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Subordination and superordination results of $p-$valent analytic functions involving a linear operator

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abstract: In this paper we derive some subordination and superordination results for certain $p-$valent analytic functions in the open unit disc, which are acted upon by a class of a linear operator. Some of our results improve and generalize previously known results.

Key Words: Analytic function, Hadamard product, differential subordination, superordination, linear operator.

Contents

1. Introduction

Let $H(U)$ denotes the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a,p]$ denotes the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + ...(a \in \mathbb{C}; p \in \mathbb{N} = \{1, 2, ..\}).$$

Also, let $\mathcal{A}(p)$ be the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N}), \tag{1.1}$$

and set $\mathcal{A} \equiv \mathcal{A}(1)$. For functions $f(z) \in \mathcal{A}(p)$, given by (1.1), and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \qquad (p \in \mathbb{N}), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

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For $f, g \in H(U)$, we say that the function f is subordinate to g, if there exists a Schwarz function w, i.e, $w \in H(U)$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. This subordination is usually denoted by $f(z) \prec g(z)$. It is well-known that, if the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [6] and [11]).

Supposing that h and k are two analytic functions in U, let

$$\phi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}.$$

If h and $\varphi(h(z), zh^{'}(z), z^2h^{''}(z); z)$ are univalent functions in U and if h satisfies the second-order superordination

$$k(z) \prec \varphi(h(z), zh^{'}(z), z^2h^{''}(z); z), \tag{1.4}$$

then h is called to be a solution of the differential superordination (1.4). A function $q \in H(U)$ is called a subordinant of (1.4), if $q(z) \prec h(z)$ for all the functions h satisfying (1.4). A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (1.4), is said to be the best subordinant.

Recently, Miller and Mocanu [12] obtained sufficient conditions on the functions k, q and φ for which the following implication holds:

$$k(z) \prec \varphi(h(z), zh^{'}(z), z^2h^{''}(z); z) \Rightarrow q(z) \prec h(z).$$

Using these results, Bulboaca [4] considered certain classes of first-order differential superordinations, as well as superordination-preserving integral operators [5]. Ali et al. [1], using the results from [4], obtained sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf^{'}(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in U.

For complex parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}; j = 1, 2, ..., s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,..., \alpha_q; \beta_1, ..., \beta_s; z)$ by (see, for example, [18, p.19])

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k}.\frac{z^k}{k!} \tag{1.5}$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}), \\ \theta(\theta + 1)....(\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \tag{1.6}$$

Let

$$\begin{aligned} h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) &= z^p {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) \\ &= z^p + \sum_{k=p+1}^{\infty} \Gamma_{p,q,s}(\alpha_1) z^k, \end{aligned}$$

where

$$\Gamma_{p,q,s}(\alpha_1) = \frac{(\alpha_1)_{k-p}...(\alpha_q)_{k-p}}{(\beta_1)_{k-p}...(\beta_s)_{k-p}(1)_{k-p}}, \tag{1.7}$$

and using the Hadamard product, El-Ashwah and Aouf [8] defined the following operator

$$I_{p,\lambda}^{m,\ell}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : A(p) \to A(p)$$

by

$$\begin{aligned} I_{p,\lambda}^{0,\ell}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) &= f(z) * h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z); \\ I_{p,\lambda}^{1,\ell}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) &= (1-\lambda)(f(z) * h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)) \\ &\quad + \frac{\lambda}{(p+\ell)z^{\ell-1}}(z^\ell f(z) * h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z))'; \end{aligned}$$

and

$$I_{p,\lambda}^{m,\ell}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = I_{p,q,s,\lambda}^{1,\ell}(I_{p,q,s,\lambda}^{m-1,\ell}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z)). \tag{1.8}$$

If $f \in A(p)$, then from (1.1) and (1.8), we can easily see that

$$I_{p,\lambda}^{m,\ell}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m \Gamma_{p,q,s}(\alpha_1) a_k z^k. \tag{1.9}$$

$$(p \in \mathbb{N};\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ell \geq 0; \lambda \geq 0; z \in U)$$

