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$\mathcal{I}$ and $\mathcal{I}^*$ convergence of multiple sequences of fuzzy numbers

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ABSTRACT: Recently, the concept of statistical convergence for multiple sequences of fuzzy numbers has been studied by Kumar *et al.*. This motivate us to extend the idea of $\mathcal{I}$-convergence to sequences of fuzzy numbers of multiplicity greater than two.

Key Words: Fuzzy number sequences; multiple sequences; ideal convergence.

Contents

1. Introduction

R. P. Agnew [1] studied the summability theory of multiple sequences and obtained certain theorems for multiple sequences which have already been proved by the author himself for double sequences. Móricz [7] continued with the study of multiple sequences and gave some remarks on the notion of regular convergence of multiple series. In 2003 [8], the author extended statistical convergence from single to multiple real sequences and obtained some results for real double sequences. Very recently, Kumar *et al.* [18] studied the concept of statistical convergence for multiple sequences of fuzzy numbers.

The notion of statistical convergence of sequence of numbers was introduced by Fast [3] and Schoenberg [24] independently in 1951 and discussed by [4,9,19] and many others. Nuray and Savaş [14] first introduced statistical convergence of sequences of fuzzy numbers. After their pioneer work, many authors have been made their contribution to study different generalizations of statistical convergence for sequences of fuzzy numbers(see [6,15,20,21,22] *etc.*).

Kostyrko, Salat and Wilczynski [5] defined $\mathcal{I}$-convergence for single sequences which is a natural generalization of statistical convergence. Tripathy and Tripathy

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[25] introduced the concept of $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy for double sequences and proved some properties related to the solidity, symmetricity, completeness and denseness. In past years, $\mathcal{I}$-convergence has also become an interesting area of research for sequences of fuzzy numbers. The credit goes to Kumar *et al*. [17] who first defined $\mathcal{I}$-convergence for sequences of fuzzy numbers. For an extensive view of this subject, we refer [2,5,10,11,12,13,16,23].

Continuing our work [18], in present paper, we study the concept of ideal convergence for sequences of fuzzy numbers having multiplicity greater than or equal to two.

2. Background and Preliminaries

Let $C(\mathbb{R})= \{A \subset \mathbb{R}$: A compact and convex$\}$. The space $C(\mathbb{R})$ has a linear structure induced by the operation $A+B = \{a+b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(\mathbb{R})$ and $\lambda \in \mathbb{R}$. The Hausdroff distance between A and B is defined as

$$\delta_\infty(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} \parallel a-b \parallel, \sup_{b\in B}\inf_{a\in A} \parallel a-b \parallel\}$$

It is well known that $(C(\mathbb{R}), \delta_\infty)$ is a complete (not separable) metric space.

Definition 2.1. *A fuzzy number is a function X from $\mathbb{R}$ to [0,1], which satisfying the following conditions*
(i) X is normal, i.e., there exists and $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$;
(ii) X is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and
$\lambda \in [0,1], X(\lambda x + (1-\lambda)y) \geq \min\{X(x), X(y)\}$;
(iii) X is upper semi-continuous;
(iv) the closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$, denoted by X^0, is compact.

The properties (i)-(iv) imply that for each $\alpha \in (0,1]$, the α-level set,
$X^\alpha = \{x \in \mathbb{R} : X(x) \geq \alpha\} = [\underline{X}^\alpha, \overline{X}^\alpha]$
is a non-empty compact convex subset of $\mathbb{R}$. Let $L(\mathbb{R})$ denotes the set of all fuzzy numbers. The linear structure of $L(\mathbb{R})$ induces an addition $X+Y$ and a scalar multiplication λX in terms of α-level sets by
$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha$ and $[\lambda X]^\alpha = \lambda[X]^\alpha \qquad (x, y \in L(\mathbb{R}), \lambda \in \mathbb{R})$
for each $\alpha \in [0,1]$.

