

SOME SETS OF χ^2 - SUMMABLE SEQUENCES OF FUZZY NUMBERS DEFINED BY A MODULUS

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ABSTRACT. In this paper we introduce the χ^2 fuzzy numbers defined by a modulus, study some of their properties and inclusion results.

This paper is dedicated to my beloved teacher Dr. Professor Umakanta Misra, Department of Mathematics, Berhampur University, Berhampur-760 007, Odissa, India, for his sixtith birth anniversary.

1. INTRODUCTION

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}) , where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinatewise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solancan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\begin{aligned}\mathcal{M}_u(t) &:= \{(x_{mn}) \in w^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty\}, \\ \mathcal{C}_p(t) &:= \{(x_{mn}) \in w^2 : p - \lim_{m,n \rightarrow \infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C}\}, \\ \mathcal{C}_{0p}(t) &:= \{(x_{mn}) \in w^2 : p - \lim_{m,n \rightarrow \infty} |x_{mn}|^{t_{mn}} = 1\},\end{aligned}$$

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$$\mathcal{L}_u(t) := \{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n \rightarrow \infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets \mathcal{M}_u , \mathcal{C}_p , \mathcal{C}_{0p} , \mathcal{L}_u , \mathcal{C}_{bp} and \mathcal{C}_{0bp} , respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α -, β -, γ - duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M -core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the M -core of x . More recently, Altay and Feyzi Başar [27] have defined the spaces \mathcal{BS} , $\mathcal{BS}(t)$, \mathcal{CS}_p , \mathcal{CS}_{bp} , \mathcal{CS}_r and \mathcal{BV} of double sequences consisting of all double series whose sequence of partial sums are in the spaces \mathcal{M}_u , $\mathcal{M}_u(t)$, \mathcal{C}_p , \mathcal{C}_{bp} , \mathcal{C}_r and \mathcal{L}_u , respectively, and also examined some properties of those sequence spaces and determined the α - duals of the spaces \mathcal{BS} , \mathcal{BV} , \mathcal{CS}_{bp} and the $\beta(\vartheta)$ - duals of the spaces \mathcal{CS}_{bp} and \mathcal{CS}_r of double series. Quite recently Feyzi Başar and Sever [28] have introduced the Banach space \mathcal{L}_q of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space \mathcal{L}_q . Quite recently Subramanian and Misra [29] have studied the space $\chi_M^2(p, q, u)$ of double sequences and gave some inclusion relations.

Spaces of strongly summable sequences were discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong A -summability with respect to a modulus where $A = (a_{n,k})$

is a nonnegative regular matrix and established some connections between strong A -summability, strong A - summability with respect to a modulus, and A - statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(1.1) \quad (a + b)^p \leq a^p + b^p$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}$ ($m, n \in \mathbb{N}$) (see[1]).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{m,n} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2 . A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \rightarrow 0$ as $m, n \rightarrow \infty$. The double gai sequences will be denoted by χ^2 . Let $\phi = \{all\ finite\ sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \mathfrak{S}_{ij}$ for all $m, n \in \mathbb{N}$; where \mathfrak{S}_{ij} denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space) X is said to have AK property if (\mathfrak{S}_{mn}) is a Schauder basis for X . Or equivalently $x^{[m,n]} \rightarrow x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \rightarrow (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞} , where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_{∞} denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced

and studied in the case $1 \leq p \leq \infty$ by Feyzi Başar and Altay in [42] and in the case $0 < p < 1$ by Altay and Feyzi Başar in [43]. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

2. DEFINITIONS AND PRELIMINARIES

Throughout a double sequence is denoted by $\langle X_{mn} \rangle$, a double infinite array of fuzzy real numbers.

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on the real line \mathbb{R} . For $X = [a_1, a_2] \in D$ and $Y = [b_1, b_2] \in D$, define

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|)$$

It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on \mathbb{R} , that is, a mapping $X : \mathbb{R} \rightarrow I (= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

The α -level set $[X]^\alpha$, of the fuzzy real number X , for $0 < \alpha \leq 1$; is defined by

$$[X]^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\}.$$

The 0-level set is the closure of the strong 0-cut that is, $cl \{t \in \mathbb{R} : X(t) > 0\}$.

A fuzzy real number X is called convex if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$, where $s < t < r$. If there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$ then, the fuzzy real number X is called normal.

