Bol. Soc. Paran. Mat. (3s.) v. 33 1 (2015): 247–256.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v33i1.22914

On the Fourier Transform of the Products of M-Wright Functions *

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ABSTRACT: In this paper, using the Bromwich's integral of the inverse Mellin transform we find a new integral representation for the M-Wright function

$$M_\alpha(x)=\sum_{k=0}^{\infty}\frac{(-x)^k}{k!\Gamma(-\alpha k+1-\alpha)},\quad \alpha=\frac{1}{2n+1}, n\in\mathbb{N},$$

and state the Fourier transform of this function. Moreover, using the new integral representations for the products of the M-Wright functions, we also get the Fourier transform of it.

Key Words: M-Wright function, Fourier transform, Mellin transform.

Contents

1. Introduction and Preliminaries

The M-Wright function or the Mainardi function [13]

$$M_\alpha(x)=\sum_{k=0}^{\infty}\frac{(-x)^k}{k!\Gamma(-\alpha k+1-\alpha)},\quad 0<\alpha<1, \tag{1.1}$$

with the well-known differential equation for $\alpha=\frac{1}{\nu}$,

$$\frac{d^{\nu-1}}{dx^{\nu-1}}M_{\frac{1}{\nu}}(x)+\frac{(-1)^\nu}{\nu}xM_{\frac{1}{\nu}}(x)=0,\quad \nu=2,3,\cdots, \tag{1.2}$$

plays an important role in fractional calculus. This function is derived from the Wright function

$$W(\alpha,\beta;x)=\sum_{k=0}^{\infty}\frac{x^k}{k!\Gamma(\alpha k+\beta)},\quad \alpha>-1,\ \beta\in\mathbb{C},\ x\in\mathbb{C}, \tag{1.3}$$

* The author would also like to thank the Center of Excellence for Mathematics, Shahrekord university for financial support.
2000 *Mathematics Subject Classification:* 33C47, 43A30

and for the first time was introduced by Mainardi at the beginning of the 1990's for expressing the fractional diffusion-wave equations. Later, other researchers stated some prominent roles of this function in analyzing partial fractional differential equations and statistical distributions. Most of these works have been surveyed using the operational calculus of this function such as the Laplace and Mellin transforms. See for example [3,4,5,6], [9], [11,12] and [15].
Therefore for the importance and significance of this function, in this paper, we get the Fourier transform of the M-Wright and its associate functions. For this purpose, using the Laplace integral we derive an integral representation for the special case $M_{\frac{1}{2n+1}}(x)$. Then, by applying the Bromwich's integral for the inverse Mellin transform we find an extended class of integrals representation for the M-Wright function. Finally, we obtain the Fourier transform of the M-Wright function and the products of M-Wright functions.
First, we consider the following ordinary differential equation of order $2n$ which has $2n$ linear independent solutions in terms of the generalized hypergeometric functions [10]

$$y^{(2n)} - \frac{x}{2n+1} y = 0, \quad x \in \mathbb{R}, n \in \mathbb{N}. \tag{1.4}$$

One of the solutions can be obtained using the Laplace integral method

$$y(x) = \int_C e^{xz} v(z) dz. \tag{1.5}$$

By inserting the relation (1.5) in (1.4), we get a first order differential equation

$$\frac{1}{2n+1} v'(z) + z^{2n} v(z) = 0, \tag{1.6}$$

and find the solution $y(x)$ as the following integral representation (except for a normalization constant)

$$y(x) = \int_C e^{xz - \frac{z^{2n+1}}{(2n+1)^2}} dz. \tag{1.7}$$

The contour C is chosen such that the function $v(z)$ must vanish at boundaries. For this purpose, the real part of z^{2n+1} must be positive and can be considered as paths between two lines of set $\{re^{\frac{2k\pi i}{2n+1}}, r > 0, k = 0, 1, \cdots, 2n\}$. Anyway, after the deformation and normalization of integral (1.7), we rewrite the function y as the $\mathcal{A}_{2n+1}(x)$ function (the generalized airy function) as follows [2]

$$M_{\frac{1}{2n+1}}(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{xz - \frac{z^{2n+1}}{(2n+1)^2}} dz, \tag{1.8}$$

or equivalently

$$\begin{aligned} M_{\frac{1}{2n+1}}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixz + i(-1)^{n+1} \frac{z^{2n+1}}{(2n+1)^2}} dz, \\ &= \frac{1}{\pi} \int_0^{\infty} \cos(xz + (-1)^{n+1} \frac{z^{2n+1}}{(2n+1)^2}) dz. \end{aligned} \tag{1.9}$$