Define for each $1 \leq q < \infty$

$$d_q(X,Y) = (\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d_\alpha)^{\frac{1}{q}}$$

and $\delta_\infty = \sup_{0\leq\alpha\leq 1} \delta_\infty(X^\alpha, Y^\alpha)$. Clearly $d_\infty(X,Y) = \lim_{q\to\infty} d_q(X,Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover, d_q is a complete, separable and locally compact metric space [1].

Throughout the paper, d will denote d_q with $1 \leq q < \infty$.
We now quote the following definitions which will be needed in the sequel.

Definition 2.2. *A double sequence $X = (X_{nk})$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for each $\epsilon > 0$ there exist a positive integer m such that*
$d(X_{nk}, X_0) < \epsilon$ *for every* $n, k \geq m$.

The fuzzy number X_0 is called the limit of the sequence (X_{nk}) and we write $\lim_{n,k\to\infty} X_{nk} = X_0$.

Definition 2.3. *A double sequence $X = (X_{nk})$ of fuzzy numbers is said to be Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer n_0 such that*
$d(X_{nk}, X_{NK}) < \epsilon$
for every $n \geq N \geq n_0, k \geq K \geq n_0$.

Definition 2.4. *A double sequence $X = (X_{nk})$ of fuzzy numbers is said to be bounded if there exists a positive number M such that*
$d(X_{nk}, \widetilde{0}) < M$ *for all* n, k.

Let l_∞^2 denote the set of all bounded triple sequences of fuzzy numbers.

Definition 2.5. *If X is a non-empty set. A family of sets $\mathfrak{I} \subset P(X)$ is called an ideal in X if and only if*
(i) $\phi \in \mathfrak{I}$;
(ii) for each $A, B \in \mathfrak{I}$ *we have* $A \cup B \in \mathfrak{I}$;
(iii) for each $A \in \mathfrak{I}$ *and* $B \subset A$ *we have* $B \in \mathfrak{I}$.

Definition 2.6. *Let X is a non-empty set. A non-empty family of sets $F \subset P(X)$ is called a filter on X if and only if*
(i) $\phi \in F$;
(ii) for each $A, B \in F$ *we have* $A \cap B \in F$;
(iii) for each $A \in F$ *and* $B \supset A$ *we have* $B \in F$.
An ideal I is called non-trivial if $\mathfrak{I} \neq \phi$ and $X \notin \mathfrak{I}$. It immediately follows that $\mathfrak{I} \subset P(X)$ is a non-trivial ideal if and only if the class $F = F(\mathfrak{I}) = X - A : A \subset \mathfrak{I}$ is a filter on X. The filter $F = F(\mathfrak{I})$ is called the filter associated with the ideal I.

Definition 2.7. *A non-trivial ideal $\mathfrak{I} \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons i.e., if it contains $\{x : x \in X\}$.*

Throughout this paper, $\mathbb{N}^2$ denotes the usual product set $\mathbb{N} \times \mathbb{N}$.

3. $\mathfrak{I}_2$-convergence

In this section, we shall state and prove our results only for double sequences. Our methods can readily be applied to sequences of fuzzy numbers of any multiplicity.

Definition 3.1. *Let $\mathfrak{I}_2 \subset P(\mathbb{N}^2)$ be a non-trivial ideal in $\mathbb{N}^2$. A double sequence $X = (X_{ij})$ of fuzzy numbers is said to be $\mathfrak{I}_2$-convergent to some fuzzy number X_0, in symbol: $\mathfrak{I}_2 - \lim X_{ij} = X_0$, if for each $\epsilon > 0$,*

$$\{(i, j) \in \mathbb{N}^2 : d(X_{ij}, X_0) \geq \epsilon\} \in \mathfrak{I}_2.$$

We shall denote the set of all $\mathcal{I}_2$-convergent double sequences of fuzzy numbers by $\mathcal{I}^2$.