A fuzzy real number X is said to be upper-semi continuous if, for each $\epsilon > 0$, $X^{-1}([0, a + \epsilon))$ is open in the usual topology of \mathbb{R} for all $a \in I$.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $L(\mathbb{R})$.

The absolute value, $|X|$ of $X \in L(\mathbb{R})$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Let $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \rightarrow \mathbb{R}$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha).$$

Then, \bar{d} defines a metric on $L(\mathbb{R})$ and it is well-known that $(L(\mathbb{R}), \bar{d})$ is a complete metric space.

A sequence $\langle X_{mn} \rangle \subset L(\mathbb{R})$ is said to be null if $\bar{d}(X_{mn}, \bar{0}) = 0$.

A double sequence $\langle X_{mn} \rangle$ of fuzzy real numbers is said to be chi in Pringsheim's sense to a fuzzy number 0 if $\lim_{m,n \rightarrow \infty} ((m+n)!X_{mn})^{1/m+n} = 0$.

A double sequence $\langle X_{mn} \rangle$ is said to chi regularly if it converges in the Pringsheim's sense and the following limits zero:

$$\lim_{m \rightarrow \infty} ((m+n)!X_{mn})^{1/m+n} = 0 \text{ for each } n \in \mathbb{N},$$

and

$$\lim_{n \rightarrow \infty} ((m+n)!X_{mn})^{1/m+n} = 0 \text{ for each } m \in \mathbb{N}.$$

A fuzzy real-valued double sequence space E^F is said to be solid if $\langle Y_{mn} \rangle \in E^F$ whenever $\langle X_{mn} \rangle \in E^F$ and $|Y_{mn}| \leq |X_{mn}|$ for all $m, n \in \mathbb{N}$.

Let $K = \{(m_i, n_i) : i \in \mathbb{N}; m_1 < m_2 < m_3 \cdots \text{ and } n_1 < n_2 < n_3 < \cdots\} \subseteq \mathbb{N} \times \mathbb{N}$ and E^F be a double sequence space. A K -step space of E^F is a sequence space $\lambda_K^E = \{\langle X_{m_i n_i} \rangle \in w^{2F} : \langle X_{mn} \rangle \in E^F\}$.

A canonical pre-image of a sequence $\langle X_{m_i n_i} \rangle \in E^F$ is a sequence $\langle Y_{mn} \rangle$ defined as follows:

$$Y_{mn} = \begin{cases} X_{mn}, & \text{if } (m, n) \in K, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E .

A sequence set E^F is said to be monotone if E^F contains the canonical pre-images of all its step spaces.

A sequence set E^F is said to be symmetric if $\langle X_{\pi(m), \pi(n)} \rangle \in E^F$ whenever $\langle X_{mn} \rangle \in E^F$, where π is a permutation of \mathbb{N} .

A fuzzy real-valued sequence set E^F is said to be convergent free if $\langle Y_{mn} \rangle \in E^F$ whenever $\langle X_{mn} \rangle \in E^F$ and $X_{mn} = \bar{0}$ implies $Y_{mn} = \bar{0}$.

We define the following classes of sequences:

$$\Lambda_f^{2F} = \left\{ \langle X_{mn} \rangle : \sup_{mn} f \left(\bar{d} \left(X_{mn}^{1/m+n}, \bar{0} \right) \right) < \infty, X_{mn} \in L(\mathbb{R}) \right\}.$$

$$\chi_f^{2F} = \left\{ \langle X_{mn} \rangle : \lim_{mn \rightarrow \infty} f \left(\bar{d} \left(((m+n)!X_{mn})^{1/m+n}, \bar{0} \right) \right) = 0 \right\}.$$

Also, we define the classes of sequences χ_f^{2FR} as follows :

A sequence $\langle X_{mn} \rangle \in \chi_f^{2FR}$ if $\langle x_{mn} \rangle \in \chi_f^{2F}$ and the following limits hold

$$\lim_{m \rightarrow \infty} f \left(\bar{d} \left(((m+n)! X_{mn})^{1/m+n}, \bar{0} \right) \right) = 0 \text{ for each } n \in \mathbb{N}.$$

$$\lim_{n \rightarrow \infty} f \left(\bar{d} \left(((m+n)! X_{mn})^{1/m+n}, \bar{0} \right) \right) = 0 \text{ for each } m \in \mathbb{N}.$$

2.1. Definition. A modulus function was introduced by Nakano [12]. We recall that a modulus f is a function from $[0, \infty) \rightarrow [0, \infty)$, such that

- (1) $f(x) = 0$ if and only if $x = 0$
- (2) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$,
- (3) f is increasing,
- (4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from here that f is continuous on $[0, \infty)$.

