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Some sets of χ^2- summable sequences of Fuzzy Numbers Defined By A Modulus

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ABSTRACT: In this paper we introduce the χ^2 fuzzy numbers defined by a modulus, study some of their properties and inclusion results.

Key Words: gai sequence, analytic sequence, modulus function, double sequences, completeness, solid space, symmetric space.

Contents

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinatewise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solancan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - |^{t_{mn}} = 1\, for\, some\, \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

2000 *Mathematics Subject Classification:* 40A05,40C05,40D05 where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Feyzi Başar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Feyzi Başar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

Spaces of strongly summable sequences were discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{mn}_{k\ell} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{allfinitesequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \le p \le \infty$ by Feyzi Başar and Altay in [42] and in the case $0 < p < 1$ by Altay and Feyzi Başar in [43]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\ge1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \le p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

2. Definitions and Preliminaries

Throughout a double sequence is denoted by $\langle X_{mn} \rangle$, a double infinite array of fuzzy real numbers.

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on the real line $\mathbb{R}$. For $X = [a_1, a_2] \in D$ and $Y = [b_1, b_2] \in D$, define

$$d(X,Y) = max(|a_1 - b_1|, |a_2 - b_2|)$$

It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on $\mathbb{R}$, that is, a mapping $X : \mathbb{R} \to I\,(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

The $\alpha-$ level set $[X]^{\alpha}$, of the fuzzy real number X, for $0 < \alpha \leq 1$; is defined by

$$[X]^{\alpha} = \{t \in \mathbb{R} : X(t) \geq \alpha\}.$$

The $0-$ level set is th closure of the strong $0-$ cut that is, $cl\,\{t \in \mathbb{R} : X(t) > 0\}$.

A fuzzy real number X is called convex if $X(t) \geq X(s) \wedge X(r) = min\,\{X(s), X(r)\}$, where $s < t < r$. If there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$ then, the fuzzy real number X is called normal.

A fuzzy real number X is said to be upper-semi continuous if, for each $\epsilon > 0, X^{-1}([0, a+\epsilon))$ is open in the usual topology of $\mathbb{R}$ for all $a \in I$.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $L(\mathbb{R})$.

The absolute value, $|X|$ of $X \in L(\mathbb{R})$ is defined by

$$|X|(t) = \begin{cases} max\,\{X(t), X(-t)\}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Let $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\bar{d}(X, Y) = sup_{0 \leq \alpha \leq 1} d([X]^{\alpha}, [Y]^{\alpha}).$$

Then, $\bar{d}$ defines a metric on $L(\mathbb{R})$ and it is well-known that $\left(L(\mathbb{R}), \bar{d}\right)$ is a complete metric space.

A sequence $\langle X_{mn} \rangle \subset L(\mathbb{R})$ is said to be null if $\bar{d}(X_{mn}, \bar{0}) = 0$.

A double sequence $\langle X_{mn} \rangle$ of fuzzy real numbers is said to be chi in Pringsheim's sense to a fuzzy number 0 if $lim_{m,n\to\infty}\left((m+n)!X_{mn}\right)^{1/m+n} = 0$.

A double sequence $\langle X_{mn} \rangle$ is said to chi regularly if it converges in the Prinsheim's sense and the following limts zero:

$$lim_{m\to\infty}\left((m+n)!X_{mn}\right)^{1/m+n} = 0 \text{ for each } n \in \mathbb{N},$$

and

$$lim_{n\to\infty}\left((m+n)!X_{mn}\right)^{1/m+n} = 0 \text{ for each } m \in \mathbb{N}.$$

A fuzzy real-valued double sequence space E^F is said to be solid if $\langle Y_{mn} \rangle \in E^F$ whenever $\langle X_{mn} \rangle \in E^F$ and $|Y_{mn}| \leq |X_{mn}|$ for all $m, n \in \mathbb{N}$.

