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Quasirecognition by prime graph of $C_n(4)$, where $n \geq 17$ is odd

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abstract: Let G be a finite group and let $\Gamma(G)$ be the prime graph of G. We assume that $n \geq 17$ is an odd number. In this paper, we show that if $\Gamma(G) = \Gamma(C_n(4))$, then G has a unique non-abelian composition factor isomorphic to $C_n(4)$. As consequences of our result, $C_n(4)$ is quasirecognizable by its spectrum.

Key Words: Quasirecognition, prime graph, simple group, element order.

Contents

1. Introduction

The spectrum $\omega(G)$ of a finite group G is the set of element orders of G, i.e. a natural number n is in $\omega(G)$ if there is an element of order n in G. A finite non-abelian simple group G is called quasirecognizable by its spectrum, if each finite group H with $\omega(G) = \omega(H)$ has a unique non-abelian composition factor isomorphic to G [1]. If G is a finite group, we denote by $\pi(G)$ the set of all prime divisors of $|G|$. The prime graph (or Gruenberg-Kegel graph) $\Gamma(G)$ of G is the graph with vertex set $\pi(G)$ where two distinct vertices p and q are adjacent by an edge (we write $(p, q) \in \Gamma(G)$) if $p.q \in \omega(G)$ and we denote by $s(G)$ the number of connected components of $\Gamma(G)$. A finite non-abelian simple group G is quasirecognizable by its prime graph, if each finite group P with $\Gamma(P) = \Gamma(G)$ has a unique non-abelian composition factor isomorphic to G [5]. The most recent lists of finite simple groups that are quasirecognizable by prime graph are presented in [2], [4], [6], [7] and [8].

In this paper, we show that the group $C_n(4)$ is quasirecognizable by its prime graph. In fact, we prove the following main theorem:

Main Theorem: Let $n \geq 17$ be an odd number. Then the simple group $C_n(4)$ is quasirecognizable by its prime graph.

Actually in this paper, we will show that how the method in [4] for $C_n(2)$ can be applied for $C_n(4)$.

Note that if a finite group G is quasirecognizable by prime graph, then it is quasirecognizable by spectrum. Thus as a consequence of main theorem, we can prove that the finite simple group $C_n(4)$ is quasirecognizable by spectrum.

2. Preliminaries

Throughout this paper, we use the following notations: we denote by $\rho(G)$ and $\rho(r, G)$ a coclique of maximal size in $GK(G)$ and a coclique of maximal size, containing r, in $GK(G)$, respectively. We put $t(G) = |\rho(G)|$ and $t(r, G) = |\rho(r, G)|$. Also we assume that $q = p^{\alpha}$, where p is a prime and α is a natural number. All further unexplained notations are standard and can be found, for example in [3].

Lemma 2.1. [9, Theorem] *Let G be a finite group whit $t(G) \geq 3$ and $t(2, G) \geq 2$. Then*

1. There exists a finite non-abelian simple group S that

$$S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$$

for the maximal normal soluble subgroup K of G.

2. For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K|.|\bar{G}/S|$. In particular, $t(S) \geq t(G) - 1$.

3. one of the following statements holds:

a) every prime $r \in \pi(G)$ non-adjacent to 2 in $\Gamma(G)$ does not divide the product $|K|.|\bar{G}/S|$. In particular, $t(2, S) \geq t(2, G)$;

b) there exists a prime $r \in \pi(G)$ non-adjacent to 2 in $\Gamma(G)$ in which case $t(G) = 3$, $t(2, G) = 2$ and $S \cong A_7$ or $A_1(q)$ for some odd q.

Lemma 2.2. *Let G be a finite group, H a subgroup of G and N a normal subgroup of G. Then:*

1. if $(p, q) \in \Gamma(H)$, then $(p, q) \in \Gamma(G)$;

2. if $(p, q) \in \Gamma(G/N)$, then $(p, q) \in \Gamma(G)$;

3. if $(p, q) \in \Gamma(G)$ and $\{p, q\} \cap \pi(N) = \emptyset$, then $(p, q) \in \Gamma(G/N)$.