3. MAIN RESULTS

3.1. Theorem. Let $N_1 = \min \left\{ n_0 : \sup_{mn \geq n_0} f \left(\bar{d} \left(((m+n)! (X_{mn} - Y_{mn}))^{1/m+n}, \bar{0} \right) \right)^{P_{mn}} < \infty \right\}$
 $N_2 = \min \{ n_0 : \sup_{mn \geq n_0} P_{mn} < \infty \}$ and $N = \max(N_1, N_2)$.

(i) $\chi_{f_p}^{2FR}$ is not a paranormed space with

$$(3.1) \quad g(X) = \lim_{N \rightarrow \infty} \sup_{mn \geq N} f \left(\bar{d} \left(((m+n)! (X_{mn} - Y_{mn}))^{1/m+n}, \bar{0} \right) \right)^{P_{mn}/M}$$

if and only if $\mu > 0$, where $\mu = \lim_{N \rightarrow \infty} \inf_{mn \geq N} P_{mn}$ and $M = \max(1, \sup_{mn \geq N} P_{mn})$

(ii) $\chi_{f_p}^{2FR}$ is complete with the paranorm (3.1).

Proof: (i) Necessity: Let $\chi_{f_p}^{2FR}$ be a paranormed space with (3.1) and suppose that $\mu = 0$. Then $\alpha = \inf_{mn \geq N} P_{mn} = 0$ for all $N \in \mathbb{N}$ and $g\langle \lambda X \rangle = \lim_{N \rightarrow \infty} \sup_{mn \geq N} |\lambda|^{P_{mn}/M} = 1$ for all $\lambda \in (0, 1]$, where $X = \langle \alpha \rangle \in \chi_{f_p}^{2FR}$ whence $\lambda \rightarrow 0$ does not imply $\lambda X \rightarrow \theta$, when X is fixed. But this contradicts to (3.1) to be a paranorm.

Sufficiency: Let $\mu > 0$. It is trivial that $g(\theta) = 0, g(-X) = g(X)$ and $g\langle X + Y, \bar{0} \rangle \leq g\langle X, \bar{0} \rangle + g\langle Y, \bar{0} \rangle$. Since $\mu > 0$ there exists a positive number β such that $P_{mn} > \beta$ for sufficiently large positive integer m, n . Hence for any $\lambda \in \mathbb{C}$, we may write $|\lambda|^{P_{mn}} \leq \max(|\lambda|^M, |\lambda|^\beta)$ for sufficiently large positive integers $m, n \geq N$. Therefore, we obtain $g\langle \lambda X, \bar{0} \rangle \leq \max(|\lambda|, |\lambda|^{\beta/M}) g\langle X \rangle$. Using this, one can prove that $\lambda X \rightarrow \theta$, whenever X is fixed and $\lambda \rightarrow 0$ or $\lambda \rightarrow 0$ and $X \rightarrow \theta$, or λ is fixed and $X \rightarrow \theta$.

Because a paranormed space is a vector space. $\chi_{f_p}^{2FR}$ is a set of sequences of fuzzy numbers. But the set $w^F = \{\langle X_{mn} \rangle : X_{mn} \in L(R)\}$ of all sequences of fuzzy numbers is not a vector space. That is why, in order to say that $\chi_{f_p}^{2FR}$ is a vector subspace (that is a sequence space) it is not sufficient to show that $\chi_{f_p}^{2FR}$ is closed under addition and scalar multiplication. Consequently since w^F is not a vector space, then $\chi_{f_p}^{2FR}$ is not a vector subspace so that not a sequence space. Therefore it can not be a paranormed space.

Proof: (ii) Let $\langle X^{k\ell} \rangle$ be a Cauchy sequence in $\chi_{f_p}^{2FR}$, where $X^{k\ell} = \langle X_{mn}^{k\ell} \rangle_{m,n \in \mathbb{N}}$. Then for every $\epsilon > 0$ ($0 < \epsilon < 1$) there exists a positive integer s_0 such that

$$(3.2) \quad g \langle X^{k\ell} - X^{rt} \rangle = \lim_{N \rightarrow \infty} \sup_{mn \geq N} f \left(\bar{d} \left(((m+n)! (X_{mn}^{k\ell} - X_{mn}^{rt}))^{1/m+n}, \bar{0} \right) \right)^{P_{mn}/M} < \frac{\epsilon}{2}$$

for all $k, \ell, r, t > s_0$.