It can be easily verified from the definition (1.9) that:

$$z(I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z))' = \alpha_1 I_{p,q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z) - (\alpha_1 - p)I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z), \tag{1.10}$$

where

$$I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f(z) = I_{p,\lambda}^{m,\ell}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z).$$

It should be remarked that the linear operator $I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)$ is a generalization of many other linear operators considered earlier. In particular, we have

$$I_{p,q,s,\lambda}^{0,\ell}(\alpha_1)f(z) = H_{p,q,s}(\alpha_1)f(z),$$

where the linear operator $H_{p,q,s}(\alpha_1)$ was investigated by Dziok and Srivastava [9] (see also [13], [10] and [2]), and also we have

$$I_{p,2,1,\lambda}^{0,\ell}(a, 1; c)f(z) = L_p(a, c)f(z)(a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-),$$

where the linear operator $L_p(a, c)$ was studied by Saitoh [16] which yields the operator $L(a, c)f(z)$ introduced by Carlson and Shaffer [7] for $p = 1$.

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and results.

Definition 2.1. [12] *Denote by* Q *the set of all functions* $f(z)$ *that are analytic and injective on* $\overline{U}\backslash E(f)$, *where*

$$E(f)=\left\{\zeta:\zeta\in\partial \text{ and } \lim_{z\to\zeta}f(z)=\infty\right\} \tag{2.1}$$

and are such that $f^{'}(\zeta)\neq 0$ *for* $\zeta\in\partial U\backslash E(f)$.

Lemma 2.2. [11] *Let the function* $q(z)$ *be univalent in the unit disc* U *and let* θ *and* φ *be analytic in a domain* D *containing* $q(U)$ *with* $\varphi(w)\neq 0$ *when* $w\in q(U)$. *Set* $Q(z)=zq^{'}(z)\varphi(q(z))$ *and* $h(z)=\theta(q(z))+Q(z)$. *Suppose that*

(i) $Q(z)$ *is starlike univalent in* U,

(ii) $\Re\left(\dfrac{zh^{'}(z)}{Q(z)}\right)>0$ *for* $z\in U$.

If p *is analytic with* $p(0)=q(0),\ p(U)\subseteq D$ *and*

$$\theta(p(z))+zp^{'}(z)\varphi(p(z))\prec\theta(q(z))+zq^{'}(z)\varphi(q(z)), \tag{2.2}$$

then $p(z)\prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 2.3. [6] *Let* $q(z)$ *be convex univalent in the unit disc* U *and let* θ *and* φ *be analytic in a domain* D *containing* $q(U)$. *Suppose that*

(i) $\Re\left\{\dfrac{\theta^{'}(q(z))}{\varphi(q(z))}\right\}>0$ *for* $z\in U$;

(ii) $zq^{'}(z)\varphi(q(z))$ *is starlike univalent in* U.

If $p(z)\in H[q(0),1]\cap Q$, *with* $p(U)\subseteq D$, *and* $\theta(p(z))+zp^{'}(z)\varphi(p(z))$ *is univalent in* U, *and*

$$\theta(q(z))+zq^{'}(z)\varphi(q(z))\prec\theta(p(z))+zp^{'}(z)\varphi(p(z)), \tag{2.4}$$

then $q(z)\prec p(z)$ *and* $q(z)$ *is the best subordinant.*

The following lemma gives us a necessary and sufficient condition for the univalence of a special function which will be used in some particular case.

Lemma 2.4. [15] *The function* $q(z)=(1-z)^{-2ab}\,(a,b\in\mathbb{C}^{*})$ *is univalent in the unit disc* U *if and only if* $|2ab-1|\leq 1$ *or* $|2ab+1|\leq 1$.

3. Main Results

Unless otherwise mentioned, we assume throughout this paper that $p\in\mathbb{N}, m\in\mathbb{N}_0, \ell\geq 0, \lambda\geq 0$ and the power understood as principal values.