Proof. The proof is straightforward. □

Let s be a prime and let m be a natural number. The s-part of m is denoted by m_s, i.e., $m_s = s^t$ if $s^t \mid m$ and s^{t+1} doesn't divide m. If $\gcd(\mathrm{s}, \mathrm{m}) = 1$ and s is odd, then by $e(s, m)$ we mean that $s \mid (m^{e(s,m)} - 1)$ but s does not divide $(m^a - 1)$ for all natural numbers a with $a < e(s, m)$. If m is odd, we put $e(2, m) = 1$, if $m \equiv 1 \pmod 4$ and $e(2, m) = 2$, if $m \equiv -1 \pmod 4$.

Lemma 2.3. [11, Corollary of Zsigmondy's theorem] *Let q be a natural number greater than 1. For every natural number m, there exists a prime r with $e(r, q) = m$, unless $q = 2$ and $m = 1$, $q = 3$ and $m = 1$, and $q = 2$ and $m = 6$.*

The prime s with $e(s, m) = n$ is called a primitive prime divisor of $m^n - 1$. It is obvious that $m^n - 1$ can have more than one primitive prime divisor. We denote by $r_n(m)$ some primitive prime divisor of $m^n - 1$. We write $A_n^{\epsilon}(q)$ and $D_n^{\epsilon}(q)$, where $\epsilon \in \{\pm\}$, and $A_n^+(q) = A_n(q)$, $A_n^-(q) =^2 A_n(q)$, $D_n^+(q) = D_n(q)$ and $D_n^-(q) =^2 D_n(q)$.

Also, $\nu(n)$ and $\eta(n)$ for an integer n, are defined in [10] as follow:

$$\nu(n) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 4; \\ \frac{n}{2}, & \text{if } n \equiv 2 \pmod 4; \\ 2n, & \text{if } n \equiv 1 \pmod 2. \end{cases}, \tag{2.1}$$

$$\eta(n) = \begin{cases} n, & \text{if } n \text{ is odd}; \\ \dfrac{n}{2}, & \text{otherwise}. \end{cases}. \tag{2.2}$$

Lemma 2.4. [10, Proposition 4.1] *Let $G = A_{n-1}(q)$ be a finite simple group of Lie type, r be a prime divisor of $q-1$ and $s \in \pi(G) - \{2, p\}$ such that $k = e(s, q)$. Then s and r are non-adjacent if and only if one of the following holds:*

(1) $k = n$, $n_r \leq (q-1)_r$, and if $n_r = (q-1)_r$, then $2 < (q-1)_r$;

(2) $k = n-1$ and $(q-1)_r \leq n_r$.

Lemma 2.5. [10, Proposition 4.2] *Let $G =^2 A_{n-1}(q)$ be a finite simple group of Lie type, r be a prime divisor of $q+1$ and s be an odd prime distinct from the characteristic. Put $k = e(s, q)$. Then s and r are non-adjacent if and only if one of the following holds:*

(1) $\nu(k) = n$, $n_r \leq (q+1)_r$, and if $n_r = (q+1)_r$, then $2 < (q+1)_r$;

(2) $\nu(k) = n-1$ and $(q+1)_r \leq n_r$.

Lemma 2.6. [10, Propositions 2.1,2.2] *Let G be a finite simple group of Lie type over a field of order q. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$.*

1. If $G = A_{n-1}(q)$ and $2 \leq k \leq l$, then r and s are non-adjacent if and only if $k + l > n$ and k does not divide l;

2. if $G =^2 A_{n-1}(q)$ and $2 \leq \nu(k) \leq \nu(l)$, then r and s are non-adjacent if and only if $\nu(k) + \nu(l) > n$ and $\nu(k)$ does not divide $\nu(l)$.