Let $K = \{(m_i, n_i) : i \in \mathbb{N}; m_1 < m_2 < m_3 \cdots and\, n_1 < n_2 < n_3 < \cdots\} \subseteq \mathbb{N} \times \mathbb{N}$ and E^F be a double sequence space. A $K-$step space of E^F is a sequence space $\lambda_K^E = \left\{\langle X_{m_i n_i} \rangle \in w^{2F} : \langle X_{mn} \rangle \in E^F\right\}$.

A canonical pre-image of a sequence $\langle X_{m_i n_i} \rangle \in E^F$ is a sequence $\langle Y_{mn} \rangle$ defined as follows:

$$Y_{mn} = \begin{cases} X_{mn}, & \text{if } (m,n) \in K, \\ \bar{0}, & \text{otherwise}. \end{cases}$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

A sequence set E^F is said to be monotone if E^F contains the canonical pre-images of all its step spaces.

A sequence set E^F is said to be symmetric if $\left\langle X_{\pi_{(m)},\pi_{(n)}}\right\rangle \in E^F$ whenever $\langle X_{mn}\rangle \in E^F$, where π is a permutation of $\mathbb{N}$.

A fuzzy real-valued sequence set E^F is said to be convergent free if $\langle Y_{mn}\rangle \in E^F$ whenever $\langle X_{mn}\rangle \in E^F$ and $X_{mn} = \bar{0}$ implies $Y_{mn} = \bar{0}$.

We define the following classes of sequences:

$\Lambda_f^{2F} = \left\{\langle X_{mn}\rangle : sup_{mn} f\left(\bar{d}\left(X_{mn}^{1/m+n}, \bar{0}\right)\right) < \infty, X_{mn} \in L(\mathbb{R})\right\}.$

$\chi_f^{2F} = \left\{\langle X_{mn}\rangle : lim_{mn\to\infty} f\left(\bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)\right) = 0\right\}.$

Also, we define the classes of sequences $\chi_f^{2F^R}$ as follows :

A sequence $\langle X_{mn}\rangle \in \chi_f^{2F^R}$ if $\langle x_{mn}\rangle \in \chi_f^{2F}$ and the following limits hold

$$lim_{m\to\infty} f\left(\bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)\right) = 0 \text{ for each } n \in \mathbb{N}.$$

$$lim_{n\to\infty} f\left(\bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)\right) = 0 \text{ for each } m \in \mathbb{N}.$$

Definition 2.1. *A modulus function was introduced by Nakano [12]. We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$, such that*
(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x+y) \le f(x) + f(y)$, for all $x \ge 0$, $y \ge 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \le f(|x-y|)$, it follows from here that f is continuous on $[0,\infty)$.

3. Main Results

Theorem 3.1. *Let*
$N_1 = min\left\{n_0 : sup_{mn\ge n_0} f\left(\bar{d}\left(((m+n)!(X_{mn}-Y_{mn}))^{1/m+n}, \bar{0}\right)\right)^{P_{mn}} < \infty\right\}$
$N_2 = min\{n_0 : sup_{mn\ge n_0} P_{mn} < \infty\}$ *and* $N = max(N_1, N_2)$.
(i) $\chi_{f_p}^{2F^R}$ *is not a paranormed space with*

$$g(X) = lim_{N\to\infty} sup_{mn\ge N} f\left(\bar{d}\left(((m+n)!(X_{mn}-Y_{mn}))^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} \tag{3.1}$$

if and only if $\mu > 0$, where $\mu = lim_{N\to\infty} inf_{mn\ge N} P_{mn}$ and
$M = max(1, sup_{mn\ge N} P_{mn})$
(ii) $\chi_{f_p}^{2F^R}$ *is complete with the paranorm (3.1).*

Proof:
(i) Necesity: Let $\chi_{f_p}^{2F^R}$ be a paranormed space with (3.1) and suppose that $\mu = 0$. Then $\alpha = inf_{mn\ge N} P_{mn} = 0$ for all $N \in \mathbb{N}$ and

$g\left\langle \lambda X\right\rangle = lim_{N\to\infty} sup_{mn\geq N} \left|\lambda\right|^{P_{mn/M}} = 1$ for all $\lambda \in (0,1]$, where $X = \langle\alpha\rangle \in \chi^{2F^R}_{f_p}$ whence $\lambda \to 0$ does not imply $\lambda X \to \theta$, when X is fixed. But this contradicts to (3.1) to be a paranorm.

