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Fractional Differintegral Operators of The Generalized Mittag-Leffler Function

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abstract: The Riemann-Liouville and Erdélyi-Kober operators of fractional integrals and derivatives,those were generalized by Saigo, are invoked to the generalized Mittag-Leffler function.Some apparent particular cases are mentioned.

Key Words: Generalized fractional calculus, generalized Wright hypergeometric function, generalized Mittag-Leffler function.

Contents

1. Introduction and Preliminaries

The function $E_\alpha(z)$ is advented and studied by [3]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(\alpha n+1)}(\alpha \in C, Re(\alpha) > 0). \tag{1.1}$$

Wiman [1] gave generalization of (1.1), expressed as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(\alpha n+\beta)}; (\alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0). \tag{1.2}$$

In 1971, Prabhakar [8] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ defined by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)n!}(z)^n(\alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0). \tag{1.3}$$

Salim and Faraj [9] have further, gave a generalization of (1.1)as

$$E^{\delta,\xi,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{qn}}{\Gamma(\alpha n+\beta)(\xi)_{pn}}(z)^n, \tag{1.4}$$

2000 *Mathematics Subject Classification:* 26A33,33E12,33C45 where $(\alpha, \beta, \delta, \xi \in C,\ Re(\alpha), Re(\beta), Re(\delta), Re(\xi) > 0, p, q > 0)$ and $q \leq Re(\alpha) + p$.For $\xi = p = q = 1$ and $\delta = \xi = p = q = 1$, it reduces to generalized Mittag-Liffler $E^{\delta}_{\alpha,\beta}(z)$ and Mittag-Liffler function $E_{\alpha,\beta}(z)$ respectively. For $z, a_i, b_j \in C$ and $\alpha_i, \beta_j \in R$ (set of real numbers) $= (-\infty, \infty)$ $\alpha_i, \beta_j \neq 0; i = 1, 2, ...p; j = 1, 2, ...q$, the generalized Wright function [2] is given by

$$ {}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ \\ (b_j, \beta_j)_{1,q} \end{matrix} : z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i n)(z)^n}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j n) n!} \tag{1.5} $$

where $\Gamma(.)$ denote the gamma function.Wright proved several theorems on the asymptotic expansion of generalized Wright function ${}_p\Psi_q(z)$ for all values of the argument z under the condition

$$ \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1. \tag{1.6} $$

The left and right-sided operators of Riemann-Liouville fractional integral operators are defined by Samko, Kilbas and Marichev [7, Sect.5]

$$ (I^{\alpha}_{0+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} f(t) dt, \tag{1.7} $$

$$ (I^{\alpha}_{-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t - x)^{\alpha - 1} f(t) dt, \tag{1.8} $$

$$ (D^{\alpha}_{0+} f)(x) = (\frac{d}{dx})^{[Re(\alpha)+1]} [I^{1-\alpha+[Re(\alpha)]}_{0+} f](x) $$

$$ = (\frac{d}{dx})^{[Re(\alpha)+1]} \frac{1}{\Gamma(1 - \alpha + [Re(\alpha)])} \int_0^x (x - t)^{[Re(\alpha)]-\alpha} f(t) dt, \tag{1.9} $$

$$ (D^{\alpha}_{-} f)(x) = (-\frac{d}{dx})^{[Re(\alpha)+1]} [I^{1-\alpha+[Re(\alpha)]}_{-} f](x) $$

$$ = (-\frac{d}{dx})^{[Re(\alpha)+1]} \frac{1}{\Gamma(1 - \alpha + [Re(\alpha)])} \int_x^{\infty} (t - x)^{[Re(\alpha)]-\alpha} f(t) dt, \tag{1.10} $$

where $x > 0$,$\alpha \in C$,$Re(\alpha) > 0$ and $[Re(\alpha)]$ is the integer part of $Re(\alpha)$

Saigo [7] introduced a generalization of the Riemann-Liouville and Erdélyi-Kober fractional integral operators in terms of Gauss hypergeometric function, those are given by

$$(I_{0+}^{\alpha,\beta,\eta}f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})f(t)dt; (Re(\alpha) > 0), \tag{1.11}$$

$$(I_{-}^{\alpha,\beta,\eta}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t})f(t)dt; (Re(\alpha) > 0), \tag{1.12}$$

$$(D_{0+}^{\alpha,\beta,\eta}f)(x) = (I_{0+}^{-\alpha-\beta,\alpha+\eta}f)(x) = (\frac{d}{dx})^m[I_{0+}^{-\alpha+m,-\beta-m,\alpha+\eta-m}f](x);$$
$$(Re(\alpha) > 0); m = [Re(\alpha)] + 1, \tag{1.13}$$

$$(D_{-}^{\alpha,\beta,\eta}f)(x) = (I_{-}^{-\alpha-\beta,\alpha+\eta}f)(x) = (-\frac{d}{dx})^m[I_{-}^{-\alpha+m,-\beta-m,\alpha+\eta}f](x);$$
$$(Re(\alpha) > 0); m = [Re(\alpha)] + 1. \tag{1.14}$$

