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On limit behavior in space-time

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abstract: This article presents an interpretation of cosmic evolution. Concepts of general topology, geometry, and topological dynamics are used in the construction of a mathematical model for limit behavior in space-time. The β-compactification plays a role in the formation of a transcendent setting which covers the limits of space-time.

Key Words: Transformation group; compactification; limit behavior; galilean space.

Contents

1. Introduction

The concept of cosmological singularity is one of the main questioning in modern science. The expansion of the universe and the evidence for black holes indicate that the singularities in fact exist. But one says that the laws of nature break down into a singularity, since it is a location where the measure of temperature and density of matter become infinite. In other words, the singularities transcend space-time.

The general theory of relativity published by Albert Einstein in 1916 is the current description of gravitation in modern physics. Einstein's theory implies the existence of black holes and gravitational singularities, where the quantities that are used to measure the gravitational field become infinite in such a way that does not depend on the coordinate system. According to general relativity, the initial state of the universe at the beginning of the Big Bang was a singularity ([11]). Another type of singularity would be formed inside a black hole ([5]). On the other hand

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of the history, the ultimate fate of the universe might also be a singularity, the Big Crunch, in which the metric expansion of space eventually reverses and the universe collapses, ultimately ending as a black hole singularity or causing a reformation of the universe starting with another Big Bang ([12], [13], [14]). If the universe is finite in extent, the cosmological principle does not apply, and the expansion speed does not exceed the escape velocity, then the mutual gravitational attraction of all its matter will eventually cause it to contract. Eventually all matter would collapse into black holes, which would then coalesce producing a unified black hole or Big Crunch singularity. The Hubble Constant measures the current state of expansion in the universe, and the strength of the gravitational force depends on the density and pressure of the matter in the universe, that is, the critical density of the universe. By assuming that there is no repulsive force such as a cosmological constant, if the density of the universe is greater than the critical density, then the strength of the gravitational force will stop the universe from expanding and the universe will collapse back on itself. Otherwise, since the nature of the dark energy that is postulated to drive the acceleration is unknown, it is still possible that the expansion of the universe might eventually reverse sign and cause a collapse ([19]).

Nevertheless, some theories such as the theory of loop quantum gravity suggest that singularities may not exist. Based on the quantum gravity effects, there is a minimum distance beyond which the force of gravity no longer continues to increase as the distance between the masses becomes shorter. Stephen Hawking showed that the Big Bang has infinite density. But Hawking later revised his position in [8] where he stated that there was no singularity at the beginning of the universe. This revision followed from quantum mechanics, in which general relativity must break down at times less than the Planck time. Hence general relativity can not be used to show a singularity. By assuming that matter carries out the dominant energy condition, which means that the energy is bigger than the pressure, Roger Penrose has stated that the universe violates the stronger dominant energy condition during inflation, and inflationary cosmologies avoid the initial Big Bang singularity, rounding them out to a smooth beginning. One can extend general relativity to a unified field theory, such as the Einstein-Maxwell-Dirac system, where no such singularities occur.

The evidence for singularities indicates that human has many things to discovery on the universe. The general theory of relativity is not complete without a specification for what happens to matter that hits a singularity. The main question concerns the infinite quantities in which the law of physics break down. However, an asymptotically compact extension for the space-time could be a theoretical hypothesis for an explanation about the infinite quantities and the singularities. The space-time may be bigger than one knows on or may be embedded into a larger space.

There are various mathematical methods of embedding the space-time in a compact space. The Stone–Čech compactification is the greatest process of embedding the space-time as a dense subset of some compact space. The central useful fact about the Stone–Čech compactification is the extension property. Assuming the galilean space-time structure, the action of the four-dimensional euclidean group on the space-time extends to an action on the Stone–Čech compactification of the space-time. By considering the asymptotical behavior on the time direction, the action in the compactification admits only one nontrivial Morse decomposition, which consists of an attractor and a repeller. The limit behavior of the action is completely determined, since all limit sets of events lie in the attractor-repeller pair. This fact means that all events start asymptotically backward at the repeller and terminate asymptotically forward at the attractor.