Theorem 3.1. *Let $q(z)$ be univalent in U such that $q(0)=1$, $q(z)\neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike in U. Let $f\in A(p)$ and suppose that f and q satisfy the next conditions:*

$$\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{\mu}\neq 0 \quad (\mu\in\mathbb{C}^*;z\in U), \tag{3.1}$$

and

$$\Re\left\{1+\frac{\zeta}{\gamma}q\left(z\right)+\frac{2\delta}{\gamma}\left[q\left(z\right)\right]^2-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right\}>0 \quad (\zeta,\delta\in\mathbb{C};\gamma\in\mathbb{C}^*;z\in U). \tag{3.2}$$

If

$$\Psi\left(z\right)\prec\chi+\zeta q\left(z\right)+\delta\left[q\left(z\right)\right]^2+\gamma\frac{zq'(z)}{q(z)}, \tag{3.3}$$

where

$$\Psi\left(z\right)=\chi+\zeta\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{\mu}+\delta\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{2\mu}$$
$$+\gamma\mu\alpha_1\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1+1)f(z)}{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}-1\right], \tag{3.4}$$

then

$$\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{\mu}\prec q\left(z\right),$$

and q is the best dominant of (3.3).

Proof: Let

$$h(z)=\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{\mu} \ (z\in U). \tag{3.5}$$

According to (3.1) the function $h(z)$ is analytic in U, and differentiating (3.5) logarithmically with respect to z, we obtain

$$\frac{zh'(z)}{h(z)}=\mu\left[\frac{z(I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z))'}{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}-p\right].$$

By using the identity (1.10), we obtain

$$\frac{zh'(z)}{h(z)}=\mu\alpha_1\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1+1)f(z)}{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}-1\right].$$

In order to prove our result we will use Lemma 2.2. In this lemma consider

$$\theta(w)=\chi+\zeta w+\delta w^{2} \quad \textit{and} \quad \varphi(w)=\frac{\gamma}{w},$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w)\neq 0$ is analytic in $\mathbb{C}^{*}$.Also, if we let

$$Q(z)=zq^{'}(z)\varphi(q(z))=\gamma\frac{zq^{'}(z)}{q(z)},$$

and

$$g(z)=\theta(q(z))+Q(z)=\chi+\zeta q\left(z\right)+\delta\left[q\left(z\right)\right]^{2}+\gamma\frac{zq^{'}(z)}{q(z)}.$$

We see that $Q(z)$ is starlike function in U. From (3.2), we also have

$$\Re\left\{\frac{zg^{'}(z)}{Q(z)}\right\}=\Re\left\{1+\frac{\zeta}{\gamma}q\left(z\right)+\frac{2\delta}{\gamma}\left[q\left(z\right)\right]^{2}-\frac{zq^{'}(z)}{q(z)}+\frac{zq^{''}(z)}{q^{'}(z)}\right\}>0 \ \ (z\in U),$$

and then, by using Lemma 2.2 we deduce that the subordination (3.3) implies $h(z)\prec q(z)$, and the function q is the best dominant of (3.3).

Putting $q=2, s=p=1, m=0, \alpha_1=a+1\left(a\in\mathbb{C}\right), \alpha_2=1$ and $\beta_1=c$ $(c\in\mathbb{C}\backslash\mathbb{Z}_0^{-})$ in Theorem 3.1, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Theorem 3]. □

Corollary 3.2. *Let $q(z)$ be univalent in U such that $q(0)=1$, $q(z)\neq 0$ and $\frac{zq^{'}(z)}{q(z)}$ is starlike in U. Let $f\in\mathcal{A}$ such that*

$$\left[\frac{L\left(a+1,c\right)f(z)}{z}\right]^{\mu}\neq 0 \quad (\mu\in\mathbb{C}^{*}; z\in U), \tag{3.6}$$

and suppose that q satisfies (3.2).If

$$\Lambda\left(z\right)\prec\chi+\zeta q\left(z\right)+\delta\left[q\left(z\right)\right]^{2}+\gamma\frac{zq^{'}(z)}{q(z)}, \tag{3.7}$$

where

$$\Lambda\left(z\right)=\chi+\zeta\left[\frac{L\left(a+1,c\right)f(z)}{z}\right]^{\mu}+\delta\left[\frac{L\left(a+1,c\right)f(z)}{z}\right]^{2\mu}$$
$$+\gamma\mu\left(a+1\right)\left[\frac{L\left(a+2,c\right)f(z)}{L\left(a+1,c\right)f(z)}-1\right], \tag{3.8}$$

then

$$\left[\frac{L\left(a+1,c\right)f(z)}{z}\right]^{\mu}\prec q\left(z\right),$$

and q is the best dominant of (3.7).