Lemma 2.7. [11] *Let G be one of the simple groups of Lie type, $B_n(q)$ or $C_n(q)$, over a field of characteristic p. Let r, s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$, and suppose that $1 \leq \eta(k) \leq \eta(l)$. Then r and s are nonadjacent if and only if $\eta(k) + \eta(l) > n$ and l/k is not an odd natural number.*

Lemma 2.8. [10, Proposition 3.1] *Let G be a finite simple classical group of Lie type defined over a field of characteristic p. Let $r \in \pi(G)$ and $r \neq p$. Then r and p are non-adjacent if and only if one of the following holds:*

1. $G = A_{n-1}(q)$, r is odd and $e(r, q) > n-2$;

2. $G =^2 A_{n-1}(q)$, r is odd and $\nu(e(r, q)) > n-2$;

3. $G = C_n(q)$, $\eta(e(r, q)) > n-1$;

4. $G = B_n(q)$, $\eta(e(r, q)) > n-1$;

5. $G = D_n^\epsilon(q)$, where $\epsilon \in \{+, -\}$, $\eta(e(r, q)) > n-2$;

6. $G = A_1(q)$, $r = 2$;

7. $G = A_2^\epsilon(q)$, $r = 3$ and $(q - \epsilon 1)_3 = 3$.

3. Proof of the main theorem

Let G be a finite group with $\Gamma(G) = \Gamma(C_n(4))$, where n is an odd number and $n \geq 17$. During the proof of the main theorem we used Tables 2-8 in [10] without the reference number [10].

By Tables 4 and 8 we have:

1. $t(C_n(4)) = [(3n+5)/4] \geq 14,\ \rho(C_n(4)) = \{r_{2i}(4) | [(n+1)/2] \leq i \leq n\ \} \cup \{r_i(4) | (n+1)/2 \leq i \leq n,\ i \equiv 1 \pmod 2\}$,
2. $t(2, C_n(4)) = 3, \rho(2, C_n(4)) = \{2, r_n(4), r_{2n}(4)\}$.

Since $t(G) = t(C_n(4)) \geq 14$ and $t(2, G) = t(2, C_n(4)) = 3$, we can apply Lemma (2.1) for G. Let S be the non-abelian simple group which is obtained in that Lemma. If S is a sporadic simple group or an exceptional simple group of Lie type, then $t(S) \leq 12$ (see Table 4 in [11] and Table 2 in [10]). But this is impossible, because by Lemma (2.1)(2), $t(S) \geq t(G) - 1 \geq 13$. Hence, S is either an alternating group or a classical group of Lie type. We will prove that $S \cong C_n(4)$ in two steps:

Step I. The simple group S can not be an alternating group $A_m, m \geq 5$.

If $S \cong A_m$, where $m \geq 5$, then since $t(S) \geq 13$, by Table 3, we can see that $m \geq 10$ and $\rho(2, S) = \tau(2, m) \cup \{2\}$, where

$$\tau(2, m) = \{\ s : s \textit{ is a prime},\ m - 3 \leq s \leq m\}.$$

Also, by Lemma (2.1)(3), we can see that $\rho(2, C_n(4)) = \rho(2, G) \subseteq \rho(2, S)$. Hence, $\{r_n(4), r_{2n}(4)\} \subseteq \tau(2, m)$. So, by the definition of $\tau(2, m)$, we can see that $r_n(4) - r_{2n}(4) = \epsilon$, where $\epsilon \in \{+2, -2\}$. But by Fermat's little theorem, $2n = e(r_{2n}(4), 4) \mid r_{2n}(4) - 1$ and $n = e(r_n(4), 4) \mid r_n(4) - 1$. Therefore, $n | \epsilon$. This implies that $n \mid 2$, which is impossible.

Step II. If S is a classical Lie type group, then we shall prove that $S \cong C_n(4)$. We prove this, with a case by case analysis.

Case 1. S can not be a simple group of type $A_{n'-1}(q)$, where $q = p^\alpha$.

If $S \cong A_{n'-1}(q)$, then since $t(S) \geq 13$, by Table 8, we can see that $n' \geq 25$. Thus by Tables 4 and 6, we have the following two subcases:

(i) If $p = 2$, then by Table 4, $\rho(2, S) = \{2, r_{n'-1}(2^\alpha), r_{n'}(2^\alpha)\}$ and since $\rho(2, C_n(4)) \subseteq \rho(2, S)$, we conclude that each number in the set

$$\{r_{n'-1}(2^\alpha), r_{n'}(2^\alpha)\}$$

is a primitive prime divisor of $4^n - 1$ or $4^{2n} - 1$.