Theorem 3.2. *If a double sequence $X = (X_{ij})$ of fuzzy numbers is $\mathcal{I}_2$-convergent to some limit, then it must be unique.*

Theorem 3.3. *Let $X = (X_{ij})$ and $Y = (Y_{ij})$ be two double sequences of fuzzy numbers, then*

(i) If $X = (X_{ij})$ is convergent to X_0, then (X_{ij}) is $\mathcal{I}_2$-convergent to X_0.

(ii) If $X = (X_{ij})$ is $\mathcal{I}_2$-convergent to X_0 and $c \in \mathbb{R}$, then (cX_{ij}) is $\mathcal{I}_2$-convergent to cX_0.

(iii) If $X = (X_{ij})$ and $Y = (Y_{ij})$ are $\mathcal{I}_2$-convergent to fuzzy numbers X_0 and Y_0 respectively, then $(X_{ij} + Y_{ij})$ is $\mathcal{I}_2$-convergent to $X_0 + Y_0$.

Theorem 3.4. *Let $X = (X_{ij})$ and $Y = (Y_{ij})$ be two double sequences of fuzzy numbers, then*

(i) $X_{ij} \leq Y_{ij}$ for every $(i,j) \in K \subset \mathbb{N}^2$ with $K \in F(\mathcal{I}_2)$

(ii) $\mathcal{I}_2 - \lim X_{ij} = X_0$ and $\mathcal{I}_2 - \lim Y_{ij} = Y_0$.

Then $X_0 \leq Y_0$.

Theorem 3.5. *Let $X = (X_{ij})$, $Y = (Y_{ij})$ and $Z = (Z_{ij})$ be three double sequences of fuzzy numbers such that*

(i) $X_{ij} \leq Y_{ij} \leq Z_{ij}$ for every $(i,j) \in K$ with $K \in F(\mathcal{I}_2)$

(ii) $\mathcal{I}_2 - \lim X_{ij} = \mathcal{I}_2 - \lim Z_{ij} = X_0$, then

$\mathcal{I}_2 - \lim Y_{ij} = X_0$.

Theorem 3.6. *Let $\mathcal{I}_2 \subset P(\mathbb{N}^2)$ be an admissible ideal in $\mathbb{N}^2$. The set $\mathcal{I}^2 \cap l^2_\infty$ is closed linear subspace of the normed linear space l^2_∞.*

Proof: Let $X^{(mn)} = (X^{(mn)}_{ij}) \in \mathcal{I}^2 \cap l^2_\infty$ and $X^{(mn)} \longrightarrow X \in l^2_\infty$. Since $X^{(mn)} \in \mathcal{I}^2 \cap l^2_\infty$, therefore there exist a fuzzy number Y_{mn} such that

$$\mathcal{I}_2 - \lim_{i,j} X^{(mn)}_{ij} = Y_{mn} \qquad (m, n = 1, 2, \ldots).$$

Furthermore, $X^{(mn)} \longrightarrow X$, implies that there exists a positive integer M such that for every $p \geq m \geq M$ and $q \geq n \geq M$

$$d(X^{(pq)}, X^{(mn)}) < \frac{\epsilon}{3} \tag{3.1}$$

Also, there exist subsets $K_{pq}, K_{mn} \subset \mathbb{N}^2$ such that $K_{pq} \in F(\mathcal{I})$ and $K_{mn} \in F(\mathcal{I}_2)$ such that

$$\lim_{(i,j)\in K_{pq}} X^{(pq)}_{ij} = Y_{pq}. \tag{3.2}$$

$$\lim_{(i,j)\in K_{mn}} X^{(mn)}_{ij} = Y_{mn}. \tag{3.3}$$

Now, the set $K_{pq} \cap K_{mn}$ is non-empty in $F(\mathcal{I}_2)$. Choose $(k_1, k_2) \in K_{pq} \cap K_{mn}$, then we have from (3.2) and (3.3)

$$d(X^{(pq)}_{k_1k_2}, Y_{pq}) < \frac{\epsilon}{3} \qquad and \qquad d(X^{(mn)}_{k_1k_2}, Y_{mn}) < \frac{\epsilon}{3} \tag{3.4}$$