2. Main Theorems

In this section, we establish some theorems and corollaries on the operator $e^{-\lambda\Phi(s)}$. First, we derive an integral representation for the operator $e^{-\lambda\Phi(s)}$.

Theorem 2.1. *Let $\Phi(s)$ be an entire function such that for even and odd functions $u(r)$ and $v(r)$, we have $\phi(c\pm ir) = h(c)(u(r)\pm iv(r))$ where h is an analytic function. Then for $c_1 < \Re(s) < c_2$ the following relation holds true*

$$e^{-\lambda\Phi(s)} = \int_{-\infty}^{\infty} e^{s\xi}\mathcal{A}(\xi,\lambda)\, d\xi, \tag{2.1}$$

where the function $\mathcal{A}(\xi,\lambda)$ is presented by

$$\mathcal{A}(\xi,\lambda) = \frac{1}{\pi}\int_0^{\infty} e^{-\lambda u(r)}\cos(r\xi + \lambda v(r))dr. \tag{2.2}$$

Proof: By definition of the inverse Mellin transform of function $e^{-\lambda\Phi(s)}$ and transition of the contour integration $\Re(s) = c$, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-\lambda\Phi(s)}t^{-s}ds &= \frac{1}{2\pi}\int_0^{\infty} e^{-\lambda\Phi(-ir)}t^{ir}dr + \frac{1}{2\pi}\int_0^{\infty} e^{-\lambda\Phi(ir)}t^{-ir}dr \\
&= \frac{1}{2\pi}\int_0^{\infty} e^{-\lambda(u(r)-iv(r))}t^{ir}dr \\
&\quad + \frac{1}{2\pi}\int_0^{\infty} e^{-\lambda(u(r)+iv(r))}t^{-ir}dr \\
&= \frac{1}{2\pi}\int_0^{\infty} e^{-\lambda u(r)}e^{+i\lambda v(r)+ir\ln(t)}dr \\
&\quad + \frac{1}{2\pi}\int_0^{\infty} e^{-\lambda u(r)}e^{-i\lambda v(r)-ir\ln(t)}dr \\
&= \frac{1}{\pi}\int_0^{\infty} e^{-\lambda u(r)}\cos(r\ln(t) + \lambda v(r))dr.
\end{aligned} \tag{2.3}$$

The relation (2.3) implies that the Mellin transform of the last integral is equal to function $e^{-\lambda\Phi(s)}$, that is

$$\begin{aligned}
e^{-\lambda\Phi(s)} &= \frac{1}{\pi}\mathcal{M}\{\int_0^{\infty} e^{-\lambda u(r)}\cos(r\ln(t) + \lambda v(r))dr; s\} \\
&= \frac{1}{\pi}\int_0^{\infty} t^{s-1}\int_0^{\infty} e^{-\lambda u(r)}\cos(r\ln(t) + \lambda v(r))dr\, dt,
\end{aligned}$$

which by setting $\ln(t) = \xi$, we get the relation (2.1). □

Corollary 2.2. *If we suppose that the function $\Phi(ir)$ can be written as the real function $\Phi_1(ir) = u(r)$ or pure imaginary function $\Phi_2(ir) = iv(r)$, then, the relation (2.1) can be changed into two special cases:*

$$e^{-\lambda\Phi_1(s)} = \int_{-\infty}^{\infty} e^{s\xi}\mathcal{A}_{\mathfrak{R}}(\xi,\lambda)d\xi. \tag{2.4}$$