If we set $\beta = -\alpha$, operators (1.11)-(1.14)reduces to Riemann-Liouville operators

$$(I_{0+}^{\alpha,-\alpha,\eta}f)(x) = (I_{0+}^{\alpha}f)(x), \tag{1.15}$$

$$(I_{-}^{\alpha,-\alpha,\eta}f)(x) = (I_{-}^{\alpha}f)(x), \tag{1.16}$$

$$(D_{0+}^{\alpha,-\alpha,\eta}f)(x) = (D_{0+}^{\alpha}f)(x), \tag{1.17}$$

$$(D_{-}^{\alpha,-\alpha,\eta}f)(x) = (D_{-}^{\alpha}f)(x). \tag{1.18}$$

2. Main Results: Fractional Calculus and Mittag-Leffler Function

This section deals with results, which established well defined ralations for fractional differintegrals (fractional differential and integral operators) and generalized Mittag-Leffler function,defined by (1.4). First two theorems are with regard to relations (1.11) and (1.12),and ${}_p\Psi_q(z)$generalized Wright function.

Theorem 2.1. *Let $\alpha, \beta, \eta, \rho, \delta, \xi \in C$, such that $Re(\alpha) > 0, Re(\rho+\eta-\beta) > 0, \nu > 0, p, q > 0, q \leq Re(\nu)+p$ and $c \in R$ and $I_{0+}^{\alpha,\beta,\eta}$ be the left-sided operator of the generalized fractional integration then there holds the relation:*

$$(I_{0+}^{\alpha,\beta,\eta}(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{\nu}))(x) = \frac{x^{\rho-\beta-1}\Gamma(\xi)}{\Gamma(\delta)}{}_3\Psi_3\left[\begin{matrix}(\rho+\eta-\beta,\nu),(\delta,q),(1,1) \\ (\rho-\beta,\nu),(\rho+\alpha+\eta,\nu),(\xi,p)\end{matrix} : cx^{\nu}\right], \tag{2.1}$$

provided each side of the equation (2.1) exists.

Proof: Applying (1.4) and (1.11)in the left-side of (2.1), we have

$$I_{0+}^{\alpha,\beta,\eta}(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{\nu}))(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{\nu}])dt. \tag{2.2}$$

We invoke Gauss hypergeometric series (see [4]) and relation (1.4). Then change the order of integration and summation (permitted under prescribed condition) and evaluate the inner integral by beta function. Finally, use Gauss summation theorem and rearrange terms,expression (2.2) yields

$$I_{0+}^{\alpha,\beta,\eta}(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{\nu}))(x) =$$

$$\frac{x^{\rho-\beta-1}\Gamma(\xi)}{\Gamma(\delta)}\sum_{r=0}^{\infty}\frac{\Gamma(\rho+\eta-\beta+\nu r)\Gamma(\delta+qr)\Gamma(1+r)}{\Gamma(\xi+pr)\Gamma(\rho+\alpha+\eta+\nu r)\Gamma(\rho-\beta+\nu r)}\frac{(cx^{\nu})^r}{r!},$$

or

$$I_{0+}^{\alpha,\beta,\eta}(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{\nu}))(x) = \frac{x^{\rho-\beta-1}\Gamma(\xi)}{\Gamma(\delta)}{}_3\Psi_3\left[\begin{matrix}(\rho+\eta-\beta,\nu),(\delta,q),(1,1) \\ (\rho-\beta,\nu),(\rho+\alpha+\eta,\nu),(\xi,p)\end{matrix} : cx^{\nu}\right]$$

The proof is completed □

Remark 1. *If we set $\beta = -\alpha$ and $\xi = p = q = 1$ in equation (2.1), we obtain the known result given by Saxena and Saigo [6, p.145, Eq.14].*

Remark 2. *If we set $\beta = -\alpha$ and $\delta = \xi = p = q = 1$in equation (2.1), we get the known result given by Samko, Kilbas and Marichev [7, table 9.1,formula(23)].*