Putting $q(z) = \frac{1+Az}{1+Bz}\,(-1 \le B < A \le 1)$ in Corollary 3.2, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Corollary 1].

Corollary 3.3. *Assume that*

$$\Re\left\{\frac{1-ABz^2}{(1+Az)(1+Bz)} + \frac{\zeta}{\gamma}\left[\frac{1+Az}{1+Bz}\right] + \frac{2\delta}{\gamma}\left[\frac{1+Az}{1+Bz}\right]^2\right\} > 0 \quad (\zeta,\delta \in \mathbb{C}; \gamma \in \mathbb{C}^*; z \in U)$$

holds. Let $f \in \mathcal{A}$ such that (3.6) *holds. If*

$$\Lambda(z) \prec \chi + \zeta\frac{1+Az}{1+Bz} + \delta\left[\frac{1+Az}{1+Bz}\right]^2 + \frac{\gamma(A-B)z}{(1+Az)(1+Bz)}, \tag{3.9}$$

where $\Lambda(z)$ is given by (3.8)*, then*

$$\left[\frac{L(a+1,c)f(z)}{z}\right]^{\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.9).

Putting $q(z) = \left(\frac{1+z}{1-z}\right)^{\vartheta}\,(0 < \vartheta \le 1)$ in Corollary 3.2, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Corollary 2].

Corollary 3.4. *Assume that*

$$\Re\left\{\frac{1-3z^2}{1-z^2} + \frac{\zeta}{\gamma}\left[\frac{1+z}{1-z}\right]^{\vartheta} + \frac{2\delta}{\gamma}\left[\frac{1+z}{1-z}\right]^{2\vartheta}\right\} > 0 \quad (\zeta,\delta \in \mathbb{C}; \gamma \in \mathbb{C}^*; z \in U)$$

holds. Let $f \in \mathcal{A}$ such that (3.6) holds. If

$$\Lambda(z) \prec \chi + \zeta\left(\frac{1+z}{1-z}\right)^{\vartheta} + \delta\left(\frac{1+z}{1-z}\right)^{2\vartheta} + \frac{2\gamma\vartheta z}{(1-z^2)} \quad (0 < \vartheta \le 1), \tag{3.10}$$

where $\Lambda(z)$ is given by (3.8)*, then*

$$\left[\frac{L(a+1,c)f(z)}{z}\right]^{\mu} \prec \left(\frac{1+z}{1-z}\right)^{\vartheta},$$

and $\left(\frac{1+z}{1-z}\right)^{\vartheta}$ is the best dominant of (3.10).

Putting $q(z) = e^{\mu Az}\,(|\mu A| < \pi)$ in Corollary 3.2, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Corollary 3].

Corollary 3.5. *Assume that*

$$\Re\left\{1+\frac{\zeta}{\gamma}e^{\mu Az}q(z)+\frac{2\delta}{\gamma}e^{2\mu Az}\right\}>0 \quad (\zeta,\delta\in\mathbb{C};\gamma\in\mathbb{C}^{*};z\in U)$$

holds. Let $f\in A$ such that (3.6) holds. If

$$\Lambda(z)\prec\chi+\zeta e^{\mu Az}+\delta e^{2\mu Az}+\gamma A\mu z \qquad (|\mu A|<\pi), \tag{3.11}$$

where $\Lambda(z)$ is given by (3.8), *then*

$$\left[\frac{L(a+1,c)\,f(z)}{z}\right]^{\mu}\prec e^{\mu Az},$$

and $e^{\mu Az}$ is the best dominant of (3.11).

