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Sufficient conditions for certain subclasses of meromorphic p-valent functions

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ABSTRACT: In the present paper, we obtain certain sufficient conditions for meromorphic p-valent functions. Several corollaries and consequences of the main results are also considered.

Key Words: Meromorphic multivalent functions, meromorphic starlike functions, meromorphic convex functions, meromorphic close-to-convex functions.

Contents

1. Introduction and definitions

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n \quad , \qquad (p \in \mathbb{N} := \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured open unit disk

$$\mathcal{U}^* = \{z : z \in \mathbb{C}; 0 < |z| < 1\} =: \mathcal{U}\backslash\{0\}.$$

where $\mathcal{U}$ is an open unit disk. A function $f(z)$ in Σ_p is said to be meromorphically p-valent starlike of order δ if and only if

$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \delta \quad (z \in \mathcal{U}^*), \tag{1.2}$$

for some $\delta(0 \leq \delta < p)$. We denote by $\Sigma_p^*(\delta)$ the class of all meromorphically p-valent starlike of order δ. Further, a function $f(z)$ in Σ_p is said to be meromorphically p-valent convex of order δ if and only if

$$\Re\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > \delta \quad (z \in \mathcal{U}^*), \tag{1.3}$$

2000 *Mathematics Subject Classification:* 30C45 for some $\delta(0 \leq \delta < p)$. We denote by $\Sigma_p^k(\delta)$ the class of all meromorphically p-valent convex of order δ. A function $f(z)$ belonging to Σ_p is said to be meromorphically p-valent close-to-convex of order δ if it satisfies

$$\Re\left(-\frac{f'(z)}{z^{-p-1}}\right) > \delta \qquad (z \in \mathfrak{U}^*), \tag{1.4}$$

for some $\delta(0 \leq \delta < p)$.We denote by $\Sigma_p^c(\delta)$ the subclass of Σ_p consisting of functions which are meromorphically p-valent close-to-convex of order δ in $\mathfrak{U}^*$.

Note that $\Sigma_1^*(\delta) = \Sigma^*(\delta)$, $\Sigma_1^k(\delta) = \Sigma^k(\delta)$ and $\Sigma_1^c(\delta) = \Sigma^c(\delta)$, where $\Sigma^*(\delta)$, $\Sigma^k(\delta)$ and $\Sigma^c(\delta)$ are subclasses of Σ_1 consisting meromorphic univalent functions which are respectively, starlike, convex and close-to-convex of order $\delta(0 \leq \delta < 1)$.

Some subclasses of $\Sigma_p = \Sigma$ when $p = 1$ were considered by (for example) Miller [12], Pommerenke [16], Clunie [7], Frasin and Darus [8] and Royster [17]. Furthermore, several subclasses of Σ_p were studied by (amongst others) Mogra *et al.* [14], Goyal and Prajapat [11], Owa *et al.* [15], Srivastava *et al.* [18], Wang and Zhang [21],Uralegaddi and Ganigi [19], Cho *et al.* [6], Aouf [1-4], and Uralegaddi Somantha [20].

The object in the present paper is to obtain some sufficient conditions for meromorphic p-valent functions.

In the proofs of our main results, we need the following Jack's Lemma [9]:

Lemma 1.1. *Let the (non constant) function $w(z)$ be analytic in $\mathfrak{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathfrak{U}$, then*

$$z_0 w'(z_0) = m w(z_0)$$

where m is a real number and $m \geq n$ where $n \geq 1$.

2. Main Results

With the aid of Lemma 1.1, we derive the next two theorems.

Theorem 2.1. *Let the function $f \in \Sigma_p$, satisfies the inequality*

$$-\Re\left[\alpha\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \frac{[2(\alpha+\beta)p + n] + \lambda\,[2(\alpha+\beta)p - n]}{2(1+\lambda)}. \tag{2.1}$$

Then

$$\Re\left[(z^p f(z))^{\alpha}\left(\frac{-z^{p+1}f'(z)}{p}\right)^{\beta}\right] > \frac{1+\lambda}{2} \tag{2.2}$$

where $(\alpha, \beta \in \mathbb{R},\ \lambda \geq 1,\ p, n \in \mathbb{N})$.