By (3.2) there exists a positive integer n_0 such that

$$(3.3) \quad \sup_{mn \geq N} f \left(\bar{d} \left(((m+n)! (X_{mn}^{k\ell} - X_{mn}^{rt}))^{1/m+n}, \bar{0} \right) \right)^{P_{mn}/M} < \frac{\epsilon}{2}$$

for all $k, \ell, r, t > s_0$ and for $N > n_0$. Hence we obtain

$$(3.4) \quad f \left(\bar{d} \left(((m+n)! (X_{mn}^{k\ell} - X_{mn}^{rt}))^{1/m+n}, \bar{0} \right) \right)^{P_{mn}/M} < \frac{\epsilon}{2} < 1$$

so that

$$(3.5) \quad f \left(\bar{d} \left(((m+n)! (X_{mn}^{k\ell} - X_{mn}^{rt}))^{1/m+n}, \bar{0} \right) \right) < f \left(\bar{d} \left(((m+n)! (X_{mn}^{k\ell} - X_{mn}^{rt}))^{1/m+n}, \bar{0} \right) \right)^{P_{mn}/M} < \frac{\epsilon}{2}$$

for all $k, \ell, r, t > s_0$. This implies that $\langle X_{mn}^{k\ell} \rangle_{k\ell \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{C} for each fixed $m, n \geq n_0$. Hence the sequence $\langle X_{mn}^{k\ell} \rangle_{k\ell \in \mathbb{N}}$ is convergent to X_{mn} say,

$$(3.6) \quad \lim_{k\ell \rightarrow \infty} X_{mn}^{k\ell} = X_{mn} \text{ for each fixed } m, n > n_0.$$

Getting X_{mn} , we define $X = \langle X_{mn} \rangle$. From (3.2) we obtain

$$(3.7) \quad g \langle X^{k\ell} - X \rangle = \lim_{N \rightarrow \infty} \sup_{mn \geq N} f \left(\bar{d} \left(((m+n)! (X_{mn}^{k\ell} - X_{mn}))^{1/m+n}, \bar{0} \right) \right)^{P_{mn}/M} < \frac{\epsilon}{2}$$

as $r, t \rightarrow \infty$, for all $k, \ell, r, t > s_0$. by (3.6). This implies that $\lim_{k\ell \rightarrow \infty} X^{k\ell} = X$.

Now we show that $X = \langle X_{mn} \rangle \in \chi_{f_p}^{2FR}$. Since $X^{k\ell} \in \chi_{f_p}^{2FR}$ for each $(k, 1) \in N \times N$ for every $\epsilon > 0$ ($0 < \epsilon < 1$) there exists a positive integer $n_1 \in N$ such that

$$(3.8) \quad f \left(\bar{d} \left(((m+n)! X_{mn})^{1/m+n}, \bar{0} \right) \right)^{P_{mn}/M} < \frac{\epsilon}{2} \text{ for every } m, n > n_1.$$

By (3.6), (3.7) and (1.1) we obtain

$$\begin{aligned} f\left(\bar{d}\left(((m+n)!(X_{mn}))^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} &\leq f\left(\bar{d}\left(((m+n)!(X_{mn}^{k\ell}))^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} + \\ f\left(\bar{d}\left(((m+n)!(X_{mn}^{k\ell} - X_{mn}))^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } k, \ell > \max(s_0, s_1) \\ \text{and } m, n > \max(n_0, n_1). &\text{ This implies that } X \in \chi_{f_p}^{2F^R}. \end{aligned}$$

3.2. Proposition. The class of sequences Λ_f^{2F} is symmetric but the classes of sequences χ_f^{2F} and $\chi_f^{2F^R}$ are not symmetric.

