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Solving Two Point Boundary Value Problems for Ordinary Differential Equations Using Exponential Finite Difference Method

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ABSTRACT: In this article, a new exponential finite difference scheme for the numerical solution of two point boundary value problems with Dirichlet's boundary conditions is proposed. The scheme is based on an exponential approximation of the discretized derivative. The local truncation error and the convergence of the scheme under appropriate condition discussed. The theoretical and numerical results show that this new scheme is efficient and at least fourth order accurate.

Key Words: Two-point Boundary value problems, Exponential finite difference method, Fourth order finite difference method.

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1. Introduction

In this article, we implement an exponential finite difference method for solving the two point non-linear boundary value problem of the form

$$y''(x) = f(x, y), \quad a < x < b \tag{1.1}$$

subject to boundary conditions $y(a) = \alpha$ and $y(b) = \beta$.
The existence and uniqueness of the solution to problem (1) is assumed. Further we assume that problem (1) is well posed with continuous derivatives and that the solution depends differentially on the boundary conditions. The specific assumption on $f(x, y)$ to ensure existence and uniqueness will not be considered [1,2,3].

In this article, we develop a new algorithm capable of solving equation of form (1). To best of our knowledge, no similar method for the solution of problem (1) has been discussed in literature so far. In this paper we discuss exponential finite difference, a new method of at least order four based on local assumption. Its development and analysis are based on Taylor and exponential series expansion.

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In the next section we discuss the derivation of the exponential finite difference method. Local truncation error and convergence of the method are discussed in Section 3. The application of the developed method to the problem (1.1) has been presented and illustrative numerical results have been produced to show the efficiency of the new method in Section 4. Discussion and conclusion on the performance of the method are presented in section 5.

2. Derivation of the method

We define$N+1$,the finite numbers of nodal points of the interval [a,b], in which the solution of the problem (1) is desired as $x_j = a + jh, \quad j = 0, 1, 2, \ldots\ldots, N$ using uniform step length where $h = \frac{b-a}{n}$, $x_0 = a$ and $x_N = b$. Suppose we have to determine a number y_j, which is the numerical approximation of the theoretical solution $y(x)$ of the problem (1) at the nodal point $x_j, \quad j = 1, 2, \ldots.., N-1$ and other similar notations like f_j defined as $f(x_j, y_j)$. Assuming the local assumption that no previous truncation error has been made in computation of solution at mesh point x_j, i.e. $y_{j\pm1} = y(x_j \pm h)$. Following the ideas in [4,5], we propose an approximation to the theoretical solution $y(x_j)$ of the problem (1)by the exponential finite difference scheme as,

$$a_0 y_{j+1} + a_1 y_j + a_2 y_{j-1} = b_0 h^2 f_j \exp(\phi(x_j + h)), \tag{2.1}$$

where a_0, a_1, a_2 and b_0 are unknown constants and $\phi(x_j + h)$, is an unknown sufficiently differentiable function of x. Let us define a function $F_j(h, y)$ and associate it with (2) as,

$$F_j(h, y) \equiv a_0 y_{j+1} + a_1 y_j + a_2 y_{j-1} - b_0 h^2 f_j \exp(\phi(x_j + h)) = 0, \tag{2.2}$$

Assume that $\phi(x_j + h)$ can be expand in Taylor series about point $x = x_j$. Hence we write $\phi(x_j + h)$ in Taylor series we have,

$$\phi(x_j + h) = \phi(x_j) + h\phi'(x_j) + \frac{h^2}{2}\phi''(x_j) + O(h^3), \tag{2.3}$$

The application of (4) in the expansion of $\exp(\phi(x_j + h))$ will provide an $O(h^3)$ approximation of the form as,

$$\begin{aligned}\exp(\phi(x_j + h)) = 1 + \phi(x_j) + \frac{1}{2}\phi^2(x_j) + h(1 + \phi(x_j))\phi'(x_j)\\ + \frac{h^2}{2}((\phi'(x_j))^2 + (1 + \phi(x_j))\phi''(x_j)) + O(h^3),\end{aligned} \tag{2.4}$$