If $r_{n'-1}(2^\alpha) = r_n(4)$, then we can see that $e(r_n(4), 2) \in \{n, 2n\}$ and hence, $\alpha(n' - 1) \in \{n, 2n\}$.

Also, $r_{n'}(2^\alpha) = r_{2n}(4)$ gives that $n'\alpha = 4n$. This give that $n' = 2$ or $n' = 4/3$, which is impossible.

If $r_{n'-1}(2^\alpha) = r_{2n}(4)$ and $r_{n'}(2^\alpha) = r_n(4)$, then we can see that $(n' - 1)\alpha = 4n$ and $n'\alpha \in \{n, 2n\}$. Therefore, $\alpha < 0$, which is impossible.

(ii) If $p \neq 2$, then gcd(4, p) = 1. Let $t = e(p, 4)$, then $t \geq 1$. Thus one of the following occurs:

a. If $t = 1$, then $p = 3$. Since $n' \geq 25$,

$$|S| = |A_{n'-1}(q)| = \frac{1}{\gcd(n', q-1)} q^{(n'-1)\frac{n'}{2}} \prod_{i=2}^{n'} (q^i - 1)$$

and $11 \mid 3^5 - 1$, we deduce that $11 = r_5(3^\alpha)$ or $r_1(3^\alpha) \in \pi(S)$. Now we are going to find $t(11, S)$. If $11 = r_5(3^\alpha)$, then by Lemmas (2.4) and (2.8)(1), we can see that $(2, 11), (r_1(q), 11)$ and $(3, 11) \in \Gamma(S)$. Therefore, if $11 \neq x \in \rho(11, S)$, then x is an odd number distinct from 3 and $r_1(q)$ and if $e(x, q) = l$, then by Lemma (2.6)(1), we conclude that $l + 5 > n'$ and 5 does not divide l. Therefore, $l \in \{n', n'-1, n'-2, n'-3, n'-4\}$. Since $\{n', n'-1, n'-2, n'-3, n'-4\}$ are five consecutive numbers, then 5 divides exactly one of them and we have exactly four choices for l. Thus, four elements of the set $\{r_{n'}(q), r_{n'-1}(q), r_{n'-2}(q), r_{n'-3}(q), r_{n'-4}(q)\}$ can be chosen for x. Also, by Lemma (2.6)(1), we can see that this set is independent. Thus, $t(11, S) = 4$. If $11 = r_1(3^\alpha)$, then by Lemma (2.4), we can see that $t(11, S) \leq 3$. On the other side, $11 \in \pi(S) \subseteq \pi(C_n(4))$ and we can consider $\rho(11, C_n(4))$. Since $\eta(e(11, 4)) = e(11, 4) = 5$, by Lemma (2.7), we can see that $\{r_{2n}(4), r_{2(n-1)}(4), r_{2(n-2)}(4), r_{2(n-3)}(4), r_{2(n-4)}(4)\} \subseteq \rho(11, C_n(4))$ and also, since 5 divides at most one of the elements of the set $\{n, n-2, n-4\}$, we obtain that at least seven elements of the set

$$\{r_n(4), r_{2n}(4), r_{2(n-1)}(4), r_{n-2}(4), r_{2(n-2)}(4), r_{2(n-3)}(4), r_{n-4}(4), r_{2(n-4)}(4)\}$$

are in $\rho(11, C_n(4))$. Hence, $t(11, C_n(4)) \geq 7$. Now by assuming $\rho = \rho(11, C_n(4))$ in Lemma (2.1)(3), we can see that $t(11, S) \geq |\rho(11, C_n(4)) \cap \pi(S)| \geq t(11, C_n(4)) - 1$ and hence, $4 \geq t(11, S) \geq t(11, C_n(4)) - 1 \geq 7 - 1 = 6$, which is impossible.