Hence, for every $p \geq m \geq M$ and $q \geq n \geq M$, we have from (3.1) to (3.4)

$$\begin{aligned} d(Y_{pq}, Y_{mn}) &\leq d(Y_{pq}, X^{(pq)}_{k_1k_2}) + d(X^{(pq)}_{k_1k_2}, X^{(mn)}_{k_1k_2}) + d(X^{(mn)}_{k_1k_2}, Y_{mn}) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This shows that (Y_{mn}) is a Cauchy sequence and hence convergent. Let,

$$\lim_{m,n \longrightarrow \infty} Y_{mn} = Y. \tag{3.5}$$

Next we show that X is $\mathcal{I}_2$-convergent to Y. Since $X^{(mn)} \longrightarrow X$, so by the structure of l^2_∞, it is also coordinate wise convergent. Therefore for each $\epsilon > 0$, there exist a positive integer $\eta_1(\epsilon)$ such that

$$d(X^{(mn)}_{ij}, X_{ij}) < \frac{\epsilon}{3}; \qquad i, j \geq \eta_1(\epsilon). \tag{3.6}$$

Also from (3.5) we have for every $\epsilon > 0$ there exists $\eta_2(\epsilon)$ such that

$$d(Y_{ij}, Y) < \frac{\epsilon}{3}; \qquad i, j \geq \eta_2(\epsilon). \tag{3.7}$$

Let $\eta_3(\epsilon) = \max\{\eta_1(\epsilon), \eta_2(\epsilon)\}$ and choose $m_0, n_0 > \eta_3(\epsilon)$. Then for any $(i, j) \in \mathbb{N}^2$

$$\begin{aligned} d(X_{ij}, Y) &\leq d(X_{ij}, X^{m_0n_0}_{ij}) + d(X^{m_0n_0}_{ij}, Y_{m_0n_0}) + d(Y_{m_0n_0}, Y) \\ &< \frac{\epsilon}{3} + d(X^{m_0n_0}_{ij}, Y_{m_0n_0}) + \frac{\epsilon}{3} \quad \text{(by using (3.6) and (3.7))} \end{aligned}$$

$$\begin{aligned} \text{Let} \quad A_{m_0n_0}\big(\frac{\epsilon}{3}\big) &= \{(i, j) \in \mathbb{N}^2 : d(X^{m_0n_0}_{ij}, Y_{m_0n_0}) \geq \frac{\epsilon}{3}\} \\ A(\epsilon) &= \{(i, j) \in \mathbb{N}^2 : d(X_{ij}, Y) \geq \epsilon\} \\ A^C_{m_0n_0}\big(\frac{\epsilon}{2}\big) &= \{(i, j) \in \mathbb{N}^2 : d(X^{m_0n_0}_{ij}, Y_{m_0n_0}) < \frac{\epsilon}{3}\} \quad \text{and} \\ A^C(\epsilon) &= \{(i, j) \in \mathbb{N}^2 : d(X_{ij}, Y) < \epsilon\}. \end{aligned}$$

So for any $(i, j) \in A^C_{m_0n_0}\big(\frac{\epsilon}{3}\big)$ we have $d(X_{ij}, Y) < \epsilon$ and therefore $A^C_{m_0n_0}\big(\frac{\epsilon}{3}\big) \subset A^C(\epsilon)$. This implies that $A(\epsilon) \subset A_{m_0n_0}\big(\frac{\epsilon}{3}\big)$. Since $A_{m_0n_0}\big(\frac{\epsilon}{3}\big) \in \mathcal{I}_2$, therefore we have $A(\epsilon) \in \mathcal{I}_2$. Hence, X is $\mathcal{I}_2$-convergent to Y and $X \in \mathcal{I}_2$. This proves that $\mathcal{I}_2 \cap l^2_\infty$ is closed linear subspace of l^2_∞. This completes the proof of the Theorem.
□