Expand $F_j(h, y)$ in Taylor series about mesh point $x = x_j$ and using (5)in it, we have

$$\begin{aligned}F_j(h, y) \equiv\ & \{(a_0 + a_1 + a_2)y_j + h(a_0 - a_2)y'_j + \frac{h^2}{2}(a_0 + a_2)f_j + \frac{h^3}{6}(a_0 - a_2)f'_j \\ & + \frac{h^4}{24}(a_0 + a_2)f''_j\} - b_0 h^2 f_j\{1 + \phi(x_j) + \frac{1}{2}\phi^2(x_j) + \\ & h(1 + \phi(x_j))\phi'(x_j) + \frac{h^2}{2}((\phi'(x_j))^2 + (1 + \phi(x_j))\phi''(x_j))\}\end{aligned} \tag{2.5}$$

where $f'_j = y_j^{(3)}$ and $f''_j = y_j^{(4)}$. On comparing the coefficients of $h^p, p = 0, 1, 2, 3, 4$ in (6), we get the following system of nonlinear equations

$$a_0 + a_1 + a_2 = 0,$$

$$a_0 - a_2 = 0,$$

$$a_0 + a_2 - 2b_0(1 + \phi(x_j) + \frac{1}{2}\phi^2(x_j)) = 0,$$

$$\frac{1}{6}(a_0 - a_2)f'_j - b_0 f_j(1 + \phi(x_j))\phi'(x_j) = 0,$$

$$\frac{1}{12}(a_0 + a_2)f''_j - b_0 f_j((\phi(x_j))^2 + (1 + \phi(x_j))\phi''(x_j)) = 0, \tag{2.6}$$

To determine the unknown constants in (7), we have to assign arbitrary values to some constants . To simplify the system of equationsin (7), we have considered the following assumptions:

$$\phi(x_j) = 0,$$

and

$$\phi'(x_j) = 0, \tag{2.7}$$

Using (8) in (7) and solved the reduced system of equations, we obtained

$$a_1 = -2a_0,$$

$$b_0 = a_0,$$

$$\phi''(x_j) = \frac{f''_j}{6f_j}. \tag{2.8}$$

Substituting the values of $\phi(x_j), \phi'(x_j)$, and $\phi''(x_j)$ from (8) and (9) in (4), we have

$$\phi(x_j + h) = \frac{h^2 f''_j}{12 f_j}. \tag{2.9}$$

Finally substitute the values of a_1, a_2, b_0 and $\phi(x_j + h)$ from (9) and (10) in (2), we obtain our proposed exponential finite difference method as

$$y_{j+1} - 2y_j + y_{j-1} = h^2 f_j \exp(\frac{h^2 f''_j}{12 f_j}). \tag{2.10}$$

For each nodal point, we will obtain the nonlinear system of equations given by (11) or a linear system of equations if the source function is f(x). For computational purpose reported in Section 4, we have used second order finite difference approximation in place of f''_j i.e.

$$h^2 f''_j = f_{j+1} - 2f_j + f_{j-1}.$$

3. Local Truncation Error and Convergence

The truncation error T_j at the nodal point x_j may be written as in [6,7,8,9],

$$T_j = y_{j+1} - 2y_j + y_{j-1} - h^2 f_j \exp(\frac{h^2 f_j''}{12 f_j})$$

$$= -\frac{h^6}{240}\{y_j^{(6)} + \frac{5}{6}\frac{(y^{(4)})^2}{y_j''}\} + O(h^7) \tag{3.1}$$

The exact solution $y(x)$ of (1) satisfies the equation

$$y(x_{j+1}) - 2y(x_j) + y(x_{j-1})$$
$$= h^2 f(x_j, y(x_j)) \exp(\frac{f(x_{j+1}, y(x_{j+1})) + f(x_{j-1}, y(x_{j-1}))}{12 f(x_j, y(x_j))} - \frac{1}{6}) + T_j, \quad i \le j \le N-1. \tag{3.2}$$