b. If $t = 2$, then $p = 5$. Since $n' \geq 25$

$$|S| = |A_{n'-1}(q)| = \frac{1}{\gcd(n', q-1)} q^{(n'-1)\frac{n'}{2}} \prod_{i=2}^{n'} (q^i - 1)$$

and $31 \mid 5^3 - 1$, we deduce that $31 = r_3(5^\alpha)$ or $r_1(5^\alpha) \in \pi(S)$. Now we are going to find $t(31, S)$. If $31 = r_3(5^\alpha)$, then by Lemmas (2.4) and (2.8)(1), we can see that $(2, 31), (r_1(q), 31)$ and $(5, 31) \in \Gamma(S)$. Therefore, if $31 \neq x \in \rho(31, S)$, then x is an odd number distinct from 5 and $r_1(q)$ and if $e(x, q) = l$, then by Lemma (2.6)(1), we conclude that $l+3 > n'$ and 3 does not divide l. Therefore, $l \in \{n', n'-1, n'-2\}$. Since $\{n', n'-1, n'-2\}$ are three consecutive numbers, then 3 divides exactly one of them and we have exactly two choices for l. Thus, two elements of the set $\{r_{n'}(q), r_{n'-1}(q), r_{n'-2}(q)\}$ can be chosen for x. Also, by Lemma (2.6)(1), we can see that this set is independent. Thus, $t(31, S) = 3$. If $31 = r_1(5^\alpha)$, then by Lemma (2.4), we can see that $t(31, S) \leq 3$. On the other side, $31 \in \pi(S) \subseteq \pi(C_n(4))$ and we can consider $\rho(31, C_n(4))$. Since $\eta(e(31, 4)) = e(31, 4)) = 5$, by Lemma (2.7), we can see that $\{r_{2n}(4), r_{2(n-1)}(4), r_{2(n-2)}(4), r_{2(n-3)}(4), r_{2(n-4)}(4)\} \subseteq \rho(31, C_n(4))$. Hence, $t(31, C_n(4)) \geq 5$. Now by assuming $\rho = \rho(31, C_n(4))$ in Lemma (2.1)(3), we can see that $t(31, S) \geq |\rho(31, C_n(4)) \cap \pi(S)| \geq t(31, C_n(4)) - 1$ and hence, $3 \geq t(31, S) \geq t(31, C_n(4)) - 1 \geq 5 - 1 = 4$, which is impossible.

c. If $t = 3, 4, 5$ and 7, then applying the argument given for Subcase (ii)(b) for $r_3(p)$ leads us to get a contradiction.

d. If $t = 6$, then $p = 13$ and since $61 \mid 13^3 - 1$, $61 \in \pi(S)$ and similar to Subcase (ii)(b), we can see that $t(61, S) \leq 3$. On the other hand, $61 \in \pi(C_n(4))$, $\eta(e(61, 4)) = e(61, 4)/2 = 15$ and $|C_n(4)| = 4^{n^2} \prod_{i=1}^{n}(4^{2i} - 1)$. Therefore $n \geq 15$ and by Lemma (2.7),

$$\{r_{2(n-1)}(4), r_{2(n-3)}(4), r_{2(n-5)}(4), r_{2(n-7)}(4), r_{2(n-9)}(4)\} \subseteq \rho(61, C_n(4)).$$

Thus $t(61, C_n(4)) \geq 5$ and hence, similar to Subcase (ii)(b), we get a contradiction.

e. For $t \geq 8$, if t is an odd number, then set

$$\rho = \{r_{2(n-1)}(4), r_{2(n-3)}(4), r_{2(n-5)}(4), r_{2(n-7)}(4)\}.$$

Since n and t are odd numbers and $n \geq 17$, by Lemma (2.7), we can see that $\rho \subseteq \rho(p, C_n(4)) \setminus \{p\}$ and since $S \leq G/K$, by Lemma (2.2)(1,2),

$$\rho \cap \pi(S) \subseteq \rho(p, S) \setminus \{p\}.$$

Thus by Table 4, $|\rho \cap \pi(S)| \leq 2$. But, by Lemma (2.1)(2), we conclude that $| \rho \cap \pi(S) | \geq | \rho | - 1 = 3$, which is a contradiction. Also, if t is an even number except 10, 14, where $t/2$ is an odd number, then similar to the previous argument, we get a contradiction. If $t = 10$, then $p = 41$ and now, repeating the argument given for Subcase (ii)(b) leads us to get a contradiction. If $t = 14$, then $p = 29$ and since $67 \mid 29^3 - 1$, $67 \in \pi(S)$ and similar to Subcase (ii)(b), we can see that $t(67, S) \leq 3$. On the other hand, $67 \in \pi(C_n(4))$, $\eta(e(67, 4)) = e(67, 4) = 33$ and