Proof: *Let the function w be defined by*

$$(z^p f(z))^{\alpha} \left(\frac{-z^{p+1} f'(z)}{p} \right)^{\beta} = \frac{1 + \lambda w(z)}{1 + w(z)}. \tag{2.3}$$

Then, clearly, w is analytic in $\mathfrak{U}$ with $w(0) = 0$. We also find from (2.3) that

$$-\left[\alpha \frac{zf'(z)}{f(z)} + \beta \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] = p(\alpha+\beta) - \frac{\lambda z w'(z)}{1 + \lambda w(z)} + \frac{z w'(z)}{1 + w(z)}, (z \in \mathfrak{U}). \tag{2.4}$$

Suppose there exists a point $z_0 \in \mathfrak{U}$ such that $|w(z_0)| = 1$ and $|w(z)| < 1$, when $|z| < |z_0|$. Then by applying Lemma 1.1, there exists $m \geq n$ such that

$$z_0 w'(z_0) = m w(z_0), \qquad \left(m \geq n \geq 1; w(z_0) = e^{i\theta}; \theta \in \mathbb{R} \right). \tag{2.5}$$

Then by using (2.4) and (2.5), it follows that

$$\begin{aligned}
& -\Re \left[\alpha \frac{zf'(z_0)}{f(z_0)} + \beta \left(1 + \frac{zf''(z_0)}{f'(z_0)} \right) \right] \\
= \; & p(\alpha + \beta) - \Re \left(\frac{\lambda m e^{i\theta}}{1 + \lambda e^{i\theta}} \right) + \Re \left(\frac{m e^{i\theta}}{1 + e^{i\theta}} \right) \\
= \; & p(\alpha + \beta) - \frac{\lambda m (\lambda + \cos\theta)}{1 + \lambda^2 + 2\lambda \cos\theta} + \frac{m}{2} \\
= \; & p(\alpha + \beta) - \frac{m(\lambda^2 - 1)}{2(1 + \lambda^2 + 2\lambda \cos\theta)} \\
\leq \; & p(\alpha + \beta) - \frac{n}{2} \left(\frac{\lambda - 1}{1 + \lambda} \right) \\
\leq \; & \frac{[2(\alpha + \beta)p + n] + \lambda [2(\alpha + \beta)p - n]}{2(1 + \lambda)}
\end{aligned}$$

which contradicts the given hypothesis. Hence $|w(z)| < 1$, which implies

$$\left| \frac{1 - (z^p f(z))^{\alpha} \left(\frac{-z^{p+1} f'(z)}{p} \right)^{\beta}}{(z^p f(z))^{\alpha} \left(\frac{-z^{p+1} f'(z)}{p} \right)^{\beta} - \lambda} \right| < 1 \tag{2.6}$$

or equivalently

$$\Re \left[(z^p f(z))^{\alpha} \left(\frac{-z^{p+1} f'(z)}{p} \right)^{\beta} \right] > \frac{1 + \lambda}{2}.$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let the function* $f \in \Sigma_p$*, satisfies the inequality*

$$-\Re\left[\alpha\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)\right] < \frac{\{(\alpha+\beta)p+n\}\,\lambda+\{2p(\alpha+\beta)+n\}}{\lambda+2}. \quad (3.1)$$

Then

$$\Re\left[\left(z^p f(z)\right)^{\alpha}\left(\frac{-z^{p+1}}{p}f'(z)\right)^{\beta}\right] > \frac{1}{2+\lambda} \quad (3.2)$$

where $(\alpha,\beta\in\mathbb{R},\ \lambda\geq 1,\ p,n\in\mathbb{N})$.

Proof: *Let the function* w *be defined by*

$$\left(z^p f(z)\right)^{\alpha}\left(\frac{-z^{p+1}}{p}f'(z)\right)^{\beta} = \frac{1}{(1+\lambda)\,w(z)+1}. \quad (3.3)$$

Then clearly w *is analytic in* $\mathfrak{U}$ *with* $w(0)=0$
Using logarithmic differentiation (3.3) *yields*

$$-\left[\alpha\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)\right] = p(\alpha+\beta)+\frac{(1+\lambda)\,zw'(z)}{1+(1+\lambda)\,w(z)}, (z\in\mathfrak{U}). \quad (3.4)$$

Suppose there exists a point $z_0\in\mathfrak{U}$ *such that* $|w(z_0)|=1$ *and* $|w(z)|<1$*, when* $|z|<|z_0|$*. Then by applying Lemma 1.1, there exists* $m\geq n$ *such that*

$$z_0 w'(z_0) = m w(z_0), \qquad \left(m\geq n\geq 1; w(z_0)=e^{i\theta}; \theta\in\mathbb{R}\right). \quad (3.5)$$

Then by using (3.4) *and* (3.5)*, it follows that*

$$\begin{aligned}
-\Re\left[\alpha\frac{zf'(z_0)}{f(z_0)}+\beta\left(1+\frac{zf''(z_0)}{f'(z_0)}\right)\right] &= (\alpha+\beta)\,p+\Re\left(\frac{(1+\lambda)z_0w'(z_0)}{(1+\lambda)w(z_0)+1}\right)\\
&= (\alpha+\beta)\,p+\Re\left(\frac{(1+\lambda)me^{i\theta}}{(1+\lambda)e^{i\theta}+1}\right)\\
&= (\alpha+\beta)\,p+\left(\frac{m(1+\lambda)(1+\lambda+\cos\theta)}{1+(1+\lambda)^2+2(1+\lambda)\cos\theta}\right)\\
&\geq \frac{\{(\alpha+\beta)p+n\}\,\lambda+\{2p(\alpha+\beta)+n\}}{\lambda+2}
\end{aligned}$$

which contradicts the hypothesis (3.1)*. It follows that* $|w(z)|<1$*, that is*

$$\left|\frac{1}{\left(z^p f(z)\right)^{\alpha}\left(\frac{-z^{p+1}}{p}f'(z)\right)^{\beta}}-1\right| < 1+\lambda.$$

This evidently completes the proof of Theorem 2.2. □

3. Corollaries and Consequences

In this concluding section, we consider some Corollaries and Consequences of our main results (Theorem 2.1 and Theorem 2.2).