$$e^{-\lambda\Phi_2(s)} = \int_{-\infty}^{\infty} e^{s\xi}\mathcal{A}_{\mathfrak{I}}(\xi,\lambda)d\xi. \tag{2.5}$$

where the functions $\mathcal{A}_{\mathcal{K}}$ and $\mathcal{A}_{\mathfrak{I}}$ are given by

$$\mathcal{A}_{\mathfrak{R}}(\xi,\lambda) = \frac{1}{\pi}\int_0^{\infty} e^{-\lambda u(r)}\cos(r\xi)dr, \tag{2.6}$$

$$\mathcal{A}_{\mathfrak{I}}(\xi,\lambda) = \frac{1}{\pi}\int_0^{\infty} \cos(r\xi+\lambda v(r))dr. \tag{2.7}$$

Example 2.3. *Setting $\Phi_1(s) = s^{2n+1}$, we have*

$$e^{\frac{s^{2n+1}}{2n+1}} = \int_{-\infty}^{\infty} e^{s\xi}\mathcal{A}_{2n+1}(\xi)d\xi, \tag{2.8}$$

where the $\mathcal{A}_{2n+1}(\xi)$ function is given by

$$\mathcal{A}_{2n+1}(\xi) = \frac{1}{\pi}\int_0^{\infty} \cos(r\xi + (-1)^{n+1}\frac{r^{2n+1}}{2n+1}))dr. \tag{2.9}$$

For more details about this function and its asymptotic expansion at infinity, see [2], [14].

Corollary 2.4. *It is obvious that the relationship between the $\mathcal{A}_{2n+1}(x)$ in (2.9) and $M_{\frac{1}{2n+1}}(x)$ in (1.9) is given by*

$$M_{\frac{1}{2n+1}}(x) = (2n+1)^{\frac{1}{2n+1}}\mathcal{A}_{2n+1}(x(2n+1)^{\frac{1}{2n+1}}). \tag{2.10}$$

Theorem 2.5. *(The Schouten-Vanderpol theorem for the Mellin transform)*
Let $F(s)$ and $\phi(s)$ be analytic functions in the strip $c_1 < \Re(s) < c_2$ and let $F(s)$ be the Mellin transform of $f(t)$, then the inverse Mellin transform of $F(\phi(s))$ is written by

$$g(t) = \mathcal{M}^{-1}\{F(\phi(s));t\} = \int_0^{\infty} \frac{f(\tau)}{\tau}\,[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-\phi(s)\ln(\frac{1}{\tau})+s\ln(\frac{1}{t})}ds]\,d\tau. \tag{2.11}$$

Proof: Using the definition of Mellin transform of $F(\phi(s))$

$$F(\phi(s)) = \int_0^{\infty} \tau^{\phi(s)-1}f(\tau)d\tau,$$

and by replacing in the inverse Mellin transform of $F(\phi(s))$ we have

$$g(t) = \mathcal{M}^{-1}\{F(\phi(s)); t\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(\phi(s))t^{-s}ds.$$

Now, by changing the order of integration, we arrive at the relation (2.11). □

Corollary 2.6. *It is obvious that by setting $\phi(s) = s^{\alpha}$, $0 < \alpha < 1$, in the relation (2.11) and using the following Bromwich's integral in terms of the M-Wright function [1]*

$$\frac{\alpha\lambda}{t^{\alpha+1}} M_{\alpha}(\lambda t^{-\alpha}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-\lambda s^{\alpha}+st}\, ds, \tag{2.12}$$

the inverse Mellin transform of $F(s^{\alpha})$ can be presented in terms of the M-Wright function as follows

$$\mathcal{M}^{-1}\{F(s^{\alpha}); t\} \;=\; \frac{\alpha}{(-\ln(t))^{\alpha+1}}\int_{-\infty}^{\infty} \tau f(e^{-\tau}) M_{\alpha}(\tau(-\ln(t))^{-\alpha})\, d\tau. \tag{2.13}$$