$$|C_n(4)| = 4^{n^2} \prod_{i=1}^{n}(4^{2i} - 1).$$

Therefore $n \geq 33$ and by Lemma (2.7), we see that

$$\{r_{2(n-1)}(4), r_{2(n-3)}(4), r_{2(n-5)}(4), r_{2(n-7)}(4), r_{2(n-9)}(4)\} \subseteq \rho(67, C_n(4)).$$

This gives that $t(67, C_n(4)) \geq 5$. Thus similar to Subcase (ii)(b), we get a contradiction. If t and $t/2$ are even, it is enough to replace ρ with the set

$$\{r_n(4), r_{2n}(4), r_{(n-2)}(4), r_{2(n-2)}(4)\}$$

in the previous argument and get a contradiction.

Hence, by (i) and (ii), we have shown that S can not be a simple group of type $A_{n'-1}(q)$. Similar argument shows that S can not be a simple group of type $D_{n'}(q)$ or ${}^2D_{n'}(q)$. We omit the details here.

Case 2. S can not be a simple group of type ${}^2A_{n'-1}(q)$.

If $S \cong {}^2 A_{n'-1}(q)$, then since $t(S) \geq 13$, by Table 8, we can see that $n' \geq 25$. Thus by Tables 4 and 6, we consider the following possibilities:

(i) If $p = 2$, then by Table 4, we can assume four different cases for n' as follows:

If $n' \equiv 0 \pmod 4$, then $\rho(2, S) = \{2, r_{2n'-2}(2^\alpha), r_{n'}(2^\alpha)\}$ and since

$$\rho(2, C_n(4)) \subseteq \rho(2, S),$$

applying the argument given for Case 1(i) shows that $\alpha \leq 0$ or $n' \leq 2$, which is impossible.

If $n' \equiv 1 \pmod 4$, then $\rho(2, S) = \{2, r_{n'-1}(2^\alpha), r_{2n'}(2^\alpha)\}$ and since

$$\rho(2, C_n(4)) \subseteq \rho(2, S),$$

we can see that $\alpha \leq 0$ or $n' \leq 2$, which is impossible.

If $n' \equiv 2 \pmod 4$, then $\rho(2, S) = \{2, r_{2n'-2}(2^\alpha), r_{n'/2}(2^\alpha)\}$ and since

$$\rho(2, C_n(4)) \subseteq \rho(2, S),$$

$\{2, r_{2n}(4), r_n(4)\} \subseteq \{2, r_{2n'-2}(2^\alpha), r_{n'/2}(2^\alpha)\}$ and hence, we can assume that either $r_n(4) = r_{2n'-2}(2^\alpha)$ and $r_{2n}(4) = r_{n'/2}(2^\alpha)$ or $r_{2n}(4) = r_{2n'-2}(2^\alpha)$ and $r_n(4) = r_{n'/2}(2^\alpha)$. Therefore, $n' \leq 2$ or $\alpha = 0$, which is impossible.

If $n' \equiv 3 \pmod 4$, then $\rho(2, S) = \{2, r_{2n'}(2^\alpha), r_{(n'-1)/2}(2^\alpha)\}$ and similar to the previous argument, we get a contradiction.

(ii) If $p \neq 2$, then by Table 6 and since $n' \geq 25$, we see that

$$2 < n'_2 = (q+1)_2$$

and $\rho(2, S) = \{2, r_{2n'-2}(q), r_{n'}(q)\}$. Similar to Case 1, we are going to get a contradiction by considering $t = e(p, 4)$ in different cases. We know that $t \geq 1$ and $t \neq 2, 4, 6, 10$ (for example if $t = 2$, then $p = 5$ and 4 does not divide $5^\alpha + 1$) and $\rho(2, S) = \rho(2, G) = \{2, r_n(4), r_{2n}(4)\}$.

