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The Generalized Difference of χ^2 over $p-$ metric spaces defined by Musielak

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abstract: In this paper, we define the sequence spaces: $\chi^{2qu}_{f\mu}(\Delta)$ and $\Lambda^{2qu}_{f\mu}(\Delta)$, where for any sequence $x = (x_{mn})$, the difference sequence Δx is given by $(\Delta x_{mn})^{\infty}_{m,n=1} = [(x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1})]^{\infty}_{m,n=1}$. We also study some properties and theorems of these spaces.

Key Words: analytic sequence, double sequences, χ^2 space, difference sequence space,Musielak - modulus function, $p-$ metric space, Lacunary sequence, ideal.

Contents

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [14], Moricz [19], Moricz and Rhoades [20], Basarir and Solankan [2],Turkmenoglu [30] and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - |^{t_{mn}} = 1 \, for\, some \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

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$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n \to \infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [8,9] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [33] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [21] and Tripathy [29] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [1] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever [3] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [28] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [18] as an extension of the definition of strongly Cesàro summable sequences. Cannor [5] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [25] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [11]-[12], and [13] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$.

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m + n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all finite sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m,n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m,n \in \mathbb{N})$ are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have:
(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality)[See[15]] \tag{1.2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri [18] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \, for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup \{ |v| u - (f_{mn})(u) : u \geq 0 \}, m, n = 1, 2, \cdots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f and its subspace h_f are defined as follows

$$t_f = \left\{ x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m, n \to \infty \right\},$$

$$h_f = \left\{ x \in w^2 : I_f \left(|x_{mn}| \right)^{1/m+n} \to 0 \, as \, m, n \to \infty \right\},$$

where I_f is a convex modular defined by

$$I_f (x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \left(|x_{mn}| \right)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d (x, y) = sup_{mn} \left\{ inf \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \left(\frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}$$

If X is a sequence space, we give the following definitions:

(i)$X' =$ the continuous dual of X;

(ii)$X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \, for \, each \, x \in X \right\};$

(iii)$X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn} \, is \, convegent, \, foreach \, x \in X \right\};$

(iv)$X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, foreachx \in X \right\};$

(v)$let \, X \, beanFK - space \supset \phi; \, then \, X^f = \left\{ f(\Im_{mn}) : f \in X' \right\};$

(vi)$X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty, \, foreach \, x \in X \right\};$

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha - (or Köthe - Toeplitz)$dual of $X, \beta - (or \, generalized - Köthe - Toeplitz) \, dual \, of \, X, \gamma - \, dual \, of \, X, \, \delta \, - \, dual \, of \, X \, respectively. X^{\alpha}$ is defined by Gupta and Kamptan [15]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z (\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_{∞} denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar in [1]. The spaces $c (\Delta), c_0 (\Delta), \ell_{\infty} (\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z (\Delta) = \left\{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation:
$\Delta^m x_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{m} (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{m+i,n+j}.$

2. Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,
(ii) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1), \ldots, d_n(x_n))\|_p = |\alpha| \, \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}$ $for 1 \leq p < \infty$; (or)
(v) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) := \sup \{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\}$,
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1), \ldots, d_n(x_n))\|_E = sup\,(|det(d_{mn}(x_{mn}))|) =$$
$$sup \left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \ldots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \ldots & d_{2n}(x_{1n}) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \ldots & d_{nn}(x_{nn}) \end{vmatrix} \right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

Let X be a linear metric space. A function $w : X \to \mathbb{R}$ is called paranorm, if
(1) $w(x) \geq 0$, for all $x \in X$;
(2) $w(-x) = w(x)$, for all $x \in X$;
(3) $w(x + y) \leq w(x) + w(y)$, for all $x, y \in X$;
(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with $w(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $w(\sigma_{mn} x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.

A paranorm w for which $w(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [32], Theorem 10.4.2, p.183).

$\eta = (\varphi_{rs})$ a nondecreasing sequence of positive reals tending to infinity and $\varphi_{11} = 1$ and $\varphi_{r+1,s+1} \leq \varphi_{rs} + 1$.
The generalized de la Vallee-Pussin means is defined by :

$$t_{rs}(x) = \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} x_{mn},$$

where $I_{rs} = [rs - \lambda_{rs} + 1, rs]$. For the set of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallee-Poussin method.