Sufficiency: Let $\mu > 0$. It is trivial that $g(\theta) = 0, g(-X) = g(X)$ and $g\left\langle X+Y,\bar{0}\right\rangle \leq g\left\langle X,\bar{0}\right\rangle + g\left\langle Y,\bar{0}\right\rangle$. Since $\mu > 0$ there exists a positive number β such that $P_{mn} > \beta$ for sufficiently large positive integer m,n. Hence for any $\lambda \in \mathbb{C}$, we may write $|\lambda|^{P_{mn}} \leq max\left(|\lambda|^M, |\lambda|^\beta\right)$ for suffciently large positive integers $m,n \geq N$. Therefore, we obtain $g\left\langle \lambda X,\bar{0}\right\rangle \leq max\left(|\lambda|, |\lambda|^{\beta/M}\right) g\left\langle X\right\rangle$. Using this, one can prove that $\lambda X \to \theta$, whenever X is fixed and $\lambda \to 0$ or $\lambda \to 0$ and $X \to \theta$, or λ is fixed and $X \to \theta$.

Because a paranormed space is a vector space. $\chi^{2F^R}_{f_p}$ is a set of sequences of fuzzy numbers. But the set $w^F = \{\langle X_{mn}\rangle : X_{mn} \in L(R)\}$ of all sequences of fuzzy numbers is not a vector space. That is why, in order to say that $\chi^{2F^R}_{f_p}$ is a vector subspace (that is a sequence space) it is not sufficient to show that $\chi^{2F^R}_{f_p}$ is closed under addition and scalar multiplication. Consequently since w^F is not a vector space, then $\chi^{2F^R}_{f_p}$ is not a vector subspace so that not a sequence space. Therefore it can not be a paranormed space.

Proof: (ii) Let $\left\langle X^{k\ell}\right\rangle$ be a Cauchy sequence in $\chi^{2F^R}_{f_p}$, where $X^{k\ell} = \left\langle X^{k\ell}_{mn}\right\rangle_{m,n\in\mathbb{N}}$. Then for every $\epsilon > 0\,(0 < \epsilon < 1)$ there exists a positive integer s_0 such that

$$g\left\langle X^{k\ell} - X^{rt}\right\rangle = lim_{N\to\infty} sup_{mn\geq N} f\left(\bar{d}\left(\left((m+n)!\left(X^{k\ell}_{mn} - X^{rt}_{mn}\right)\right)^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} < \frac{\epsilon}{2} \tag{3.2}$$

for all $k,\ell,r,t > s_0$.

By (3.2) there exists a positive integer n_0 such that

$$sup_{mn\geq N} f\left(\bar{d}\left(\left((m+n)!\left(X^{k\ell}_{mn} - X^{rt}_{mn}\right)\right)^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} < \frac{\epsilon}{2} \tag{3.3}$$

for all $k,\ell,r,t > s_0$ and for $N > n_0$. Hence we obtain

$$f\left(\bar{d}\left(\left((m+n)!\left(X^{k\ell}_{mn} - X^{rt}_{mn}\right)\right)^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} < \frac{\epsilon}{2} < 1 \tag{3.4}$$

so that

$$f\left(\bar{d}\left(\left((m+n)!\left(X^{k\ell}_{mn} - X^{rt}_{mn}\right)\right)^{1/m+n}, \bar{0}\right)\right) < f\left(\bar{d}\left(\left((m+n)!\left(X^{k\ell}_{mn} - X^{rt}_{mn}\right)\right)^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} < \frac{\epsilon}{2} \tag{3.5}$$

for all $k,\ell,r,t > s_0$. This implies that $\left\langle X^{k\ell}_{mn}\right\rangle_{k\ell\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for each fixed $m,n \geq n_0$. Hence the sequence $\left\langle X^{k\ell}_{mn}\right\rangle_{k\ell\in\mathbb{N}}$ is convergent to X_{mn} say,

$$lim_{k\ell\to\infty} X^{k\ell}_{mn} = X_{mn}\, for\, each\, fixed\, m,n > n_0. \tag{3.6}$$