a. If $t = 3$, then $p = 7$ and since $2 < n'_2 = (q+1)_2$, we have 4 divides $7^\alpha+1$ and n' and hence, α is odd. Thus by Table 4, $t(7, S) = 3$ and $\rho(7, S) - \{7\} = \rho(2, S) - \{2\}$, which shows that $\rho(7, S) = \{7, r_n(4), r_{2n}(4)\}$. Now applying Lemma (2.7) shows that $\{7, r_{2n}(4), r_n(4), r_{2(n-1)}(4), r_{2(n-2)}(4)\} \subseteq \rho(7, G)$, because $e(7, 4) = 3$. Thus Lemma (2.1)(2) forces $3 = t(7, S) \geq t(7, G) - 1 \geq 5 - 1 = 4$, which is a contradiction. Also, by the same procedure and those of used in Case 1(ii), we conclude that $t \notin \{1, 5, 7, 14\}$.

b. If $t \geq 8$ and $t \neq 14$, then by Table 4, $t(p, S) = 3$ and similar to Subcase (ii)(a) of Case 2, we get a contradiction. Hence, by (i) and (ii), we have shown that S can not be a simple group of type ${}^2A_{n'-1}(q)$.

Case 3. If $S \cong C_{n'}(q)$, then $t(S) \geq 13$ and $t(2, S) \geq 3$, so by Tables 4,6 and 8, we have the following:

1. n' is odd and $n' \geq 17$,
2. $p = 2$ and $\rho(2, S) = \{2, r_{n'}(2^\alpha), r_{2n'}(2^\alpha)\}$.

Since $\rho(2, C_n(4)) \subseteq \rho(2, S)$, $\{r_{2n'}(2^\alpha), r_{n'}(2^\alpha)\} = \{r_n(4), r_{2n}(4)\}$.

If $r_n(4) = r_{2n'}(2^\alpha)$ and $r_{2n}(4) = r_{n'}(2^\alpha)$, then $n = 0$, which is impossible. Therefore, $r_n(4) = r_{n'}(2^\alpha)$, which implies that $n'\alpha = 2n$.

On the contrary, suppose that $\alpha \neq 2$. Let

$$\rho = \{r_{2(n-1)}(4), r_{2(n-2)}(4), r_{2(n-4)}(4)\}.$$

By Lemma (2.7), $\rho \subseteq \rho(C_n(4))$. We claim that $\rho \cap \pi(S) = \emptyset$:

We know that $|S| = 2^{\alpha n'^2} \prod_{i=1}^{n'}(2^{2\alpha i} - 1)$. If $r_{2(n-1)}(4) \in \pi(S)$, then there exists an integer $0 \leq m < n'$ such that $r_{2(n-1)}(4) \mid 2^{2(n'-m)\alpha} - 1$. We can see that $e(r_{2(n-1)}(4), 2) = 4(n-1)$. Thus $4(n-1) \mid 2(n'-m)\alpha$. But $n'\alpha = 2n$. So $4(n-1) \mid 4n - 2m\alpha$ and hence

$$2(n-1) \mid 2n - m\alpha = 2(n-1) - (m\alpha - 2) \Longrightarrow 2(n-1) \mid m\alpha - 2.$$

But $m < n'$ and $n'\alpha = 2n$, so we obtain that $m\alpha = 2$. Since $2n = n'\alpha$ and n and n' are odd, we deduce that α is even, so "$\alpha \neq 2$" forces $\alpha = 0$, which is a contradiction. Therefore $r_{2(n-1)}(4) \notin \pi(S)$. Also, by the same argument, we can see that $r_{2(n-2)}(4), r_{2(n-4)}(4) \notin \pi(S)$. Thus $|\rho| = 3$ and $\rho \cap \pi(S) = \emptyset$, which is a contradiction with Lemma (2.1)(2). This contradiction shows that $\alpha = 2$ and hence, $n = n'$ which forces $S \cong C_n(4)$. So theorem follows. □

Corollary 3.1. *Let $n \geq 17$ be an odd number. Then the simple group $C_n(4)$ is quasirecognizable by its spectrum.*

Proof. Let G be a finite group such that $\omega(G) = \omega(C_n(4))$. Then it is easy to see that $\Gamma(G) = \Gamma(C_n(4))$, so corollary follows from the main theorem. □

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