Corollary 2.7. *By setting $F(s) = e^{-\lambda\Phi(s)}$ in the relation (2.1) and combining with (2.13), we get a new integral representation for the fractional exponential operator $e^{-\lambda\Phi(s^{\alpha})}$ as follows*

$$e^{-\lambda\Phi(s^{\alpha})} = \int_{-\infty}^{\infty} e^{s\xi}\mathcal{A}_{\alpha}(\xi,\lambda)\, d\xi, \quad 0 < \alpha < 1, \tag{2.14}$$

where the function $\mathcal{A}_{\alpha}(\xi,\lambda)$ is given by

$$\mathcal{A}_{\alpha}(\xi,\lambda) = -\frac{\alpha}{\pi\xi^{\alpha+1}}\int_{-\infty}^{\infty} \tau\mathcal{A}(\tau,\lambda) M_{\alpha}(\tau(-\xi)^{-\alpha})\, d\tau. \tag{2.15}$$

3. The Fourier Transform of The M-Wright Function

In this section, using the obtained relations in previous section we get the Fourier transform of the M-Wright function and its associate functions. First, we get an integral representation for the products of $\mathcal{A}_{2n+1}(x)$ functions.

Lemma 3.1. *The following identity holds for the product of $\mathcal{A}_{2n+1}(x)$ functions [2]*

$$\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(v) = \frac{2^{-\frac{6n+2}{2n+1}}}{\pi^2}\int_{-\infty}^{\infty} e^{2^{-\frac{1}{2n+1}}(v-u)iz}\mathcal{A}^{*}_{2n+1}(2^{-\frac{1}{2n+1}}(u+v), z)dz, \tag{3.1}$$

where the $\mathcal{A}^{*}_{2n+1}(x,z)$ function in two variables is given by

$$\begin{aligned}\mathcal{A}^{*}_{2n+1}(x,z) \;&=\; \int_{0}^{\infty} \cos((-1)^{n+1}\frac{t^{2n+1}}{2n+1} + t(x+(-1)^{n+1}z^{2n}) \\ &+\; \frac{(-1)^{n+1}}{2(2n+1)}\sum_{j=1}^{n-1}\binom{2n+1}{2j} z^{2j}t^{2n+1-2j})dt.\end{aligned} \tag{3.2}$$

Proof: With the help of the integral representation (1.9), we consider the following product of the $\mathcal{A}_{2n+1}(x)$ functions

$$\mathcal{A}_{2n+1}(2^{-\frac{2n}{2n+1}}(a - k)) \times \mathcal{A}_{2n+1}(2^{-\frac{2n}{2n+1}}(a+k)) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(a-k)x+i(a+k)y+i(-1)^{n+1}\frac{x^{2n+1}+y^{2n+1}}{2n+1}}dxdy. \tag{3.3}$$

Now, using the following change of variables

$$\begin{cases} x = 2^{-\frac{1}{2n+1}}(t-z) \\ y = 2^{-\frac{1}{2n+1}}(t+z) \end{cases} \tag{3.4}$$

the integral (3.3) leads to the inverse Fourier transform of the $\mathcal{A}^*_{2n+1}(x,z)$ function in two variables

$$\mathcal{A}_{2n+1}(2^{-\frac{2n}{2n+1}}(a-k))\mathcal{A}_{2n+1}(2^{-\frac{2n}{2n+1}}(a+k)) = \frac{2^{-\frac{6n+2}{2n+1}}}{\pi^2}\int_{-\infty}^{\infty} e^{ikz}\mathcal{A}^*_{2n+1}(a,z)dz, \tag{3.5}$$

which by setting $2^{-\frac{2n}{2n+1}}(a-k) = u, 2^{-\frac{2n}{2n+1}}(a+k) = v$, the relation (3.1) is derived. □