Getting X_{mn}, we define $X = \langle X_{mn} \rangle$. From (3.2) we obtain

$$g\left\langle X^{k\ell} - X\right\rangle = lim_{N\to\infty} sup_{mn\geq N} f\left(\bar{d}\left(\left((m+n)!\left(X_{mn}^{k\ell} - X_{mn}\right)\right)^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} < \frac{\epsilon}{2} \tag{3.7}$$

as $r, t \to \infty$, for all $k, \ell, r, t > s_0$. by (3.6). This implies that $lim_{k\ell\to\infty} X^{k\ell} = X$. Now we show that $X = \langle X_{mn} \rangle \in \chi_{f_p}^{2F^R}$. Since $X^{k\ell} \in \chi_{f_p}^{2F^R}$ for each $(k, 1) \in N \times N$ for every $\epsilon > 0\,(0 < \epsilon < 1)$ there exists a positive integer $n_1 \in N$ such that

$$f\left(\bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} < \frac{\epsilon}{2}\, for\, every\, m, n > n_1. \tag{3.8}$$

By (3.6),(3.7) and (1.1) we obtain
$f\left(\bar{d}\left(((m+n)!\,(X_{mn}))^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} \leq f\left(\bar{d}\left(((m+n)!\,(X_{mn}^{k\ell}))^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} +$ $f\left(\bar{d}\left(((m+n)!\,(X_{mn}^{k\ell} - X_{mn}))^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for $k, \ell > max\,(s_0, s_1)$ and $m, n > max\,(n_0, n_1)$. This implies that $X \in \chi_{f_p}^{2F^R}$. □

Proposition 3.2. *The class of sequences* Λ_f^{2F} *is symmetric but the classes of sequences* χ_f^{2F} *and* $\chi_f^{2F^R}$ *are not sysmmetric.*

Proof: Obviously the class of sequences Λ_f^{2F} is symmetric. For the other classes of sequences consider the following example □

Example: Consider the class of sequences χ_f^{2F}. Let $f(X) = X$ and consider the sequence $\langle X_{mn} \rangle$ be defined by

$$X_{1n}(t) = \begin{cases} \frac{(-t+1)^{1+n}}{(1+n)!}, & \text{for } t = -1, \\ \frac{(t-1)^{1+n}}{(1+n)!}, & \text{for } t = 1, \\ 0, & \text{otherwise}. \end{cases}$$

and for $m > 1$,

$$X_{mn}(t) = \begin{cases} \frac{(t+2)^{m+n}}{(m+n)!}, & \text{for } t = -2, \\ \frac{(-t-1)^{m+n}}{(m+n)!}, & \text{for } t = -1, \\ 0, & \text{otherwise}. \end{cases}$$

Let $\langle Y_{mn} \rangle$ be a rearrangement of $\langle X_{mn} \rangle$ defined by

$$Y_{nn}(t) = \begin{cases} \frac{(-t+1)^{2n}}{(2n)!}, & \text{for } t = -1, \\ \frac{(t-1)^{2n}}{(2n)!}, & \text{for } t = 1, \\ 0, & \text{otherwise}. \end{cases}$$

and for $m \neq n$,

$$Y_{mn}(t) = \begin{cases} \frac{(t+2)^{m+n}}{(m+n)!}, & \text{for } t = -2, \\ \frac{(-t-1)^{m+n}}{(m+n)!}, & \text{for } t = -1, \\ 0, & \text{otherwise} . \end{cases}$$

Then, $\langle X_{mn} \rangle \in \chi_f^{2F}$ but $\langle Y_{mn} \rangle \notin \chi_f^{2F}$. Hence, χ_p^{2F} is not symmetric. Similarly other sequence also not symmetric.