The notion of $\lambda-$ double gai and double analytic sequences as follows: Let $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$ be a strictly increasing sequences of positive real numbers tending to infinity, that is

$$0 < \lambda_{00} < \lambda_{11} < \cdots \, and \, \lambda_{mn} \to \infty \, as \, m, n \to \infty$$

and said that a sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0, called a the $\lambda-$ limit of x, if $\mu_{mn}(x) \to 0 \, as \, m, n \to \infty$, where

$$\mu_{mn}(x) = \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left(\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n} + \Delta^{m-1}\lambda_{m+1,n+1}\right) |x_{mn}|^{1/m+n}.$$

The sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ double analytic if $sup_{uv} |\mu_{mn}(x)| < \infty$. If $lim_{mn} x_{mn} = 0$ in the ordinary sense of convergence, then

$$lim_{mn} \left(\frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left(\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n} + \Delta^{m-1}\lambda_{m+1,n+1}\right) \left((m+n)! \, |x_{mn} - 0|\right)^{1/m+n} \right) = 0.$$

This implies that

$$lim_{mn} |\mu_{mn}(x) - 0| = \lim_{mn} \left| \left(\frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left(\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n} + \Delta^{m-1}\lambda_{m+1,n+1}\right) \left((m+n)! \, |x_{mn} - 0|\right)^{1/m+n} \right) \right| = 0.$$

which yields that $lim_{uv} \mu_{mn}(x) = 0$ and hence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0.

Let $f = (f_{mn})$ be a Musielak-modulus function and $\Big(X, \|(d(x_1), d(x_2), \cdots,$

$d(x_{n-1}))\|_p\Big)$ be a $p-$metric space, $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers and $u = (u_{mn})$ be any sequence such that $u_{mn} \neq 0\,(m,n = 1,2,\cdots)$. By $w^2(p-X)$ we denote the space of all sequences defined over $\Big(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\Big)$. The following inequality will be used throughout the paper. If $0 \leq q_{mn} \leq supq_{mn} = H, K = max\left(1, 2^{H-1}\right)$ then

$$|a_{mn} + b_{mn}|^{q_{mn}} \leq K\left\{|a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}}\right\} \tag{2.1}$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also $|a|^{q_{mn}} \leq max\left(1, |a|^H\right)$ for all $a \in \mathbb{C}$.
In the present paper we define the following sequence spaces:

$$\left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V =$$
$$\left\{r, s \in I_{rs} : lim_{rs} u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V =$$
$$\left\{r, s \in I_{rs} : sup_{rs} u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} < \infty\right\},$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V =$$
$$\left\{r, s \in I_{rs} : lim_{rs} u_{mn}\left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V =$$
$$\left\{r, s \in I_{rs} : sup_{rs} u_{mn}\left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} < \infty\right\},$$

If we take $q = (q_{mn}) = 1$, we get

$$\left[\chi^{2u}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V =$$
$$\left\{r, s \in I_{rs} : u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^V = 0\right\},$$

$$\left[\Lambda^{2u}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V =$$
$$\left\{r, s \in I_{rs} : u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] < \infty\right\},$$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces. $\left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ and $\left[\Lambda^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ which we shall discuss in this paper.

3. Main Results

Theorem 3.1. *Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence spaces $\left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ and $\left[\Lambda^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ are linear spaces.*

Proof: It is routine verification. Therefore the proof is omitted. □

Theorem 3.2. *Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence space $\left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ is a paranormed space with respect to the paranorm defined by*

$g(x) = inf$

$$\left\{u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq 1\right\} = 0.$$

Proof: Clearly $g(x) \geq 0$ for $x = (x_{mn}) \in \left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$inf\left\{u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq 1\right\} = 0$$

Suppose that $\mu_{mn}(x) \neq 0$ for each $m, n \in \mathbb{N}$. Then $\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p \to \infty$. It follows that $\left(u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H} \to \infty$ which is a contradiction. Therefore $\mu_{mn}(x) = 0$. Let

$$\left(u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1$$

and

$$\left(u_{mn}\left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1$$

Then by using Minkowski's inequality, we have

$$\left(u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq$$

$$\left(u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H} +$$

$$\left(u_{mn}\left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H}.$$

So we have

$$g(x+y) = inf\left\{u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq 1\right\} \leq$$

$$inf\left\{u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq 1\right\} +$$

$inf\left\{u_{mn}\left[f_{mn}\left(\|\mu_{mn}(y),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\leq 1\right\}$
Therefore,

$$g(x+y)\leq g(x)+g(y).$$

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,
$g(\lambda x)=inf\left\{u_{mn}\left[f_{mn}\left(\|\mu_{mn}(\lambda x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\leq 1\right\}.$
Then
$g(\lambda x)=inf$
$\left\{((|\lambda|\,t)^{q_{mn}/H}:u_{mn}\left[f_{mn}\left(\|\mu_{mn}(\lambda x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\leq 1\right\}$
where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}}\leq max\left(1,|\lambda|^{supp_{mn}}\right)$, we have
$g(\lambda x)\leq max\left(1,|\lambda|^{supp_{mn}}\right)inf$
$\left\{t^{q_{mn}/H}:u_{mn}\left[f_{mn}\left(\|\mu_{mn}(\lambda x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\leq 1\right\}$
This completes the proof. □