Putting $q=s+1,\alpha_i=1(i=1,..,s+1),\beta_j=1(j=1,..,s),m=\zeta=\delta=0,\chi=p=1,\gamma=\frac{1}{ab}(a,b\in\mathbb{C}^{*}),\mu=a$, and $q(z)=(1-z)^{-2ab}$ in Theorem 3.1, then combining this together with Lemma 2.4 we obtain the next result due to Obradovic et al. [14, Theorem 1].

Corollary 3.6. [14] *Let $a,b\in\mathbb{C}^{*}$ such that $|2ab-1|\leq 1$ or $|2ab+1|\leq 1$. Let $f\in A$ and suppose that $\frac{f(z)}{z}\neq 0$ for all $z\in U$. If*

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z}\right)^{a}\prec(1-z)^{-2ab} \tag{3.12}$$

and $(1-z)^{-2ab}$ is the best dominant of (3.12).

Remark 3.7. *For $a=1$, Corollary 3.6 reduces to the recent result of Srivastava and Lashin* [19].

Putting $q=s+1,\alpha_i=1(i=1,..,s+1),\beta_j=1(j=1,..,s),m=\zeta=\delta=0,\chi=p=\gamma=1$, and $q(z)=(1+Bz)^{\frac{\mu(A-B)}{B}}$ in Theorem 3.10, and using Lemma 2.3 we obtain the next result.

Corollary 3.8. *Let $-1\leq A<B\leq 1$ with $B\neq 0$, and suppose that $\left|\frac{\mu(A-B)}{B}-1\right|\leq 1$ or $\left|\frac{\mu(A-B)}{B}+1\right|\leq 1$. Let $f\in A$ such that $\frac{f(z)}{z}\neq 0$ for all $z\in U$, and let $\mu\in\mathbb{C}^{*}$. If*

$$1+\mu\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+[B+\mu(A-B)]z}{1+Bz},$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu}\prec(1+Bz)^{\frac{\mu(A-B)}{B}}, \tag{3.13}$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ is the best dominant of (3.13).

Putting $q = s+1, \alpha_i = 1 (i = 1,..,s+1), \beta_j = 1 (j = 1,..,s), m = \zeta = \delta = 0, \chi = p = 1, \gamma = \frac{e^{i\tau}}{ab\cos\tau} (a, b \in \mathbb{C}^*; |\tau| < \frac{\pi}{2}), \mu = a$, and $q(z) = (1-z)^{-2ab\cos\tau e^{-i\tau}}$ in Theorem 3.1, we obtain the following result due to Aouf et al. [3, Theorem 1].

Corollary 3.9. [3] *Let $a, b \in \mathbb{C}^*, |\tau| < \frac{\pi}{2}$ and let $\left|2ab\cos\tau e^{-i\tau} - 1\right| \leq 1$ or $\left|2ab\cos\tau e^{-i\tau} + 1\right| \leq 1$. Let $f \in \mathcal{A}$ and suppose that $\frac{f(z)}{z} \neq 0$ for all $z \in U$. If*

$$1 + \frac{e^{i\tau}}{b\cos\tau}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z}$$

then

$$\left(\frac{f(z)}{z}\right)^a \prec (1-z)^{-2ab\cos\tau e^{-i\tau}} \tag{3.14}$$

and $(1-z)^{-2ab\cos\tau e^{-i\tau}}$ is the best dominant of (3.14).

Theorem 3.10. *Let q be convex in U such that $q(0) = 1$ and $\frac{zq'(z)}{q(z)}$ is starlike in U. Further assume that*

$$\Re\left\{(\zeta + 2\delta q(z))\frac{q(z)q'(z)}{\gamma}\right\} > 0 \quad (\zeta, \delta \in \mathbb{C}; \gamma \in \mathbb{C}^*). \tag{3.15}$$

Let $f \in \mathcal{A}(p)$ such that

$$0 \neq \left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^\mu \in H[q(0), 1] \cap Q. \tag{3.16}$$

If $\Psi(z)$ given by (3.4) is univalent in U and satisfies the following superordination condition

$$\chi + \zeta q(z) + \delta[q(z)]^2 + \gamma\frac{zq'(z)}{q(z)} \prec \Psi(z), \tag{3.17}$$

then

$$q(z) \prec \left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^\mu,$$

and q is the best subordinant of (3.17).