Proposition 3.3. *The classes of sequences* $\Lambda_f^{2F}, \chi_f^{2F}$ *and* $\chi_f^{2F^R}$ *are solid.*

Proof: Consider the class of sequences χ_f^{2F}. Let $\langle X_{mn} \rangle$ and $\langle Y_{mn} \rangle \in \chi_f^{2F}$ be such that $\bar{d}\left(\left((m+n)!Y_{mn}\right)^{1/m+n}, \bar{0}\right) \leq \bar{d}\left(\left((m+n)!X_{mn}\right)^{1/m+n}, \bar{0}\right)$. As f is non-decreasing, we have
$lim_{mn\to\infty} f\left(\bar{d}\left(\left((m+n)!Y_{mn}\right)^{1/m+n}, \bar{0}\right)\right) \leq lim_{mn\to\infty} f\left(\bar{d}\left(\left((m+n)!X_{mn}\right)^{1/m+n}, \bar{0}\right)\right)$
Hence, the class of sequence χ_f^{2F} is solid. Simlarly it can be shown that the other classes of sequences are also solid. □

Proposition 3.4. *The classes of sequences* χ_f^{2F} *and* $\chi_f^{2F^R}$ *are not monotone and hence not solid.*

Proof: The result follows from the following example.
Example: Consider the class of sequences χ_f^{2F} and $f(X) = X$. Let $J = \{(m,n) : m \geq n\} \subseteq N \times N$. Let $\langle X_{mn} \rangle$ be defined by

$$X_{mn}(t) = \begin{cases} \frac{(t+3)^{m+n}}{(m+n)!}, & \text{for } -3 < t \leq -2, \\ \frac{(mt)^{m+n}}{(3m-1)^{m+n}(m+n)!} + \frac{(3m)^{m+n}}{(3m-1)^{m+n}(m+n)!}, & \text{for } -2 \leq t \leq -1 + \frac{1}{m}, \\ \bar{0}, & \text{otherwise} . \end{cases}$$

for all $m, n \in N$.
Then $\langle X_{mn} \rangle \in \chi_f^{2F}$. Let $\langle Y_{mn} \rangle$ be the canonical pre-image of $\langle X_{mn} \rangle_J$ for the subsequence J of $N \times N$. Then

$$Y_{mn} = \begin{cases} X_{mn}, & \text{for } (m,n) \in J, \\ \bar{0}, & \text{otherwise} . \end{cases}$$

Then, $\langle Y_{mn} \rangle \notin \chi_f^{2F}$. Hence χ_f^{2F} is not monotone. Similarly, it can be shown that the other classes of sequences are also not monotone. Hence, the classes of sequences χ_f^{2F} and $\chi_f^{2F^R}$ are not solid. □

Proposition 3.5. *(i)* $\chi_{f_1}^{2F} \bigcap \chi_{f_2}^{2F} \subseteq \chi_{f_1+f_2}^{2F}$, *(ii)* $\chi_{f_1}^{2F^R} \bigcap \chi_{f_2}^{2F^R} \subseteq \chi_{f_1+f_2}^{2F^R}$

Proof: It is easy, so omitted. □

Proposition 3.6. *Let f and f_1 be two modulus functions, then,(i) $\chi^{2F}_{f_1} \subseteq \chi^{2F}_{f\circ f_1}$ (ii)$\chi^{2F^R}_{f_1} \subseteq \chi^{2F^R}_{f\circ f_1}$ (iii)$\Lambda^{2F}_{f_1} \subseteq \Lambda^{2F}_{f\circ f_1}$*

Proof: We prove the result for the case $\chi^{2F}_{f_1} \subseteq \chi^{2F}_{f\circ f_1}$, the other cases similar. Let $\epsilon > 0$ be given. As f is continuous and non-decreasing, so there exists $\eta > 0$, such that $f(\eta) = \epsilon$. Let $\langle X_{mn}\rangle \in \chi^{2F}_{f_1}$. Then, there exist $m_0, n_0 \in \mathbb{N}$, such that

$$f_1\left(\bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)\right) < \eta, \text{ for all } m \geq m_0, n \geq n_0,$$

$$\Rightarrow f \circ f_1\left(\bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)\right) < \epsilon, \text{ for all } m \geq m_0, n \geq n_0.$$