Subtracting (13) from (12) and applying the mean value theorem, we obtained

$$-2(y_j - y(x_j)) = h^2\{f_j \exp(\frac{f_{j+1} + f_{j-1}}{12 f_j} - \frac{1}{6})$$
$$- f(x_j, y(x_j)) \exp(\frac{f(x_{j+1}, y(x_{j+1})) + f(x_{j-1}, y(x_{j-1}))}{12 f(x_j, y(x_j))} - \frac{1}{6})\} - T_j$$

$$= h^2 \exp(\frac{-1}{6}(f(x_j, y_j) - f(x_j, y(x_j))) - T_j$$
$$= h^2 \exp(\frac{-1}{6}(y_j - y(x_j))\frac{\partial f(\theta_j)}{\partial y} - T_j, \quad y_j < \theta_j < y(x_j). \tag{3.3}$$

Let us substitute $\varepsilon_j = y_j - y(x_j)$ in (14) and simplify, we have

$$(2 + h^2 \exp(\frac{-1}{6})\frac{\partial f(\theta_j)}{\partial y})\varepsilon_j = T_j. \tag{3.4}$$

Let us write (15) in matrix form

$$\mathbf{(J+Q)E=T},$$

where $\mathbf{E} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_{N-1})^T, \mathbf{T} = (T_1, T_2, ...T_{N-1})^T$,

$$\mathbf{Q} = h^2 \exp(\tfrac{-1}{6}) \begin{pmatrix} \frac{\partial f(\theta_1)}{\partial y} & & 0 \\ & \frac{\partial f(\theta_2)}{\partial y} & \\ & & \ddots \\ 0 & & \frac{\partial f(\theta_{N-1})}{\partial y} \end{pmatrix} \text{ and } \mathbf{J} = \begin{pmatrix} 2 & & 0 \\ & 2 & \\ & \ddots & \\ 0 & & 2 \end{pmatrix}.$$

We also have $\frac{\partial f(\theta_j)}{\partial y} > 0, \quad j = 1, 2, ..., N-1$, so $\mathbf{Q} > 0$ and $\mathbf{J}+\mathbf{Q} > \mathbf{J}$. So we have

$$0 < (\mathbf{J}+\mathbf{Q})^{-1} < \mathbf{J}^{-1}$$

$$\|\mathbf{E}\| \leq \|(\mathbf{J}+\mathbf{Q})^{-1}\|\|\mathbf{T}\| \leq \|\mathbf{J}^{-1}\|\|\mathbf{T}\|$$

$$\|\mathbf{E}\| \leq \frac{h^6}{480}\{y_j^{(6)} + \frac{5}{6}\frac{(y^{(4)})^2}{y_j''}\} \tag{3.5}$$

It follows from (16) that $\|\mathbf{E}\| \to 0$ as $h \to 0$. Thus we conclude that method (11) converges and the order of convergence is at least four.

4. Numerical Results

To illustrate our method and demonstrate its computationally efficiency, we consider some model problems. In each case we took uniform step size h. In table 1 - table 4, we have shown maximum absolute error (MAU) and l_2 -norm of the error (ERR), computed for different values of N and these are defined as,

$$MAU = \max_{1 \leq j \leq N-1 |y(x_j) - y_j|}$$

$$ERR = \frac{1}{N-1}\sqrt{\sum_{j=1}^{N-1}(y(x_j) - y_j)^2}$$

We have used Newton-Raphson iteration method to solve the system of nonlinear equations and Gauss Seidel iterative method to solve linear system of equations, both arised from equation (11). All computations were performed on a MS Window 2007 professional operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Duo Core 2.20 Ghz PC .The solutions are computed on N-1 nodes and iteration is continued until either maximum difference between two successive iterates is less than $10^{(-16)}$ or number of iteration reached 10^3. Table 1- Table 4 also showed the number of iterations performed to achieve desired accuracy for each model problem.

Problem 1. The first model problem is a nonlinear problem given by

$$y''(x) = \frac{3}{2}y^2(x), \quad y(0) = 4, \quad y(1) = 1, \quad x \in [0, 1].$$

The analytical solution is $y(x) = \frac{4.0}{(1+x)^2}$. The MAU and ERR for different values of N are presented in Table 1.

Problem 2. The second model problem is linear problem

$$y''(x) = -K^2y(x) + (K^2 - \pi^2)\sin(\pi x), \quad y(0) = 0, \quad y(1) = 0, \quad x \in [0, 1].$$

The analytical solution is found to be $y(x) = \sin(\pi x)$.For $K^2 = 10$ the MAU and ERR for different values of N are presented in Table 2.