Theorem 2.2. *Let $\alpha, \beta, \eta, \rho, \delta, \xi \in C$, such that $Re(\alpha) > 0, Re(\alpha+\rho) > max[-Re(\beta), -Re(\eta)], Re(\beta) \neq Re(\eta), \nu > 0, p, q > 0, q \leq Re(\nu) + p$ and $c \in R$ and $I_{-}^{\alpha,\beta,\eta}$be the right-sided operator of the generalized fractional integration then there holds the relation:*

$$I_{-}^{\alpha,\beta,\eta}(t^{-\alpha-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{-\nu}))(x) = \frac{x^{-\alpha-\beta-\rho}\Gamma(\xi)}{\Gamma(\delta)}{}_4\Psi_4\left[\begin{matrix}(\alpha+\beta+\rho,\nu),(\alpha+\eta+\rho,\nu),(\delta,q),(1,1)\\ (\rho,\nu),(\alpha+\rho,\nu),(2\alpha+\beta+\eta+\rho,\nu),(\xi,p)\end{matrix} : cx^{-\nu}\right], \tag{2.3}$$

provided each side of the equation (2.3) exists.

Proof: Applying (1.4) and (1.12) in the left-side of (2.3), we have

$$I_{-}^{\alpha,\beta,\eta}(t^{-\alpha-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{-\nu}))(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t})(t^{-\alpha-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{-\nu}])dt.$$

Now proceeding similarly as of the proof of theorem 2.1, we get

$$I_{-}^{\alpha,\beta,\eta}(t^{-\alpha-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{-\nu}))(x) = \frac{x^{-\alpha-\beta-\rho}\Gamma(\xi)}{\Gamma(\delta)}\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+\beta+\rho+\nu r)\Gamma(\alpha+\eta+\rho+\nu r)\Gamma(\delta+qr)\Gamma(1+r)}{\Gamma(\rho+\nu r)\Gamma(\alpha+\rho+\nu r)\Gamma(2\alpha+\beta+\rho+\nu r)\Gamma(\xi+pr)}\frac{(cx^{-\nu})^r}{r!},$$

or

$$I_{-}^{\alpha,\beta,\eta}(t^{-\alpha-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{-\nu}))(x) = \frac{x^{-\alpha-\beta-\rho}\Gamma(\xi)}{\Gamma(\delta)}{}_4\Psi_4\left[\begin{matrix}(\alpha+\beta+\rho,\nu),(\alpha+\eta+\rho,\nu),(\delta,q),(1,1)\\ (\rho,\nu),(\alpha+\rho,\nu),(2\alpha+\beta+\eta+\rho,\nu),(\xi,p)\end{matrix} : cx^{-\nu}\right].$$

The proof is completed □

Remark 3. *If we put $\beta = -\alpha$ and $\xi = p = q = 1$ put in equation (2.3), we get the known result given by Saxena and Saigo [6, p.147, Eq. (23)].*

Remark 4. *If we put $\beta = -\alpha$and $\delta = \xi = p = q = 1$ in equation (2.3), we obtain the known result [6, p.148, formula (24)].*

Now, in the two theorems that follow, we invoke operators given by (1.13) and (1.14). However, the concept of analysis remains (almost) the same as that used in preceding two theorems.

Theorem 2.3. *Let $\alpha, \beta, \eta, \rho, \delta, \xi \in C$, such that $Re(\alpha) > 0, Re(\rho+\beta+\eta) > 0, \nu > 0, p, q > 0, q \leq Re(\nu) + p$ and $c \in R$ and$D_{0+}^{\alpha,\beta,\eta}$be the left-sided operator of the generalized fractional differentiation then there holds the formula:*

$$D_{0+}^{\alpha,\beta,\eta}(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{\nu}))(x) = \frac{x^{\rho+\beta-1}\Gamma(\xi)}{\Gamma(\delta)} {}_3\Psi_3\left[\begin{matrix}(\alpha+\beta+\eta+\rho,\nu),(\delta,q),(1,1) \\ (\rho+\eta,\nu),(\rho+\beta,\nu),(\xi,p)\end{matrix} : cx^{\nu}\right], \tag{2.4}$$

provided each side of the equation (2.4) exists.