Corollary 3.2. *From the relation (3.2), we can derive some special cases*

i) $u = v$

$$\mathcal{A}^2_{2n+1}(u) = \frac{2^{-\frac{6n+2}{2n+1}}}{\pi^2}\int_{-\infty}^{\infty}\mathcal{A}^*_{2n+1}(2^{-\frac{1}{2n+1}}(u+v), z)dz, \tag{3.6}$$

ii) $u = -v$

$$\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(-u) = \frac{2^{-\frac{6n+2}{2n+1}}}{\pi^2}\int_{-\infty}^{\infty} e^{2^{\frac{2n}{2n+1}}iuz}\mathcal{A}^*_{2n+1}(0,z)dz, \tag{3.7}$$

iii) $u = v^*$

$$\begin{aligned}\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(u^*) &= |\mathcal{A}_{2n+1}(u)|^2 \\ &= \frac{2^{-\frac{6n+2}{2n+1}}}{\pi^2}\int_{-\infty}^{\infty} e^{2^{\frac{2n}{2n+1}}i\Im(u)z}\mathcal{A}^*_{2n+1}(2^{\frac{2n}{2n+1}}\Re(u), z)dz,\end{aligned} \tag{3.8}$$

iv) $u = 0$

$$\mathcal{A}_{2n+1}(v) = \frac{2^{-\frac{6n+2}{2n+1}}}{\mathcal{A}_{2n+1}(0)\pi^2} \int_{-\infty}^{\infty} e^{2^{-\frac{1}{2n+1}}ivz} \mathcal{A}^*_{2n+1}(2^{-\frac{1}{2n+1}}v, z)dz. \tag{3.9}$$

Lemma 3.3. *The function* $\frac{d^{2n}}{du^{2n}}[\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(-u)]$ *has* $2n$ *vanishing moments as follows [2]*

$$\int_{-\infty}^{\infty} u^k \frac{d^{2n}}{du^{2n}}[\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(-u)]du = 0, \quad k = 0, 1, \cdots, 2n-1. \tag{3.10}$$

Proof: By the identity (3.6), we take $2n$ derivatives of the function $\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(-u)$ as follows

$$\begin{aligned} \frac{d^{2n}}{du^{2n}}[\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(-u)] &= \frac{(-1)^n 2^{\frac{4n^2-6n-2}{2n+1}}}{\pi^2} \int_{-\infty}^{\infty} e^{2^{\frac{2n}{2n+1}}iuz} \mathcal{A}^*_{2n+1}(0, z) z^{2n} dz \\ &= \frac{(-1)^n 2^{-\frac{8n+2}{2n+1}}}{\pi^2} \int_{-\infty}^{\infty} e^{iuw} \mathcal{A}^*_{2n+1}(0, \frac{w}{2^{-\frac{2n}{2n+1}}}) w^{2n} dw. \end{aligned} \tag{3.11}$$

The above relation can be expressed as the Fourier transform of the following function

$$\begin{aligned} \frac{(-1)^n 2^{-\frac{6n+1}{2n+1}}}{\pi} \mathcal{A}^*_{2n+1}(0, \frac{w}{2^{-\frac{2n}{2n+1}}}) w^{2n} &= \int_{-\infty}^{\infty} e^{-iuw} \frac{d^{2n}}{du^{2n}}[\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(-u)]du \\ &= \mathcal{F}\{\frac{d^{2n}}{du^{2n}}[\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(-u)]; w\}, \end{aligned} \tag{3.12}$$

which implies that $2n-1$ first moments of $\frac{d^{2n}}{du^{2n}}[\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(-u)]$ are zero, that is

$$\int_{-\infty}^{\infty} u^k \frac{d^{2n}}{du^{2n}}[\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(-u)]du = 0, \quad k = 0, 1, \cdots, 2n-1. \tag{3.13}$$

□

Theorem 3.4. *The following relation have been held for the Fourier transform of* *the M-Wright function and its associate functions*

$$\begin{aligned}
\mathcal{F}\{\mathcal{A}(\xi,\lambda);w\} &= e^{-\lambda\Phi(-iw)}, && (3.14)\\
\mathcal{F}\{M_{\frac{1}{2n+1}}(\frac{\xi}{(2n+1)^{\frac{1}{2n+1}}});w\} &= (2n+1)^{\frac{1}{2n+1}}e^{\frac{(-iw)^{2n+1}}{2n+1}}, && (3.15)\\
\mathcal{F}\{\mathcal{A}_\alpha(\xi,\lambda);w\} &= e^{-\lambda\Phi((-iw)^\alpha)}, && (3.16)\\
\mathcal{F}\{\mathcal{A}^*_{2n+1}(0,z);2^{\frac{2n}{2n+1}}w\} &= \frac{\pi^2}{(2n+1),^{\frac{2}{2n+1}}2^{-\frac{6n+2}{2n+1}}}M_{\frac{1}{2n+1}}(\frac{w}{(2n+1)^{\frac{1}{2n+1}}})\\
&\times M_{\frac{1}{2n+1}}(-\frac{w}{(2n+1)^{\frac{1}{2n+1}}}). && (3.17)
\end{aligned}$$