Hence, $\langle X_{mn}\rangle \in \chi^{2F}_{f\circ f_1}$. Thus, $\chi^{2F}_{f_1} \subseteq \chi^{2F}_{f\circ f_1}$. □

Proposition 3.7. *(i) $\chi^{2F}_{f} \subseteq \Lambda^{2F}_{f}$ (ii) $\chi^{2F^R}_{f} \subseteq \Lambda^{2F}_{f}$. The inclusion are strict.*

Proof: The inclusion (i) $\chi^{2F}_{f} \subseteq \Lambda^{2F}_{f}$ (ii) $\chi^{2F^R}_{f} \subseteq \Lambda^{2F}_{f}$ is obvious. For establishing that the inclusions are proper, consider the following example.
Example: We prove the result for the case $\chi^{2F}_{f} \subseteq \Lambda^{2F}_{f}$, the other case similar. Let $f(X) = X$. Let the sequence $\langle X_{mn}\rangle$ be defined by for $m > n$,

$$X_{mn}(t) = \begin{cases} \frac{(mt-m-1)^{m+n}(m-1)^{-(m+n)}}{(m+n)!}, & \text{for } 1+\frac{1}{m} \leq t \leq 2, \\ \frac{(3-t)^{m+n}}{(m+n)!}, & \text{for } 2 < t \leq 3, \\ 0, & \text{otherwise} . \end{cases}$$

and for $m < n$

$$X_{mn}(t) = \begin{cases} \frac{(mt-1)^{m+n}(m-1)^{-(m+n)}}{(m+n)!}, & \text{for } \frac{1}{m} \leq t \leq 1, \\ \frac{(-t+2)^{m+n}}{(m+n)!}, & \text{for } 1 \leq t \leq 2, \\ 0, & \text{otherwise} . \end{cases}$$

Then, $\langle X_{mn}\rangle \in \Lambda^{2F}_{f}$ but $\langle X_{mn}\rangle \notin \chi^{2F}_{f}$. □

Proposition 3.8. *The classes of sequences $\Lambda^{2F}_{f}, \chi^{2F}_{f}$ and $\chi^{2F^R}_{f}$ are not convergent free.*

Proof: The result follows from the following example.
Example: Consider the classes of sequences χ^{2F}_{f}. Let $f(X) = X$ and consider the sequence $\langle X_{mn}\rangle$ defined by $((1+n)!X_{1n})^{1/1+n} = \bar{0}$, and for other values,

$$X_{mn}(t) = \begin{cases} \frac{1^{m+n}}{(m+n)!}, & \text{for } 0 \leq t \leq 1, \\ \frac{(-mt)^{m+n}(m+1)^{-(m+n)}+(2m+1)^{m+n}(1+m)^{-(m+n)}}{(m+n)!}, & \text{for } 1 < t \leq 2+\frac{1}{m}, \\ 0, & \text{otherwise} . \end{cases}$$

Let the sequence $\langle Y_{mn}\rangle$ be defined by $((1+n)!Y_{1n})^{1/1+n} = \bar{0}$, and for other values,

$$Y_{mn}(t) = \begin{cases} \frac{1^{m+n}}{(m+n)!}, & \text{for } 0 \le t \le 1, \\ \frac{(m-t)^{m+n}(m-1)^{-(m+n)}}{(m+n)!}, & \text{for } 1 < t \le m, \\ 0, & \text{otherwise .} \end{cases}$$

Then, $\langle X_{mn}\rangle \in \chi_f^{2F}$ but $\langle Y_{mn}\rangle \notin \chi_f^{2F}$. Hence, the clases of sequences χ_f^{2F} is not convergent free. Simlarly, the other spaces are also ot convergent free. □

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