2. The Stone-Čech compactification

In this section we recall the definition of Stone-Čech compactification. We refer to [7] and [20] for the details of this process of compactification.

Let X be a Tychonoff space, that is, a completely regular and Hausdorff space. Let $C^{*}(X)$ denote the collection of all bounded continuous real-valued functions on X. The range of each $f \in C^{*}(X)$ can be taken as a compact interval I_f in the real line $\mathbb{R}$. By the Tychonoff theorem, the product $\prod_{f\in C^{*}(X)} I_f$ is a compact space. Define the evaluation map $e : X \longrightarrow \prod_{f\in C^{*}(X)} I_f$ by $[e(x)]_f = f(x)$. Since X is Tychonoff, the collection $C^{*}(X)$ separates points from closed sets in X and thus, the evaluation map is an embedding of X into $\prod_{f\in C^{*}(X)} I_f$.

Definition 2.1. *The* Stone-Čech compactification *(or* β-compactification*) of X is the closure βX of $e(X)$ in the product $\prod_{f\in C^{*}(X)} I_f$.*

The Stone-Čech compactification is the greatest compactification of a Tychonoff space because of the extension property, given by the following theorem.

Theorem 2.2. *If K is a compact Hausdorff space and $f : X \to K$ is continuous, there is a continuous $F : \beta X \to K$ such that $F \circ e = f$.*

Note that the evaluation map is an embedding of X into βX. Then X is often written for $e(X)$, so that $X \subset \beta X$, and the above theorem becomes: *every continuous function from X to a compact space K can be extended to βX*. Actually, Theorem 2.2 characterizes the Stone-Čech compactification, up to what is called topological equivalence.

2.1. The filter description of βX

A zero set in X is a set of the form $f^{-1}(0)$ for $f : X \to [0,1]$ continuous. A nonempty collection $\mathcal{F}$ of nonempty zero sets in X is a *z-filter* on X if

1. $Z_1, Z_2 \in \mathcal{F}$ implies $Z_1 \cap Z_2 \in \mathcal{F}$,
2. $Z \in \mathcal{F}$ and Z' is a zero set containing Z, then $Z' \in \mathcal{F}$.

A *z-ultrafilter* is a z-filter which is contained in no strictly z-filter. Let BX be the set of all z-ultrafilters in X. For each zero set $Z \subset X$, define $h(Z) =$

$\{u \in BX : Z \in u\}$. The sets of the form $h(Z)$ can be used as a base for closed sets to obtain a topology on BX. This topology is compact and Hausdorff. For each $x \in X$, let u_x be the unique z-ultrafilter converging to x, that is, the z-ultrafilter containing all the neighborhoods of x. Then $x \in X \longrightarrow u_x \in BX$ is an embedding of X as a dense subset of BX. Moreover, each continuous map f of X into a compact Hausdorff space K can be extended to BX. Hence, BX has the extension property, and therefore BX is topologically equivalent to the Stone-Čech compactification βX. As usual in topology, we may consider $BX = \beta X$.

If X is a T_4 (normal Hausdorff) space, then we can use closed sets instead of zero sets to define the ultrafilters. In this case, βX identifies with the space of all closed ultrafilters in X.

3. Limit behavior of group actions

We now present the basic definitions and results which enable the study of limit behavior of group actions. We introduce the notions of limit sets, attractors, and chain recurrence. We refer to papers [2], [3], [6], [17], and [18] for unexplained dynamical concepts for general semigroup actions.

Definition 3.1. *Let M be a topological space and let G be a topological group with identity e. A* right group action (M, G) *is defined by a jointly continuous in each variable separately map $\sigma : M \times G \to M$: $(x, g) \to xg$, satisfying (i) $xe = x$, and (ii) $x(gh) = (xg)h$, for all $x \in M$ and $g, h \in G$.*

Let (M, G) be a fixed group action. For given sets $X \subset M$ and $S \subset G$, we define the set $XS = \{xg : x \in X, g \in S\}$. A set $X \subset M$ is said to be S-*invariant* if $XS \subset X$.

