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Common-Neighbourhood of a Graph

P. Dundar, A. Aytac and E. Kilic

ABSTRACT: The vulnerability measures on a connected graph which are mostly used and known are based on the Neighbourhood concept. Neighbour-integrity, edge-integrity and accessibility number are some of these measures. In this work we define and examine Common-neighbourhood of a connected graph as a new global connectivity measure. Our measure examines the neighbourhoods of all pairs of vertices of any connected graph. We show that, for connected graphs G_1 and G_2 of same order, if the dominating number of G_1 is bigger than the dominating number of G_2, then the common-neighbourhood of G_1 is less than the common-neighbourhood of G_2. We give some theorems and obtain some results on common-neighbourhood of a graph.

Key Words: Vertex-neighbourhood, connectivity, stability, common- neighbourhood.

Contents

1. Introduction

The stability and reliability of a network are of prime importance to network designers. The vulnerability value of a communication network shows the resistance of the network after the disruption of some centres or connection lines until the communication breakdown. As the network begins losing connection lines or centres, eventually, there is a loss its effectiveness. If the communication network is modelled as a simple, undirected, connected and unweighted graph G, deterministic measures tend to provide a worst-case analysis of some expects of the overall disconnection process.

A graph G is denoted by $G = (V\ , \mathrm{E})$, where V and E are vertex and edge sets of G, respectively. n denotes the number of vertices and m denotes the number of edges of the graph G. The reliability of a graph can be measured by various parameters. The best known reliability measure of a graph is connectivity, defined as the minimum number of vertices whose deletion results in a disconnected or

2000 *Mathematics Subject Classification:* 05C40, 05C69, 05C85, 05C12, 68M10 trivial graph. $k(G)$ denotes the connectivity of graph G. This parameter has been extensively studied.

Let $G = (V, E)$ be a graph and v a vertex in G. The *open neighborhood* of $v \in V$ is $N(v) = \{ u\colon u \in V \mid uv \in E \}$ and the *closed neighborhood* of v is $N[v] = \{ v \} \cup N(v)$. For a set $S \subseteq V$, its open neighborhood $N(S) = \bigcup_{v \in S} N(v)$and its closed neighborhood $N[S] = N(S) \cup S$ [20].

The connectivity is considered as a worst-case measure since it does not always reflect what happens throughout the graph. But other measures such as integrity, toughness, neighbour-integrity give more information about the reliability of a graph [1]- [19]. For example, a tree and a graph obtained by adding an end-vertex to complete graph both have connectivity 1. But other vulnerability values of these graphs differ from each other. Recent interest in the vulnerability and reliability of networks has given rise to the importance of other measures, some of which are more global in nature. In this paper we investigate the common-neighbourhood, a new measure for reliability and stability of a graph.

Other measures have been found to be more useful than the corresponding measures, such as average connectivity, average degree and average distance of a graph [7] in some circumstances. For example, the average distance between vertices in a graph was introduced as a tool in architecture and later turned out to be more valuable than the diameter when analyzing transportation networks.

While the ordinary connectivity is the minimum number of vertices whose removal separates the graph into at least one connected pair of vertices and a isolated vertex, the average connectivity is a measure for the expected number of vertices that have to be removed to separate a randomly chosen pair of vertices.

Let $G = (V, E)$ be a simple graph of order n and let u and v be two distinct vertices of G. Two vertices u and v in a graph G are said to be k-neighbour, if there are k distinct vertices which are neighbours of both u and v. The k-neighbour of u and v vertices of G is denoted by $N(u, v)$.

The common-neighbourhood is a measure for vulnerability and reliability. The common-neighbourhood gives the expected number of vertices to constitute transitive neighbourhood between a randomly chosen pair of vertices which are non-adjacent. Although other global measures of reliability, such as the toughness and integrity of a graph, are NP hard, the common-neighbourhood can be computed in polynomial time, this makes it much more attractive for applications.