Theorem 3.3. *(i) If the sequence (f_{mn}) satisfies uniform Δ_2- condition, then*
$\left[\chi^{2qu}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|^{\varphi}_p\right]^{V\alpha}=$
$\left[\chi^{2qu\mu}_{g},\|\mu_{uv}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|^{\varphi}_p\right]^{V}.$
(ii) If the sequence (g_{mn}) satisfies uniform Δ_2- condition, then
$\left[\chi^{2qu\mu}_{g},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|^{\varphi}_p\right]^{V\alpha}=$
$\left[\chi^{2qu}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|^{\varphi}_p\right]^{V}$

Proof: Let the sequence (f_{mn}) satisfies uniform Δ_2- condition, we get

$$\left[\chi^{2qu\mu}_{g},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|^{\varphi}_p\right]^{V}\subset$$
$$\left[\chi^{2qu}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|^{\varphi}_p\right]^{V\alpha} \quad (3.1)$$

To prove the inclusion
$\left[\chi^{2qu}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|^{\varphi}_p\right]^{V\alpha}\subset$

$\left[\chi^{2qu\mu}_{g},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|^{\varphi}_p\right]^{V},$
let $a\in\left[\chi^{2qu}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|^{\varphi}_p\right]^{V}$. Then for all $\{x_{mn}\}$ with $(x_{mn})\in\left[\chi^{2qu}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|^{\varphi}_p\right]^{V}$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}a_{mn}|<\infty. \quad (3.2)$$

Since the sequence (f_{mn}) satisfies uniform Δ_2- condition, then
$(y_{mn}) \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$, we get
$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{\varphi_{rs}y_{mn}a_{mn}}{\Delta^m\lambda_{mn}(m+n)!}\right| < \infty$. by (3.2). Thus
$(\varphi_{rs}a_{mn}) \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V} =$
$\left[\chi^{2qu\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$ and hence
$(a_{mn}) \in \left[\chi^{2qu\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$. This gives that

$$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V\alpha} \subset \left[\chi^{2qu\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V} \tag{3.3}$$

we are granted with (3.1) and (3.3)
$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V\alpha} =$
$\left[\chi^{2qu\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$
(ii) Similarly, one can prove that
$\left[\chi^{2qu\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V\alpha} \subset$
$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$ if the sequence (g_{mn}) satisfies uniform Δ_2- condition. □

Proposition 3.4. *If $0 < q_{mn} < p_{mn} < \infty$ for each m and n, then*
$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V} \subseteq$
$\left[\Lambda^{2pu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$

Proof: The proof is standard, so we omit it. □

Proposition 3.5. *(i) If $0 < inf q_{mn} \leq q_{mn} < 1$ then*
$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V} \subset$
$\left[\Lambda^{2u}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$.
(ii) If $1 \leq q_{mn} \leq sup q_{mn} < \infty$, then
$\left[\Lambda^{2u}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V} \subset$
$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$

Proof: The proof is standard, so we omit it. □

Proposition 3.6. *Let* $f^{'} = \left(f^{'}_{mn}\right)$ *and* $f^{''} = \left(f^{''}_{mn}\right)$ *are sequences of Musielak functions, we have*
$\left[\Lambda^{2qu}_{f'\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V} \bigcap$
$\left[\Lambda^{2qu}_{f''\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V} \subseteq$
$\left[\Lambda^{2qu}_{f'+f''\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$

Proof: The proof is easy so we omit it. □

Proposition 3.7. *For any sequence of Musielak functions* $f = (f_{mn})$ *and* $q = (q_{mn})$ *be double analytic sequence of strictly positive real numbers. Then*
$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V} \subset$
$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}.$

Proof: The proof is easy so we omit it. □

Proposition 3.8. *The sequence space*
$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$ *c is solid*

Proof: Let $x = (x_{mn}) \in \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$, (i.e)

$$sup_{mn}\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V} < \infty.$$

Let (α_{mn}) be double sequence of scalars such that $|\alpha_{mn}| \leq 1$ for all $m, n \in \mathbb{N} \times \mathbb{N}$. Then we get
$sup_{mn}\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(\alpha x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V} \leq$
$\sup_{mn}\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}.$
This completes the proof. □

Proposition 3.9. *The sequence space*
$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{V}$ *is monotone*

Proof: The proof follows from Proposition 3.8. □

Proposition 3.10. *If* $f = (f_{mn})$ *be any Musielak function. Then*
$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{*}}_{p}\right]^{V} \subset$
$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]^{V}$ *if and only if* $sup_{r,s\geq 1}\frac{\varphi^{*}_{rs}}{\varphi^{**}_{rs}} < \infty$.