4. $\mathcal{I}_2^*$-convergence

In [19] Salat proved that a sequence $x = (x_n)$ of real numbers is statistically convergent to ξ if and only if there exists a subset $K = \{m_1 < m_2 < ... < m_k...\} \subset \mathbb{N}$ with $\delta(K) = 1$ such that $\lim_{n\to\infty} x_{m_k} = \xi$. Savas [20] proved the same result for sequence of fuzzy numbers as follows: A sequence $X = (X_k)$ of fuzzy numbers is statistically convergent to X_0 if and only if there exists a subset $K = \{m_1 < m_2 < ... < m_k...\} \subset \mathbb{N}$ with $\delta(K) = 1$ such that $\lim_{n\to\infty} X_{m_k} = X_0$. We use this result to introduce the concept of $\mathcal{I}_2^*$-convergence for sequences of fuzzy numbers as follows.

Definition 4.1. *A double sequence $X = (X_{ij})$ of fuzzy numbers is $\mathcal{I}_2^*$-convergent to a fuzzy number X_0, if and only if, there exists a subset $K = \{(i_n, j_n)\} \subset \mathbb{N}^2, n = 1, 2, \ldots$ such that $K \in F(\mathcal{I}_2)$ and $\lim_{n\to\infty} X_{i_n j_n} = X_0$.*

Theorem 4.2. *Let $\mathcal{I}_2$ be an admissible ideal. If $\mathcal{I}_2^* - \lim X_{ij} = X_0$, then $\mathcal{I}_2 - \lim X_{ij} = X_0$.*

Proof: Let $X = X_{ij}$ be a double sequence of fuzzy numbers such that $\mathcal{I}_2^* - \lim X_{ij} = X_0$. Then, by definition, there exist a set $K = \{(i_n, j_n)\} \subset \mathbb{N}^2$ such that $K \in F(\mathcal{I}_2)(i.e.\mathbb{N}^2 - K = H \in \mathcal{I}_2)$ and $d(X_{i_n j_n}, X_0) \to 0 \quad \text{as} \, n \to \infty$.
Then for $\epsilon > 0$, there exists a positive integer n_1 such that
$d(X_{i_n j_n}, X_0) < \epsilon$ for all $n > n_1$.
Since the set
$A(\epsilon) = \{(i_n, j_n) \in K : d(X_{i_n j_n}, X_0) \geq \epsilon\} \subset A \cup H$,
where $A = \{i_1 < i_2 < ... < i_{n_1}, j_1 < j_2 < ... < j_{n_1}\}$ and $\mathcal{I}_2$ is admissible, therefore $A \cup H \in \mathcal{I}_2$. Hence, we conclude that $\mathcal{I}_2 - \lim X_{ij} = X_0$. □

Definition 4.3. *An admissible ideal $\mathcal{I}_2 \subset P(\mathbb{N}^2)$ is said to be satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$ there exists a countable family $\{B_1, B_2, ...\}$ in $\mathcal{I}_2$ such that $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i \in \mathcal{I}_2$.*

Theorem 4.4. *If the ideal $\mathcal{I}_2$ satisfy the property (AP), then $\mathcal{I}_2$-convergence implies $\mathcal{I}_2^*$-convergence for sequence of fuzzy numbers.*

Proof: Suppose that the ideal $\mathcal{I}_2$ satisfies the condition (AP). Let $X = (X_{ij})$ be a double sequence of fuzzy numbers such that $\mathcal{I}_2 - \lim X_{ij} = X_0$. Then for each $\epsilon > 0$, the set $A(\epsilon) = \{(i, j) \in \mathbb{N}^2 : d(X_{ij}, X_0) \geq \epsilon\}$ belongs to $\mathcal{I}_2$. For $n \in \mathbb{N}$, we define the set A_n as follows: Put

$$A_1 = \{(i, j) \in \mathbb{N}^2 : d(X_{ij}, X_0) \geq 1\} \in \mathcal{I}_2 \text{ and}$$
$$A_n = \{(i, j) \in \mathbb{N}^2 : \frac{1}{n} \leq d(X_{ij}, X_0) < \frac{1}{n-1}\} \in \mathcal{I}_2$$