If the order of G is n, then the common-neighbourhood of G is denoted by $\bar{N}(G)$, is defined to be

$$\bar{N}(G) = \frac{\sum_{u,v \in V(G)} N(u,v)}{n-1} \textit{for } \mathrm{n} > 3$$

where $\sum_{u,v \in V(G)} N(u,v)$ is equal to the number of paths of length 2 occurring in the graph G. For any vertex v there exist exactly $\begin{pmatrix} \deg(v) \\ 2 \end{pmatrix}$ such paths, i.e. paths of the form $u_1\ v\ u_2$ with the vertex v in the middle. In order to determine the value $\bar{N}(G)$ one only needs $O(n^2)$ steps since

$$\bar{N}\,(G) = \frac{\sum_{v\in V(G)} \binom{\deg(v)}{2}}{n-1}.$$

We consider the two graphs in Figure 1. The connectivity number of these graphs are 1, but the second graph would be a more reliable communication network than the first one. This is obvious from the common-neighbourhood since $\bar{N}(G_1)$=3/4 and $\bar{N}(G_2)$=15/4.

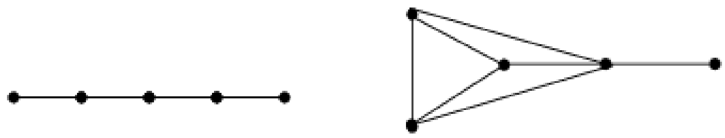

Figure 1: P_5=G_1 G_2

The two graphs in Figure 1 have the same number vertices as well as the same connectivity, but not the same number of edges. The difference in common- neighbourhood is in connection with increased number of edges.
In Figure 2 we show two graphs with the same numbers of vertices and edges, but $\bar{N}(G_1)$= 6/4 while $\bar{N}(G_2)$= 7/4.

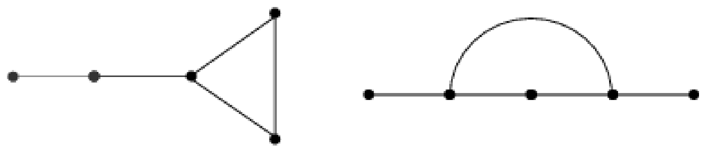

Figure 2: G_1 G_2

Definition 1.1. [20,21] *For a connected graph G, let the nodes of G be labelled as $v_1, v_2, ..., v_p$. The adjacency matrix A=A(G)= $[a_{i,j}]$ of G is the binary matrix of order p*

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \text{ is adjacent with } v_j \\ 0, \text{ otherwise} \end{cases}$$

Definition 1.2. [20,21] *For a connected graph G, we define the distance d(u, v) between two vertices u and v as the minimum of the lengths of the u-v paths in G. Under the distance function, the set V (G) is a metric space.*

The eccentricity $e(v)$ of a vertex v of connected graph G is the number $\max_{u\in V(G)} d(v,u)$. The radius *rad* (G) is defined as $\min_{v\in V(G)} e(v)$ while the diameter *diam* (G) is $\max_{v\in V(G)} e(v)$.
It follows that *diam* $(G) = \max_{v,u\in V(G)} d(v,u)$.

Definition 1.3. [20,21] *An independent set of vertices of a graph G is a set whose elements are pair wise nonadjacent. The independence number $\beta(G)$ of G is the maximum cardinality among all independent sets of vertices of G.*

Definition 1.4. [20,21] *A vertex is said to cover other vertices in a graph G if it is incident to these vertices in G. A cover in G is a set of vertices that covers all edges of G. The minimum cardinality of a cover in a graph G is called the covering number of G is denoted by $\alpha(G)$.*

The order of n in a graph is defined by $\alpha(G) + \beta(G) = n$.

Definition 1.5. [20,21] *A vertex dominating set for a graph G is a set S of vertices such that every vertex of G belongs to S or is adjacent to a vertex of S. The minimum cardinality of a vertex dominating set in a graph G is called the vertex dominating number of G and is denoted by $\sigma(G)$. For every graph G, $\sigma(G) \leq \beta(G)$.*

Lemma 1.6. *Let u and v be two vertices of a connected graph G.*

a) If the distance d(u, v)> 2 then $N(u,v) = 0$ and if d(u, v) = 2 then $N(u,v) \geq 1$.

b) If G is a connected graph, then $0 \leq N(u,v) \leq n-2$.