Proof: Using the definition of the Fourier transform

$$\mathcal{F}\{f(x);w\} = \int_{-\infty}^{\infty} e^{-iwx}f(x)dx,$$

and applying the relations (2.1), (2.8), (2.14) and (3.6) respectively, the above relations can be easily obtained. □

Theorem 3.5. *The following orthogonality relation holds for the function* $M_{\frac{1}{2n+1}}(\xi)$

$$\int_{-\infty}^{\infty} M_{\frac{1}{2n+1}}(\xi+a)M_{\frac{1}{2n+1}}(\xi+b)d\xi = \delta(b-a), \quad a,b\in\mathbb{R},\ a\neq b, \tag{3.18}$$

where $\delta(x)$ *is the Dirac delta function.*

Proof: Using the definition of the $M_{\frac{1}{2n+1}}(\xi)$ function in (2.8)

$$I = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[(-1)^{n+1}\frac{r^{2n+1}}{(2n+1)^2}+(-1)^{n+1}\frac{r'^{2n+1}}{(2n+1)^2}+ar+br']}drdr'\int_{-\infty}^{\infty} e^{i\xi(r+r')}d\xi, \tag{3.19}$$

and applying the following fact for the Dirac delta function

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(r+r')}d\xi = \delta(r+r'), \tag{3.20}$$

we get the desired result

$$\begin{aligned}
I &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i[(-1)^{n+1}\frac{r^{2n+1}}{(2n+1)^2}+(-1)^{n+1}\frac{(-r)^{2n+1}}{(2n+1)^2}+ar-br]}dr\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(a-b)r}dr\\
&= \delta(b-a). && (3.21)
\end{aligned}$$

□

Corollary 3.6. *Using the relation (3.18) for the orthogonality condition of the $M_{\frac{1}{2n+1}}(\xi)$ function, we can get a new integral transform (which we denote as the $M_{\frac{1}{2n+1}}$-transform) with its inversion formula as*

$$F(y) = \int_{-\infty}^{\infty} f(x) M_{\frac{1}{2n+1}}(y-x)dx, \quad (3.22)$$

$$f(x) = \int_{-\infty}^{\infty} F(y) M_{\frac{1}{2n+1}}(y-x)dy. \quad (3.23)$$

Example 3.7. *For stating a property of the above integral transform, by setting $f(x) = x^m$ in the relation (3.22), we get the well-known Kampé de Fériét polynomials of order m [7]*

$$F(y) = (2n+1)^{\frac{1}{2n+1}} H_m^{2n+1}((2n+1)^{\frac{1}{2n+1}}x, y), \quad (3.24)$$

where

$$H_m^{2n+1}(x,y) = e^{y(\frac{\partial}{\partial x})^{2n+1}}\{x^m\} = m! \sum_{r=0}^{[\frac{m}{2n+1}]} \frac{x^{m-(2n+1)r} y^r}{(m-(2n+1)r)! r!}. \quad (3.25)$$

4. Concluding Remarks

This paper provides new results on the operational calculus of the M-Wright function. Theses results have been obtained using the new integral representations of the exponential function in some special cases. These integral representations led to the Fourier transform of the M-Wright function and its associate functions. Also, we found the Fourier transform of the products of the M-Wright functions according to the relation (3.6) and we saw that this product is orthogonal according to the relation (3.18). Also, we showed that $2n-1$ first moments of the function $\frac{d^{2n}}{du^{2n}}[\mathcal{A}_{2n+1}(u)\mathcal{A}_{2n+1}(-u)]$ are zero.

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