Putting $q = 2, s = p = 1, m = 0, \alpha_1 = a + 1 (a \in \mathbb{C}), \alpha_2 = 1$ and $\beta_1 = c$ $(c \in \mathbb{C}\backslash\mathbb{Z}_0^-)$ in Theorem 3.10, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Theorem 4].

Corollary 3.11. *Let q be convex in U such that $q(0) = 1$ and $\frac{zq'(z)}{q(z)}$ is starlike in U. Further assume that*

$$\Re\left\{(\zeta + 2\delta q(z))\frac{q(z)q'(z)}{\gamma}\right\} > 0 \quad (\zeta, \delta \in \mathbb{C}; \gamma \in \mathbb{C}^*). \tag{3.18}$$

Let $f \in \mathcal{A}$ such that

$$0 \neq \left[\frac{L(a+1,c)\,f(z)}{z}\right]^{\mu} \in H[q(0),1] \cap Q. \tag{3.19}$$

If $\Lambda(z)$ given by (3.8) *is univalent in U and satisfies the following superordination condition*

$$\chi + \zeta q(z) + \delta [q(z)]^2 + \gamma \frac{zq'(z)}{q(z)} \prec \Lambda(z), \tag{3.20}$$

then

$$q(z) \prec \left[\frac{L(a+1,c)\,f(z)}{z}\right]^{\mu},$$

and q is the best subordinant of (3.20).

Combining Theorems 3.1 and 3.10, we obtain the following two sandwich results:

Theorem 3.12. *Let q_i be two convex functions in U such that $q_i(0) = 1$ and $\frac{zq_i'(z)}{q_i(z)}$ $(i = 1, 2)$ is starlike in U. Suppose that $q_1(z)$ satisfies* (3.18) *and $q_2(z)$ satisfies* (3.2). *Let $f \in \mathcal{A}(p)$ and suppose that $\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{\mu} \in H[q(0),1] \cap Q$. If $\Psi(z)$ given by* (3.4) *is univalent in U, and*

$$\chi + \zeta q_1(z) + \delta [q_1(z)]^2 + \gamma \frac{zq_1'(z)}{q_1(z)} \prec \Psi(z) \prec \chi + \zeta q_2(z) + \delta [q_2(z)]^2 + \gamma \frac{zq_2'(z)}{q_2(z)}, \tag{3.21}$$

then

$$q_1(z) \prec \left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p}\right]^{\mu} \prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant of (3.21).

Putting $q = 2, s = p = 1, m = 0, \alpha_1 = a + 1\,(a \in \mathbb{C}), \alpha_2 = 1$ and $\beta_1 = c$ $(c \in \mathbb{C}\backslash\mathbb{Z}_0^-)$ in Theorem 3.12, we obtain the following result which improves the corresponding work of Shammugam et al. [17, Theorem 5].

Corollary 3.13. *Let q_i be two convex functions in U such that $q_i(0) = 1$ and $\frac{zq_i'(z)}{q_i(z)}$ $(i = 1, 2)$ is starlike in U. Suppose that $q_1(z)$ satisfies* (3.18) *and $q_2(z)$ satisfies* (3.2). *Let $f \in \mathcal{A}$ and suppose that $\left[\frac{L(a+1,c)f(z)}{z}\right]^{\mu} \in H[q(0),1] \cap Q$. If $\Lambda(z)$ given by* (3.8) *is univalent in U, and*

$$\chi + \zeta q_1(z) + \delta [q_1(z)]^2 + \gamma \frac{zq_1'(z)}{q_1(z)} \prec \Lambda(z) \prec \chi + \zeta q_2(z) + \delta [q_2(z)]^2 + \gamma \frac{zq_2'(z)}{q_2(z)}, \tag{3.22}$$

then

$$q_1(z) \prec \left[\frac{L(a+1,c)\,f(z)}{z}\right]^{\mu} \prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant of (3.22).

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