Lemma 1.7. *For any connected graph with n vertices, $n-2 \leq \sum_{u,v\in V(G)} N(u,v) \leq \frac{n\,(n-1)\,(n-2)}{2}$ for n >2.*

Theorem 1.8. *Let G be a graph of order n≥3.*

a) $\bar{N}(G)=0$ if and only if G is a null graph.

b) $\bar{N}(G) > 0$ if and only if G is a connected graph at least of order 3.

Proof is clear.

Lemma 1.9. *Common–neighbourhood takes its minimum value at P_n path and its maximum value at K_n complete graph. It can be easily seen from Lemma 1.6 b.*

Theorem 1.10. *For a connected graph G, the common-neighbourhood of G is $\frac{1}{2} < \bar{N}(G) \leq \frac{n(n-2)}{2}$*

Proof: From the *Lemma 1.6* and *Theorem 1.8* G must be connected graph and at least the path P_3. Then $\bar{N}(P_3) = \frac{1}{2}$. G can be the complete graph order of n, K_n at most. In complete graph K_n, $N(u,v)=n$-2 for each u, $v\in V(K_n)$. It is obtained from the definition of common- neighbourhood and *Lemma 1.7*

$$\overline{N}\left(K_n\right)=\frac{\binom{n}{2}(n-2)}{n-1}=\frac{n\left(n-2\right)}{2}.$$

Consequently, for any connected graph G, its common-neighbourhood is $\frac{1}{2}<\overline{N}(G)\leq\frac{n(n-2)}{2}$. □

Theorem 1.11. *Let G be connected graph with n vertices which includes $K_{1,n-1}$ as a spanning subgraph then*

$$\frac{\binom{n-1}{2}}{n-1}\leq\bar{N}\left(G\right).$$

Proof: The left side of the inequality can be seen from $K_{1,n-1}$. In $K_{1,n-1}$, the number of the vertices $(u,\ v)$ pairs which have the property $d(u,v)=2$ is $\binom{n-1}{2}$. And from the definition of common-neighbourhood it can be obtained.

□

Theorem 1.12. $\bar{N}(G)\leq\bar{N}(G+e)$

Proof: From the definition of $(G+e)$, to add an e edge between any vertices, these v_i and v_j verticesmust be disjoint. If we add an edge $e=(v_i, v_j)$ to G, then $N(v_i, v_j)$ value increases at least one, for all vertices v of G.
In the definition common-neighbourhood, if $\sum_{u,v\in V(G)}N(u,\ v)$ increases, then the $\bar{N}(G)$increases also. Hence, $\bar{N}(G)\leq\bar{N}(G+e)$ □

Theorem 1.13. *Let G_1 be a graph of order n and P_n be a path graph. If diam (G_1) < diam (P_n), then*

$$\bar{N}(G_1)>\bar{N}(P_n).$$

Proof: Let $G_1$and P_n be two graphs whose orders are the same and *diam* (G_1) $<$ *diam* (P_n), from *Lemma 1.9* it is obvious that P_n has the minimum value of common-neighbourhood. Hence, this shows that the k-neighbourhoods in G_1 are grater than in P_n. By the definition of common-neighbourhood, if the number of neighbourhoods in G_1 is higher, then the number of N $(u,\ v)$ in G_1 will be high. Then

$$\bar{N}(G_1)>\bar{N}(P_n).$$

□

Theorem 1.14. *Let G be connected graph except tree. Then,* $\bar{N}(G) < \frac{\sum_{i=1}^{n} \deg(v_i)}{2(n-1)}$, *for all* v_i *vertices.*

Proof: For the graph G $\max(\sum_{i=1}^{n} \deg(v_i)) = 2m$, where m denotes the number of edges of G. m gets its maximum value $\frac{n(n-1)}{2}$ only in complete graph.

$$\sum N(u,v) = (n-2)\frac{n(n-1)}{2} = \frac{(n-2)}{2}\sum_{i=1}^{n} \deg(v_i).$$