Proof: Obviously the class of sequences Λ_f^{2F} is symmetric. For the other classes of sequences consider the following example

Example: Consider the class of sequences χ_f^{2F} . Let $f(X) = X$ and consider the sequence $\langle X_{mn} \rangle$ be defined by

$$X_{1n}(t) = \begin{cases} \frac{(-t+1)^{1+n}}{(1+n)!}, & \text{for } t = -1, \\ \frac{(t-1)^{1+n}}{(1+n)!}, & \text{for } t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

and for $m > 1$,

$$X_{mn}(t) = \begin{cases} \frac{(t+2)^{m+n}}{(m+n)!}, & \text{for } t = -2, \\ \frac{(-t-1)^{m+n}}{(m+n)!}, & \text{for } t = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\langle Y_{mn} \rangle$ be a rearrangement of $\langle X_{mn} \rangle$ defined by

$$Y_{nn}(t) = \begin{cases} \frac{(-t+1)^{2n}}{(2n)!}, & \text{for } t = -1, \\ \frac{(t-1)^{2n}}{(2n)!}, & \text{for } t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

and for $m \neq n$,

$$Y_{mn}(t) = \begin{cases} \frac{(t+2)^{m+n}}{(m+n)!}, & \text{for } t = -2, \\ \frac{(-t-1)^{m+n}}{(m+n)!}, & \text{for } t = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\langle X_{mn} \rangle \in \chi_f^{2F}$ but $\langle Y_{mn} \rangle \notin \chi_f^{2F}$. Hence, χ_p^{2F} is not symmetric. Similarly other sequence also not symmetric.

3.3. Proposition. The classes of sequences Λ_f^{2F} , χ_f^{2F} and $\chi_f^{2F^R}$ are solid.

Proof: Consider the class of sequences χ_f^{2F} . Let $\langle X_{mn} \rangle$ and $\langle Y_{mn} \rangle \in \chi_f^{2F}$ be such that $\bar{d}\left(((m+n)!Y_{mn})^{1/m+n}, \bar{0}\right) \leq \bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)$. As f is non-decreasing, we have

$$\lim_{mn \rightarrow \infty} f \left(\bar{d} \left(((m+n)! Y_{mn})^{1/m+n}, \bar{0} \right) \right) \leq \lim_{mn \rightarrow \infty} f \left(\bar{d} \left(((m+n)! X_{mn})^{1/m+n}, \bar{0} \right) \right)$$

Hence, the class of sequence χ_f^{2F} is solid. Similarly it can be shown that the other classes of sequences are also solid.

3.4. Proposition. The classes of sequences χ_f^{2F} and χ_f^{2FR} are not monotone and hence not solid.

Proof: The result follows from the following example.

Example: Consider the class of sequences χ_f^{2F} and $f(X) = X$. Let $J = \{(m, n) : m \geq n\} \subseteq N \times N$. Let $\langle X_{mn} \rangle$ be defined by

$$X_{mn}(t) = \begin{cases} \frac{(t+3)^{m+n}}{(m+n)!}, & \text{for } -3 < t \leq -2, \\ \frac{(mt)^{m+n}}{(3m-1)^{m+n}(m+n)!} + \frac{(3m)^{m+n}}{(3m-1)^{m+n}(m+n)!}, & \text{for } -2 \leq t \leq -1 + \frac{1}{m}, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

for all $m, n \in N$.

Then $\langle X_{mn} \rangle \in \chi_f^{2F}$. Let $\langle Y_{mn} \rangle$ be the canonical pre-image of $\langle X_{mn} \rangle_J$ for the subsequence J of $N \times N$. Then

$$Y_{mn} = \begin{cases} X_{mn}, & \text{for } (m, n) \in J, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Then, $\langle Y_{mn} \rangle \notin \chi_f^{2F}$. Hence χ_f^{2F} is not monotone. Similarly, it can be shown that the other classes of sequences are also not monotone. Hence, the classes of sequences χ_f^{2F} and χ_f^{2FR} are not solid.

3.5. Proposition. (i) $\chi_{f_1}^{2F} \cap \chi_{f_2}^{2F} \subseteq \chi_{f_1+f_2}^{2F}$, (ii) $\chi_{f_1}^{2FR} \cap \chi_{f_2}^{2FR} \subseteq \chi_{f_1+f_2}^{2FR}$

Proof: It is easy, so omitted.