Now it is clear that $\{A_1, A_2, ...\}$ is a countable family of mutually disjoint sets belonging to $\mathcal{I}_2$ and therefore by the condition (AP) there is a countable family of sets $\{B_1, B_2, ...\}$ in $\mathcal{I}_2$ such that $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{I}_2$. Since $B \in \mathcal{I}_2$ so there is set K in $F(\mathcal{I}_2)$ such that $K = \mathbb{N} - B$. Now to prove the result it is sufficient to prove that $\lim_{(i,j)\in K} X_{ij} = X_0$. Let $\eta > 0$ be given. Choose a positive integer q such that $\eta > \frac{1}{q+1}$. Then, we have

$$\{(i,j) \in \mathbb{N}^2 : d(X_{ij}, X_0) \geq \eta\} \subset \{(i,j) \in \mathbb{N}^2 : d(X_{ij}, X_0) \geq \frac{1}{q+1}\} \in \bigcup_{i=1}^{q+1} A_i$$

Since $A_i \Delta B_i, i = 1, 2, ...q+1$ are finite, there exists $(i_0, j_0) \in \mathbb{N}^2$ such that

$$\{\{\bigcup_{i=1}^{q+1} B_i\} \bigcap \{(i,j) : i \geq i_0, j \geq j_0\}\} = \{\{\bigcup_{i=1}^{q+1} A_i\} \bigcap \{(i,j) : i \geq i_0, j \geq j_0\}\}$$

If $(i \geq i_0, j \geq j_0)$ and $(i,j) \in K$ then $(i,j) \notin \bigcup_{i=1}^{q+1} B_i$. Therefore, by (3.3), we have $(i,j) \notin \bigcup_{i=1}^{q+1} A_i$. Hence, for every $(i \geq i_0, j \geq j_0)$ and $(i,j) \in K$, we have $d(X_{ij}, X_0) < \eta$. This completes the proof. $\square$

5. $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy sequences of fuzzy numbers

In the present section we define $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy sequence of fuzzy numbers as follows.

Definition 5.1. *A double sequence $X = (X_{ij})$ of fuzzy numbers is said to be $\mathcal{I}_2$-Cauchy if for each $\epsilon > 0$ there exists integers $M = M(\epsilon)$ and $N = N(\epsilon)$, such that $i, p \geq M$ and $j, q \geq N$,*

$$\{(i,j) \in \mathbb{N}^2 : d(X_{ij}, X_{pq}) \geq \epsilon\} \in \mathcal{I}_2.$$

Definition 5.2. *A double sequence $X = (X_{ij})$ of fuzzy numbers is said to be $\mathcal{I}_2^*$-Cauchy if there exists a subset $K = \{(i_n, j_n)\} \subset \mathbb{N}^2, n = 1, 2, \ldots$ such that $K \in F(\mathcal{I}_2)$ and the subsequence $X_{i_n j n}$ is an ordinary Cauchy sequence.*

Theorem 5.3. *Let $\mathcal{I}_2$ be an admissible ideal. If a double sequence (X_{ij}) is $\mathcal{I}_2^*$-Cauchy, then it is $\mathcal{I}_2$-Cauchy.*

Theorem 5.4. *If the ideal $\mathcal{I}_2$ satisfy the property (AP) and (X_{ij}) is a $\mathcal{I}_2$-Cauchy sequences then it is also $\mathcal{I}_2^*$-Cauchy sequence of fuzzy numbers.*
The proofs of above theorems goes on similar lines as for the theorems 4.1 and 4.2, so are omitted here.