Thus,

$$\bar{N}(G) = \frac{\left(\frac{n-2}{2}\right)\sum_{i=1}^{n} \deg(v_i)}{2(n-1)} = \frac{(n-2)\sum_{i=1}^{n} \deg(v_i)}{2(n-1)}.$$

This value is the maximum value in K_n complete graph. If we remove any $e_i = (u_i, v_i)$ edge from K_n , the neighbourhood values of the vertices u_i and v_i decrease 1. Consequently, for any connected graph G, the value of common-neighbourhood $\bar{N}(G) > \frac{\sum_{i=1}^{n} \deg(v_i)}{2(n-1)}$.
Especially if G is a star then this inequality turns to equality. $\bar{N}(K_{1,n-1}) = \frac{\sum_{i=1}^{n} \deg(v_i)}{2(n-1)}$ □

2. Common-Neighbourhood and Other Measures on Graphs

Certainly other measures provide bounds on the common-neighbourhood of a graph. In this section we give some theorems relating to common-neighbourhood and graph parameters.

Definition 2.1. *For any non-regular graph G,* $\Delta(G)$ *denotes maximum vertex degree and* $\delta(G)$ *denotes minimum vertex degree of the graph G.*

Theorem 2.2. *Let G be tree graph,* $\bar{N}(G) \geq \frac{\beta(G)}{n-1}$.

Proof: From the *Lemma 1.6*, If $d(u, v) = 2$, then $N(u, v) \geq 1$. For all (u,v) pairs of the maximum independent set of G, $d(u,v) \geq 2$. From *Theorem 1.8* if u and v is adjacent then $N(u, v) = 0$.
Then, in the definition $\overline{N}(G) = \frac{\sum_{u,v \in V(G)} N(u,v)}{n-1}$, $\sum_{u,v \in V(G)} N(u,v) \geq \beta(G)$
We divide the both sides of this inequality by n-1, we obtain $\frac{\sum_{u,v \in V(G)} N(u,v)}{n-1} \geq \frac{\beta(G)}{n-1}$. □

Theorem 2.3. *Let* G_1 *and* G_2 *be graphs with n vertices .If* $\sigma(G_1) < \sigma(G_2)$, *then* $\overline{N}(G_1) > \overline{N}(G_2)$.

Proof: Let $S_1$and S_2 be the minimum vertex dominating sets of graphs G_1 and G_2, respectively. If $\sigma(G_1) < \sigma(G_2)$, then $|S_1| < |S_2|$. From the definition the

dominating number, the number of vertices of G_1 dominated by S_1 is bigger than the number of vertices of G_2 dominated by S_2. Then, it can be easily seen that for each vertex u of S_1, $N(u,\, v)$ is more than $N(u^*,\, v)$ for each vertex u^* of S_2. Consequently, $\sum_{u\in S_1} N(u,v) > \sum_{u\in S_2} N(u^*,v)$.
If we divide the both sides of the inequality by n-1, we obtain the following inequality.

$$\frac{\sum_{u\in S_1} N(u,v)}{n-1} > \frac{\sum_{u^*\in S_2} N(u^*,v)}{n-1}$$

and by the definition of common-neighbourhood, $\overline{N}\,(G_1) > \overline{N}\,(G_2)$. □

3. Results on Common-Neighbourhood

In the following results for the common-neighbourhood of a variety of families of graphs can be seen clearly.
1) The path P_n is $\overline{N}(P_n) = \frac{n-2}{n-1} = 1 - \frac{1}{n-1}$
2) The cycle C_n is $\overline{N}(C_n) = \frac{n}{n-1} = 1 + \frac{1}{n-1}$
3) The complete graph K_n is $\overline{N}(K_n) = \frac{(n-2)\binom{n}{2}}{n-1}$
4) The star graph $K_{1,n-1}$ is $\overline{N}(K_{1,n-1}) = \binom{n-1}{2}\frac{1}{n-1}$
5) The bipartite graph $K_{m,n}$ is $\overline{N}(K_{m,n}) = \frac{\frac{1}{2}\cdot\binom{m}{2}n + \frac{1}{2}\binom{n}{2}m}{m+n-1}$
6) The wheel $\mathrm{W}_{1,n}$ is $\overline{N}(W_{1,n-1}) = \frac{\binom{n-1}{2}+2(n-1)}{n-1}$