3.6. Proposition. Let f and f_1 be two modulus functions, then, (i) $\chi_{f_1}^{2F} \subseteq \chi_{f \circ f_1}^{2F}$

(ii) $\chi_{f_1}^{2FR} \subseteq \chi_{f \circ f_1}^{2FR}$ (iii) $\Lambda_{f_1}^{2F} \subseteq \Lambda_{f \circ f_1}^{2F}$

Proof: We prove the result for the case $\chi_{f_1}^{2F} \subseteq \chi_{f \circ f_1}^{2F}$, the other cases similar. Let $\epsilon > 0$ be given. As f is continuous and non-decreasing, so there exists $\eta > 0$, such that $f(\eta) = \epsilon$. Let $\langle X_{mn} \rangle \in \chi_{f_1}^{2F}$. Then, there exist $m_0, n_0 \in \mathbb{N}$, such that

$$\begin{aligned} f_1 \left(\bar{d} \left(((m+n)! X_{mn})^{1/m+n}, \bar{0} \right) \right) &< \eta, \text{ for all } m \geq m_0, n \geq n_0, \\ \Rightarrow f \circ f_1 \left(\bar{d} \left(((m+n)! X_{mn})^{1/m+n}, \bar{0} \right) \right) &< \epsilon, \text{ for all } m \geq m_0, n \geq n_0. \end{aligned}$$

Hence, $\langle X_{mn} \rangle \in \chi_{f \circ f_1}^{2F}$. Thus, $\chi_{f_1}^{2F} \subseteq \chi_{f \circ f_1}^{2F}$.

3.7. Proposition. (i) $\chi_f^{2F} \subseteq \Lambda_f^{2F}$ (ii) $\chi_f^{2FR} \subseteq \Lambda_f^{2F}$. The inclusion are strict.

Proof: The inclusion (i) $\chi_f^{2F} \subseteq \Lambda_f^{2F}$ (ii) $\chi_f^{2FR} \subseteq \Lambda_f^{2F}$ is obvious. For establishing that the inclusions are proper, consider the following example.

Example: We prove the result for the case $\chi_f^{2F} \subseteq \Lambda_f^{2F}$, the other case similar. Let $f(X) = X$. Let the sequence $\langle X_{mn} \rangle$ be defined by for $m > n$,

$$X_{mn}(t) = \begin{cases} \frac{(mt-m-1)^{m+n}(m-1)^{-(m+n)}}{(m+n)!}, & \text{for } 1 + \frac{1}{m} \leq t \leq 2, \\ \frac{(3-t)^{m+n}}{(m+n)!}, & \text{for } 2 < t \leq 3, \\ 0, & \text{otherwise .} \end{cases}$$

and for $m < n$

$$X_{mn}(t) = \begin{cases} \frac{(mt-1)^{m+n}(m-1)^{-(m+n)}}{(m+n)!}, & \text{for } \frac{1}{m} \leq t \leq 1, \\ \frac{(-t+2)^{m+n}}{(m+n)!}, & \text{for } 1 \leq t \leq 2, \\ 0, & \text{otherwise .} \end{cases}$$

Then, $\langle X_{mn} \rangle \in \Lambda_f^{2F}$ but $\langle X_{mn} \rangle \notin \chi_f^{2F}$.

3.8. Proposition. The classes of sequences $\Lambda_f^{2F}, \chi_f^{2F}$ and χ_f^{2FR} are not convergent free.

Proof: The result follows from the following example.

Example: Consider the classes of sequences χ_f^{2F} . Let $f(X) = X$ and consider the sequence $\langle X_{mn} \rangle$ defined by $((1+n)!X_{1n})^{1/1+n} = \bar{0}$, and for other values,

$$X_{mn}(t) = \begin{cases} \frac{1^{m+n}}{(m+n)!}, & \text{for } 0 \leq t \leq 1, \\ \frac{(-mt)^{m+n}(m+1)^{-(m+n)} + (2m+1)^{m+n}(1+m)^{-(m+n)}}{(m+n)!}, & \text{for } 1 < t \leq 2 + \frac{1}{m}, \\ 0, & \text{otherwise .} \end{cases}$$

Let the sequence $\langle Y_{mn} \rangle$ be defined by $((1+n)!Y_{1n})^{1/1+n} = \bar{0}$, and for other values,

$$Y_{mn}(t) = \begin{cases} \frac{1^{m+n}}{(m+n)!}, & \text{for } 0 \leq t \leq 1, \\ \frac{(m-t)^{m+n}(m-1)^{-(m+n)}}{(m+n)!}, & \text{for } 1 < t \leq m, \\ 0, & \text{otherwise .} \end{cases}$$

Then, $\langle X_{mn} \rangle \in \chi_f^{2F}$ but $\langle Y_{mn} \rangle \notin \chi_f^{2F}$. Hence, the clases of sequences χ_f^{2F} is not convergent free. Similarly, the other spaces are also ot convergent free.

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