Proof: Applying (1.4) and (1.13)in the left-side of (2.4),we have

$$D_{0+}^{\alpha,\beta,\eta}(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{\nu}))(x) = (\frac{d}{dx})^m[I_{0+}^{-\alpha+m,-\beta-m,\alpha+\eta-m}](t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{\nu}])(x)$$

$$= (\frac{d}{dx})^m \frac{x^{\alpha+\beta}}{\Gamma(-\alpha+m)} \int_0^x (x-t)^{-\alpha+m-1} {}_2F_1(-\alpha-\beta, -\eta-\alpha+m; -\alpha+m; 1-\frac{t}{x})$$
$$(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{\nu}])dt$$

$$= \frac{\Gamma(\xi)}{\Gamma(\delta)} \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+\beta+\eta+\rho+\nu r)\Gamma(\delta+qr)\Gamma(1+r)}{\Gamma(\rho+\eta+\nu r)\Gamma(\rho+\beta+\nu r)\Gamma(\xi+pr)} \frac{c^r x^{\rho+\beta+\nu r-1}}{r!}$$

$$D_{0+}^{\alpha,\beta,\eta}(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{\nu}))(x) = \frac{x^{\rho+\beta-1}\Gamma(\xi)}{\Gamma(\delta)} {}_3\Psi_3\left[\begin{matrix}(\alpha+\beta+\eta+\rho,\nu),(\delta,q),(1,1) \\ (\rho+\eta,\nu),(\rho+\beta,\nu),(\xi,p)\end{matrix} : cx^{\nu}\right].$$

The theorem is established □

Remark 5. *If we put $\beta = -\alpha$ and $\xi = p = q = 1$ put in equation (2.4), we get the known result given by Saxena and Saigo [6, p.149, Eq. (29)].*

Remark 6. *If we put $\beta = -\alpha$ and $\delta = \xi = p = q = 1$ put in equation (2.4), we obtain the known result [6, p.149, Eq. (30)].*

Theorem 2.4. *Let $\alpha, \beta, \eta, \rho, \delta, \xi \in C$, such that $Re(\alpha) > 0, Re(\rho) > max[Re(\alpha+\beta)+m, -Re(\eta)], Re(\alpha+\beta+\eta)+m \neq 0$ (where $m = [Re(\alpha)] + 1$), $Re(\nu) > 0, p, q > 0, q \leq Re(\nu) + p$ and $c \in R$ and $D_{-}^{\alpha,\beta,\eta}$ be the right-sided operator of the generalized fractional differentiation then there holds the relation:*

$$D_{-}^{\alpha,\beta,\eta}(t^{\alpha-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{-\nu}))(x) = \frac{x^{\alpha+\beta-\rho}\Gamma(\xi)}{\Gamma(\delta)}{}_4\Psi_4\left[\begin{matrix}(\rho+\eta,\nu),(\rho-\alpha-\beta,\nu),(\delta,q),(1,1)\\ \\(\rho,\nu),(\rho-\alpha,\nu),(\rho+\eta-\alpha-\beta,\nu),(\xi,p)\end{matrix} : cx^{-\nu}\right] \quad (2.5)$$

provided each side of the equation (2.5) exists.

Proof: Applying (1.4) and (1.14) in the left-side of (2.5),we have

$$D_{-}^{\alpha,\beta,\eta}(t^{\alpha-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{-\nu}))(x) = (-\frac{d}{dx})^m(I_{-}^{-\alpha+m,-\beta-m,\alpha+\eta})(t^{\alpha-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{-\nu}])(x)$$

$$= (-\frac{d}{dx})^m\frac{1}{\Gamma(-\alpha+m)}\int_x^\infty(t-x)^{-\alpha+m-1}t^{\alpha+\beta}{}_2F_1(-\alpha-\beta,-\alpha-\eta;-\alpha+m;1-\frac{x}{t})$$
$$(t^{\alpha-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{-\nu}])dt$$

Implying the simplification process used for providing preceding theorems, we obtain

$$D_{-}^{\alpha,\beta,\eta}(t^{\alpha-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}(ct^{-\nu}))(x) = \frac{x^{\alpha+\beta-\rho}\Gamma(\delta)}{\Gamma(\xi)}{}_4\Psi_4\left[\begin{matrix}(\rho+\eta,\nu),(\rho-\alpha-\beta,\nu),(\delta,q),(1,1)\\ \\(\rho,\nu),(\rho-\alpha,\nu),(\rho+\eta-\alpha-\beta,\nu),(\xi,p)\end{matrix} : cx^{-\nu}\right].$$

This completes the proof. □

Remark 7. *If we put $\beta = -\alpha$ and $\xi = p = q = 1$in equation (2.5), we can produce the known result [6, p.150, Eq. (35)].*

Remark 8. *If we put $\beta = -\alpha$ and $\delta = \xi = p = q = 1$ in equation (2.5), we obtained the known result [6, p.151, Eq. (36)].*

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