Proof: Let $x \in \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^*}\right]^V$ and $N = sup_{r,s\geq 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty$. Then we get

$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^{**}_{rs}}\right]^V =$

$N\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^*_{rs}}\right]^V = 0.$

Thus $x \in \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^{**}}\right]^V$. Conversely, suppose that

$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^*}\right]^V \subset$

$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^{**}}\right]^V$ and

$x \in \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^*}\right]^V$. Then

$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^*}\right]^V < \epsilon$, for every $\epsilon > 0$. Suppose that $sup_{r,s\geq 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} = \infty$, then there exists a sequence of members (rs_{jk}) such that $lim_{j,k\to\infty}\frac{\varphi^*_{jk}}{\varphi^{**}_{jk}} = \infty$. Hence, we have

$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^*_{rs}}\right]^V = \infty$. Therefore

$x \notin \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^{**}}\right]^V$, which is a contradiction. This completes the proof. □

Proposition 3.11. *If $f = (f_{mn})$ be any Musielak function. Then*

$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^*}\right]^V =$

$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^{**}}\right]^V$ *if and only if* $sup_{r,s\geq 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty, sup_{r,s\geq 1}\frac{\varphi^{**}_{rs}}{\varphi^*_{rs}} > \infty$.

Proof: It is easy to prove so we omit. □

Proposition 3.12. *The sequence space*

$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^V$ *is not solid*

Proof: The result follows from the following example.
Example: Consider

$$x = (x_{mn}) = \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1 & 1 & ... & 1 \end{pmatrix} \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^V.$$

Let

$$\alpha_{mn} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\ . & & & \\ . & & & \\ . & & & \\ -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \end{pmatrix}, \text{ for all } m,n \in \mathbb{N}.$$

Then $\alpha_{mn}x_{mn} \notin \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$. Hence $\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ is not solid. □

Proposition 3.13. *The sequence space* $\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ *is not monotone*

Proof: The proof follows from Proposition 3.12.

A sequence $x = (x_{mn})$ is said to be $\varphi-$ statistically convergent or $s_\varphi-$ statistically convergent to 0 if for every $\epsilon > 0$,

$$lim_{rs}\left|\left\{u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right| \geq \epsilon\right\} = 0$$

where the vertical bars indicates the number of elements in the enclosed set. In this case we write $s_\varphi - limx = 0$ or $x_{mn} \to 0\,(s_\varphi)$ and $s_\varphi = \{x : \exists 0 \in \mathbb{R} : s_\varphi - limx = 0\}$. □

Proposition 3.14. *For any sequence of Musielak functions $f = (f_{mn})$ and $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. Then* $\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V \subset \left[s^{2qu}_{\varphi f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$.

Proof: Let $x \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ and $\epsilon > 0$. Then $u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq \left|\left\{u_{mn}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right| \geq \epsilon\right\}$ from which it follows that $x \in \left[s^{2qu}_{\varphi f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$.

To show that $\left[s^{2qu}_{\varphi f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ strictly contain $\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$. We define $x = (x_{mn})$ by $(x_{mn}) = mn$ if $rs - \left[\sqrt{\varphi_{rs}}\right] + \leq mn \leq rs$ and $(x_{mn}) = 0$ otherwise. Then $x \notin \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^V$ and for every $\epsilon\,(0 < \epsilon \leq 1)$,

$$\left|\left\{u_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p\right)\right]^{q_{mn}}\right| \geq \epsilon\right\} = \frac{\left[\sqrt{\varphi_{rs}}\right]}{\varphi_{rs}} \to 0 \text{ as } r,s \to \infty$$

i.e $x \to 0\left(\left[s^{2qu}_{\varphi f\mu}, \left\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|^{\varphi}_p\right]^V\right)$, where [] denotes the greatest integer function. On the other hand,

$$u_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p\right)\right]^{q_{mn}} \to \infty \text{ as } r,s \to \infty$$

i.e $x_{mn} \not\to 0\left[\chi^{2qu}_{f\mu}, \left\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|^{\varphi}_p\right]^V$. This completes the proof. □

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