The results of the common-neighbourhood of the above graphs relating with α, β, Δ and d are given in the following.
1) $\overline{N}(P_n) = \frac{d-1}{n-1}$
2) $\overline{N}(P_n) \leq \frac{\beta+d}{n-1}$
3) $\overline{N}(C_n) = \frac{\alpha+\beta}{n-1}$
4) $\overline{N}(C_n) = \begin{cases} \frac{\Delta d+1}{n-1}, & \text{if } n \text{ is odd} \\ \frac{\Delta d}{n-1}, & \text{if } n \text{ is even} \end{cases}$
5) $\overline{N}(K_{1,n}) = \begin{cases} \frac{\lceil\frac{\beta}{3}\rceil\Delta}{n-1}, & \text{if } n \text{ is odd} \\ \frac{\lceil\frac{\beta}{2}\rceil\Delta-3}{n-1}, & \text{if } n \text{ is even} \end{cases}$

Theorem 3.1. *Let G be $K_{1,n-1}$ graph.* $\bar{N}\,(G) \geq \frac{\Delta(G)-\delta(G)}{2}$.

Proof: If we put this value in the common –neighbourhood definition for $K_{1,n-1}$, we obtain the following equality. $\bar{N}(K_{1,\,n-1}) = \frac{n-2}{2} = \frac{n-1-1}{2} = \frac{\Delta(G)-\delta(G)}{2}$.

And this value is the minimum value that common –neighbourhood can get. To be far away from $K_{1,n-1}$ when we add edges to $K_{1,n-1}$, the value of $\Delta(G)$is still n-1 however $\delta(G)$ increases. From *Theorem 1.12*, $\overline{N}\,(G)$ increases and finally $\bar{N}\,(G) \geq \frac{\Delta(G)-\delta(G)}{2}$ is obtained. □

4. Algorithm for the Common-Neighbourhood Number of a Graph

In this section, we offer an algorithm for the common-neighbourhood number of a graph. The complexity of this algorithm is O(n^2). Data of this algorithm are adjacency matrix and the order of the graph.

A [i, j]: The adjacency matrix of the graph.
CN: Common-Neighbourhood Number of the graph
n: the order of the graph G

```
sumneigh⟵0
For i⟵1 to n do
    degv⟵0
    For j⟵1 to n do
    degv⟵degv+A [i, j]
    fact1⟵1
    fact2⟵1
    For j⟵1 to degv do
    fact1⟵fact1*j
    For j⟵1 to (degv -2) do
    fact2⟵fact2*j
    sumneigh⟵sumneigh + (fact1)/(2-fact2)
Repeat
CN ⟵ sumneigh / (n-1)
END.
```

5. Conclusion

If we want to design a communications network, we wish it to be as stable as possible. Then, we model any communication network by a connected graph. In graph theory, we have many stability measures called as toughness, neighbour-integrity, edge-neighbour integrity. These stability measures also take neighbourhood concept into consideration. But they are not interested in the total neighbourhoods of the graph. In this paper we define a new stability measure .We called it common-neighbourhood of the graph. It takes account the neighbourhoods of all pairs of vertices. We prove that, for two connected graph G_1 and G_2, if the dominating number of G_1 is bigger than the dominating number of G_2 then the common-neighbourhood of G_1 is less than the common-neighbourhood of G_2. Also, we offer an algorithm whose complexity is O (n^2), to find the common-neighbourhood of a graph. In the design of two networks having the same number of processors, if we want to choose the more stable one from these, we take their graph models and it is enough to choose the model whose common-neighbourhood is greater.

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P. Dundar, A. Aytac and E. Kilic
Faculty of Science
Department of Mathematics
Ege University PostalCode 35100
Bornova-IZMIR, Turkey
E-mail address: pinar.dundar@ege.edu.tr
E-mail address: aysun.aytac@ege.edu.tr
E-mail address: elgin.kilic@ege.edu.tr