Theorem 5.5. *A double sequence $X = (X_{ij})$ of fuzzy numbers is $\mathcal{I}_2$-convergent, if and only if, it is $\mathcal{I}_2$-Cauchy.*

Proof: Let $X = (X_{ij})$ be $\mathcal{I}_2$-convergent to X_0. By definition, for each $\epsilon \geq 0$ we have

$$A = \{(i,j) \in \mathbb{N}^2 : d(X_{ij}, X_0) \geq \frac{\epsilon}{2}\} \in \mathcal{I}_2.$$

This implies that the set $A^C = \{(i,j) \in \mathbb{N}^2 : d(X_{ij}, X_0) < \frac{\epsilon}{2}\} \in F(\mathcal{I}_2)$ is non-empty. So, we can choose $(p,q) \in A^C$ such that $d(X_{pq}, X_0) < \epsilon$ If we denote

$$B = \{(i,j) \in \mathbb{N}^2 : d(X_{ij}, X_{pq}) \geq \epsilon\},$$

we need to prove that $B \subset A$. Let $(m,n) \in B$, then we have
$\epsilon < d(X_{mn}, X_{pq}) \leq d(X_{mn}, X_0) + d(X_{pq}, X_0) \leq d(X_{mn}, X_0) + \frac{\epsilon}{2}$.
This implies that $\frac{\epsilon}{2} < d(X_{mn}, X_0)$ and therefore $(m,n) \in A$ and hence $B \subset A$. Since $A \in \mathcal{I}_2$, therefore $B \in \mathcal{I}_2$. This completes the proof.

Conversely:- Suppose that $X = (X_{ij})$ is a $\mathcal{I}_2$-Cauchy. We shall prove that (X_{ij}) is $\mathcal{I}_2$-convergent. To this effect, let $(\epsilon_p : p = 1, 2, \ldots)$ be strictly decreasing sequence of numbers converging to zero. Since $X = (X_{ij})$ is $\mathcal{I}_2$-Cauchy, therefore there exist three strictly increasing sequences (M_p) and (N_p) of positive integers such that

$$\{(i,j) \in \mathbb{N}^2 : d(X_{ij}, X_{M_p N_p}) \geq \epsilon_p\} \in \mathcal{I}_2. \tag{5.1}$$

This implies that

$$\{(i,j) \in \mathbb{N}^2 : d(X_{ij}, X_{M_p N_p}) < \epsilon_p\} \in F(\mathcal{I}_2). \tag{5.2}$$

Clearly, for each pair p and q $(p \neq q)$ of positive integers, we can select $(i_{pq}, j_{pq}) \in \mathbb{N}^2$ such that

$$d(X_{i_{pq} j_{pq}}, X_{M_p N_p}) \leq \epsilon_p \quad and \quad d(X_{i_{pq} j_{pq}}, X_{M_q N_q}) \leq \epsilon_q.$$

It follows that

$$\begin{aligned} d(X_{M_p N_p}, X_{M_q N_q}) &\leq d(X_{i_{pq} j_{pq}}, X_{M_p N_p}) + d(X_{i_{pq} j_{pq}}, X_{M_q N_q}) \\ &\leq \epsilon_p + \epsilon_q \to 0 \quad as \qquad p, q \to \infty. \end{aligned}$$

Thus $(X_{M_p N_p} : p = 1, 2, ...)$ is Cauchy sequence and satisfies the Cauchy convergence criterion. Let, $(X_{M_p N_p})$ converges to X_0. Since $(\epsilon_p : p = 1, 2, \ldots) \longrightarrow 0$ so for $\epsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that

$$\epsilon_{p_0} < \frac{\epsilon}{2} \qquad and \qquad d(X_{M_p N_p}, X_0) < \frac{\epsilon}{2}, \qquad p \geq p_0. \tag{5.3}$$

