) = [e_{11}, x]$. It is straightforward to check that F satisfies the property: $(F([x,y])^m = [x,y]_n$ for all $x, y \in I$. However, R is not commutative.*

3. The case: R a semiprime ring

Theorem 3.1. *Let R be a semiprime ring and m, n fixed positive integers. If R admits a generalized derivation F associated with a derivation d such that $(F([x,y])^m = [x,y]_n$ for all $x, y \in R$, then there exists a central idempotent element e in U such that on the direct sum decomposition $R = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.*

Proof: Since R is semiprime and F is a generalized derivation of R, by Lee [13, Theorem 3], $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. We are given that $(a[x,y] + d([x,y]))^m = [x,y]_n$ for all $x, y \in R$. By Lee [15, Theorem 3], R and U satisfy the same differential identities, then $(a[x,y] + d([x,y]))^m = [x,y]_n$ for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. Since U is a B-algebra orthogonal complete [6, p.42] and MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. For all $\overline{x}, \overline{y} \in \overline{U}$, $(\overline{a}[\overline{x},\overline{y}] + \overline{d}([\overline{x},\overline{y}]))^m = [\overline{x},\overline{y}]_n$. It is obvious that $\overline{U}$ is prime. Therefore by

Theorem 2.1, we have either $\overline{U}$ is commutative or $\overline{d} = 0$, that is either $d(U) \subseteq MU$ or $[U, U] \subset MU$. Hence $d(U)[U, U] \subseteq MU$, where MU runs over all prime ideals of U. Since $\cap_M MU = 0$, we obtain $d(U)[U, U] = 0$.

By using the theory of orthogonal completion for semiprime rings(see [3, Chapter 3]), it is clear that there exists a central idempotent element e in U such that on the direct sum decomposition $R = eU \oplus (1 - e)U$, d vanishes identically on eU and the ring $(1 - e)U$ is commutative. This completes the proof of the theorem. □

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References

1. M. Ashraf and N. Rehman, On commutativity of rings with derivations, Results Math., 42 (2002), no.1-2, 3-8.
2. N. Argac and H. G. Inceboz, Derivations of prime and semiprime rings, J. Korean Math. Soc., 46(2009), no.5, 997-1005.
3. K. I. Beidar, W. S. Martindale and V. Mikhalev, Rings with generalized identities, Monographs and Textbooks in Pure and Applied Mathematics, 196. Marcel Dekker, Inc., New York, 1996.
4. M. Bresar, On the distance of the composition of two derivations to be the generalized derivations, Glasgow Math. J., 33(1991), 89-93.
5. C. L. Chuang, GPIs having coefficents in Utumi quotient rings, Proc. Amer. Math. Soc., 103(1988), no.3, 723-728.
6. C. L. Chuang, Hypercentral derivations, J. Algebra, 161(1994), 37-71.
7. M. N. Daif and H. E. Bell, Remarks on derivations on semiprime rings, Internt. J. Math. & Math. Sci., 15(1992), 205-206.
8. J. S. Erickson, W. S. Martindale III and J. M. Osborn, Prime nonassociative algebras, Pacific J. Math., 60(1975), no.1, 49-63.
9. B. Hvala, Generalized derivations in prime rings, Comm. Algebra, 26(1998), no.4, 1147-1166.
10. S. L. Huang, On generalized derivations of prime and semiprime rings, Taiwanese Journal of Mathematics (to appear).
11. I. N. Herstein, Center-like elements in prime rings, J. Algebra, 60(1979), 567-574.
12. V. K. Kharchenko, Differential identities of prime rings, Algebra and Logic, 17(1978), 155-168.
13. T. K. Lee, Generalized derivations of left faithful rings, Comm. Algebra, 27(1998), no.8, 4057-4073.
14. C. Lanski, An engel condition with derivation, Proc. Amer. Math. Soc., 118(1993), no.3, 731-734.
15. T. K. Lee, Semiprime rings with differential identities, Bull. Inst. Math. Acad. Sinica, 20(1)(1992), 27-38
16. W. S. Martindale III, Prime rings satisfying a generalized polynomial identity, J. Algebra, 12(1969), 176-584.

17. J. H. Mayne, Centralizing mappings of prime rings, Canad. Math. Bull., 27(1984), no.1, 122-126.

18. M. A. Quadri, M. S. Khan and N. Rehman, Generalized derivations and commutativity of prime rings, Indian J.pure appl. Math., 34(2003), no.98, 1393-1396.

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