Problem 3. The third model problem is highly sensitive nonlinear, well known as the Troesch's equation [10],which is given by

$$y''(x) = \lambda \sinh(\lambda y \lambda(x)) + g(x), \quad y(0) = 1, \quad y(1) = \frac{\sinh(\lambda)}{\lambda}1, \quad x \in [0,1].$$

The analytical solution is $y(x) = \frac{\sinh(\lambda x)}{\lambda}$. For each value of $\lambda = 10, 20, 30$ the MAU for different values of N are presented in Table 3.

Problem 4. The fourth and final model problem is a nonlinear problem given by

$$y''(x) = y^2(x) + 2\pi^2 \cos(2\pi x) - \sin^4(\pi x) \quad y(0) = 0, \quad y(1) = 0, \quad x \in [0,1].$$

The analytical solution is considered to be $y(x) = \sin^2(\pi x)$. The MAU and ERR for different values of N are presented in Table 4.

Table 1: Maximum absolute error in $y(x) = \frac{4.0}{(1+x)^2}$ for problem 1.

N	Method(11)		
	MAU	*ERR*	*Iterations*
4	.35644150(-2)	.39054255(-2)	12
8	.23323059(-3)	.26143758(-3)	27
16	.13647080(-4)	.11079033(-4)	38
32	.12704658(-6)	.13229610(-7)	2

Table 2: Maximum absolute error in $y(x) = \sin(\pi x)$ for problem 2.

N	Method(11)		
	MAU	*ERR*	*Iterations*
4	.27641535(-1)	.10031415(0)	123
8	.12654066(-2)	.44306965(-2)	259
16	.60677528(-4)	.12922547(-3)	296
32	.59604645(-7)	.88322334(-8)	3

Table 3: Maximum absolute errors in $y(x) = \frac{\sinh(\lambda x)}{\lambda}$ for problem 3.

N	Method(11)					
	$\lambda = 10$		$\lambda = 20$		$\lambda = 30$	
	MAU	$Iterations$	MAU	$Iterations$	MAU	$Iterations$
8	.33518672(-3)	4	*****	*****	*****	*****
16	.34719706(-4)	7	.16759336(-3)	3	.52547678(-2)	1
32	.10840595(-5)	15	.17359893(-4)	6	.78815967(-4)	4
64	.11920929(-6)	12	.54202974(-6)	14	.27474016(-5)	8
128	.29802322(-7)	5	.59604645(-7)	12	.83819032(-7)	21

Table 4: Maximum absolute error in $y(x) = \sin^2(\pi x)$ for problem 4.

N	Method(11)		
	MAU	ERR	$Iterations$
4	.39065361(-1)	.50859000(-1)	20
8	.25463104(-2)	.37111132(-2)	54
16	.16093254(-3)	.24020411(-3)	129
32	.53644180(-5)	.34110020(-5)	79
64	.11992816(-6)	.21216829(-7)	17

5. Conclusion

A new approach to obtain the numerical solution of second order boundary value problems has been developed. The new scheme has advantages and disadvantages when considered individually. For example the scheme based on exponential approximation, if the source function is $f(x)$ then the system of equation from (11)is linear otherwise we will obtain nonlinear system of equations, which is always difficult to be solved. On the other hand the new method has a good rate high order of convergence which yield smaller discretization error. We can deduce the royal Numerov method from this new method by second order expansion of exponential function. It is an advantage of the method that we consider exponential function without any approximation in computation in contrasts to Numerov method. This means method depends on machine epsilon and software used in computation when solving the problem . It may be noted that method can be avoided in the case where the source function vanishes in the computational domain of the problem. The decision to use a certain difference scheme does not only depend on the given order of the method but also on its computational efficiency. The numerical results for model problems show that the new method is computationally efficient. Also it is observed from the results that method has high accuracy i.e. small discretization error. In the present article finite difference method of high order has been derived on the basis of exponential function and local assumption. It is not clear how this local assumption affect the overall solution of the problem. Investigation in this direction will be done in the future. The new method lead to possibility to develop difference methods to solve third order and forth order boundary value problems in ODEs. Work in this direction is in progress.

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