Upon setting $\alpha = 0$ and $\beta = 1$ in Theorem 2.1, we get

Corollary 3.1. *If the function $f \in \Sigma_p$ satisfies the inequality*

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{(2p+n)+\lambda\,(2p-n)}{2\,(1+\lambda)} \qquad (\lambda \geq 1,\ p,n \in \mathbb{N})$$

then

$$\Re\left(\frac{-z^{p+1}f'(z)}{p}\right) > \frac{1+\lambda}{2}.$$

Setting $p = n = 1$ in Corollary 3.1, the result reduces to

Corollary 3.2. *If the function $f \in \Sigma$ satisfies the inequality*

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{3+\lambda}{2\,(1+\lambda)} \qquad (\lambda \geq 1)$$

then

$$\Re\left[-z^2 f'(z)\right] > \frac{1+\lambda}{2},$$

or equivalently,

$$f \in \Sigma^c\left(\frac{1+\lambda}{2}\right).$$

Setting $\alpha = 0$ and $\beta = 1$, Theorem 2.1 gives

Corollary 3.3. *Let the function $f \in \Sigma_p$, satisfies the inequality*

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{(2p+n)+\lambda\,(2p-n)}{2\,(1+\lambda)} \qquad (\lambda \geq 1,\ p,n \in \mathbb{N}).$$

Then

$$\Re\left(z^p f(z)\right) > \frac{1+\lambda}{2}.$$

Setting $p = n = 1$ in Corollary 3.3, the result reduces to

Corollary 3.4. *Let the function $f \in \Sigma$, satisfies the inequality*

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{3+\lambda}{2(1+\lambda)} \qquad (\lambda \geq 1).$$

Then

$$\Re\left(zf(z)\right) > \frac{1+\lambda}{2}.$$

Setting $\alpha = 1-\gamma$ and $\beta = \gamma$; $\gamma \in \mathbb{R}$ in Theorem 2.2, we obtain the following special case:

Corollary 3.5. *Let the function $f \in \Sigma_p$, satisfies the inequality*

$$-\Re\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > p + \frac{n}{2}\left(\frac{1-\lambda}{1+\lambda}\right) \qquad (\lambda \geq 1,\ p, n \in \mathbb{N}).$$

Then

$$\Re\left[(z^p f(z))\left(\frac{-zf'(z)}{pf(z)}\right)^{\gamma}\right] > \frac{1+\lambda}{2}.$$

Setting $\alpha = 0$ and $\beta = 1$ in Theorem 2.2, we get

Corollary 3.6. *If the function $f \in \Sigma_p$ satisfies the inequality*

$$-\Re\left[1+\frac{zf''(z)}{f'(z)}\right] < \frac{(p+n)\lambda + (2p+n)}{\lambda+2} \qquad (\lambda \geq 1,\ p, n \in \mathbb{N})$$

then

$$\Re\left[\left(\frac{-z^{p+1}}{p}f'(z)\right)\right] > \frac{1}{2+\lambda}.$$

Setting $p = n = 1$ in Corollary 3.6, the result reduces to

Corollary 3.7. *If the function $f \in \Sigma$ satisfies the inequality*

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{2\lambda+3}{\lambda+2} \qquad (\lambda \geq 1)$$

then

$$\Re\left[(-z^2 f'(z))\right] > \frac{1}{2+\lambda},$$

or equivalently,

$$f \in \Sigma^c\left(\frac{1}{2+\lambda}\right).$$

Setting $\alpha = 0$ and $\beta = 1$, Theorem 2.2, it gives

Corollary 3.8. *Let the function $f \in \Sigma_p$, satisfies the inequality*

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{(p+n)\lambda + (2p+n)}{\lambda+2} \qquad (\lambda \geq 1,\ p, n \in \mathbb{N}).$$

Then

$$\Re\left[(z^p f(z))\right] > \frac{1}{2+\lambda}.$$

Setting $p = n = 1$ in Corollary 3.8, the result reduces to

Corollary 3.9. *Let the function $f \in \Sigma$, satisfies the inequality*

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{3+2\lambda}{2+\lambda} \qquad (\lambda \geq 1).$$

Then

$$\Re\left[(zf(z))\right] > \frac{1}{2+\lambda}.$$

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