Now, we prove that the set $\{(i,j) \in \mathbb{N}^2 : d(X_{ij}, X_0) \geq \epsilon\} \subset A_{p_0}$. Consider arbitrary $(i,j) \in \mathbb{N}^2$. By (5.3)

$$\begin{aligned} d(X_{ij}, X_0) &\leq d(X_{ij}, X_{M_{p_0} N_{p_0}}) + d(X_{M_{p_0} N_{p_0}}, X_0) \\ &\leq d(X_{ij}, X_{M_{p_0} N_{p_0}}) + \frac{\epsilon}{2} \end{aligned}$$

and by first half of 10, $\epsilon_{p_0} < d(X_{ij}, X_{M_{p_0} N_{p_0}})$. This implies $(i,j) \in A_{p_0}$ and therefore $A \subset A_{p_0}$. Since $A_{p_0} \in \mathcal{I}_2$, so that $\mathcal{I}_2$. Hence, (X_{ij}) is $\mathcal{I}_2$-convergent. $\square$

6. Multiple Sequences of Fuzzy Numbers

The concepts and results presented in the last sections can be extended to d-multiple sequences of fuzzy numbers where d is a fixed positive integer.

Let $\mathbb{N}^d = \{(k_1, k_2, \dots k_d) : k_i \in \mathbb{N}, \forall\ i\}$.

The d-tuple $\mathbf{k} \neq \mathbf{n}$, where $\mathbf{k} = (k_1, k_2, \dots k_d)$ and $\mathbf{n} = (n_1, n_2, \dots n_d)$, if and only if, $n_j \neq k_j$ for at least one j. Furthermore, the partial order on $\mathbb{N}^d$ is defined as follows. For $\mathbf{k}, \mathbf{n} \in \mathbb{N}^d$, we say that $\mathbf{k} \leq \mathbf{n}$, if and only if, $k_j \leq n_j$ for each j. In this section, we study the concept of ideal convergence of d-multiple sequences of fuzzy numbers. Throughout the section, we take $\mathcal{I}^d$ as a nontrivial ideal in $\mathbb{N}^d$. With the help of a nontrivial ideal $\mathcal{I}^d$, the notions of $\mathcal{I}_d$-convergence and $\mathcal{I}_d$-Cauchy for multiple sequences of fuzzy numbers can be define as follows.

Definition 6.1. *A d-tuple sequence $(\mathbf{X} = \mathbf{X_k} : \mathbf{k} \in \mathbb{N}^d)$ of fuzzy numbers is said to be $\mathcal{I}_d$-convergent to some fuzzy number $\mathbf{X_0}$ if for each $\epsilon > 0$*

$$\{\mathbf{k} \in \mathbb{N}^d : d(\mathbf{X_k}, \mathbf{X_0}) \geq \epsilon\} \in \mathcal{I}^d.$$

Definition 6.2. *A d-tuple sequence $(\mathbf{X} = \mathbf{X_k} : \mathbf{k} \in \mathbb{N}^d)$ of fuzzy numbers is said to be $\mathcal{I}_d$-Cauchy if for each $\epsilon > 0$, there exist $\mathbf{m} = (m_1, m_2, \dots m_d) \in \mathbb{N}^d$ such that and*

$$\{\mathbf{k} \leq \mathbb{N} : d(\mathbf{X_k}, \mathbf{X_m}) \geq \epsilon\} \in \mathcal{I}^d.$$

Definition 6.3. *A d-tuple sequence $(\mathbf{X} = \mathbf{X_k} : \mathbf{k} \in \mathbb{N}^d)$ of fuzzy numbers is said to be $\mathcal{I}_d^*$-Cauchy if there exists a subset $K = \{(k_1, k_2, \dots k_d) : k_i \in \mathbb{N}, \forall\ i\} \subset \mathbb{N}^d$ such that $K \in F(\mathcal{I}_d)$ and the subsequence $\mathbf{X}_K$ is an ordinary Cauchy sequence.*

All the results presented in previous sections remain true for d-multiple sequences, as well.

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