Definition 3.2. *The subsemigroup $S \subset G$ is said to be* centric *if $sS = Ss$ for every $s \in S$.*

If the semigroup S is centric and generates G, then $G = S^{-1}S$, which is well-known as *Ore's conditions* (see [4]). In this case, a subset $X \subset M$ is G-invariant if and only if it is S-invariant and S^{-1}-invariant. Note that any subsemigroup of G is centric if G is abelian.

From now on, there is a fixed generating centric subsemigroup $S \subset G$.

Definition 3.3. *The following relation in S is defined:*

$$\text{for } t, s \in S, \text{ let } t \geqslant s \text{ if and only if } t = s \text{ or } t \in Ss.$$

The relation $\geqslant$ is the reverse of the well-known Green's $\mathcal{L}$-preorder of semigroup theory: $t \leq_{\mathcal{L}} s$ if and only if $t = s$ or $t \in Ss$ ([4]). Since S is centric, the preorder $\geqslant$ is directed, because the family of translates $\mathcal{F} = \{St : t \in S\}$ is a filter basis on the subsets of S. We consider the limit behavior of (M, G) in this direction, which means that $\mathcal{F}$ is used to define the dynamical concepts, as follows.

Definition 3.4. *The* ω-limit set *of a nonempty subset* $X \subset M$ *is defined as*

$$\omega\left(X\right)=\bigcap_{t\in S}\operatorname{cls}\left(XSt\right),$$

and the ω^{*}-limit set *(or* α-limit set*) of* X *as*

$$\omega^{*}\left(X\right)=\bigcap_{t\in S}\operatorname{cls}\left(XS^{-1}t^{-1}\right).$$

If M is a compact Hausdorff space, then the limit sets $\omega\left(X\right)$ and $\omega^{*}\left(X\right)$ are nonempty, compact, and G-invariant (see [2, Propositions 2.10, 2.12]).

Definition 3.5. *An* attractor *for the group action* (M,G) *is a set* $\mathcal{A}$ *which admits a neighborhood* V *such that* $\omega\left(V\right)=\mathcal{A}$*. A* repeller *is a set* $\mathcal{R}$ *that has a neighborhood* U *with* $\omega^{*}\left(U\right)=\mathcal{R}$*. The neighborhoods* V *and* U *are called* attractor neighborhood *of* $\mathcal{A}$ *and* repeller neighborhood *of* $\mathcal{R}$*, respectively. We consider both the empty set and the whole set* M *as trivial attractors and repellers.*

For an attractor $\mathcal{A}$, the *complementary repeller* of $\mathcal{A}$ is define as $\mathcal{A}^{*}=\{x\in M:\omega\left(x\right)\cap\mathcal{A}=\emptyset\}$. The pair $(\mathcal{A},\mathcal{A}^{*})$ is called an *attractor-repeller pair*. The main property of an attractor-repeller pair is stated in the following result ([2, Proposition 3.6]).

Proposition 3.6. *Assume that* M *is a compact space. Let* $\mathcal{A}$ *be an attractor and suppose that* $x\notin\mathcal{A}\cup\mathcal{A}^{*}$*. Then* $\omega^{*}\left(x\right)\subset\mathcal{A}^{*}$ *and* $\omega\left(x\right)\subset\mathcal{A}$*.*

Now we define the concept of chain recurrence. We refer to [16] for the definition and properties of admissible family of open coverings of M.

Definition 3.7. *For* $x,y\in M$*, an open covering* $\mathcal{U}$ *of* M *and* $t\in S$*, we define a* $(\mathcal{U},t)$-chain *from* x *to* y *as a sequence* $x_{0}=x,x_{1},...,x_{n}=y$ *in* M*, elements* $t_{0},...,t_{n-1}\geqslant t$ *and open sets* $U_{0},...,U_{n-1}\in\mathcal{U}$*, such that* $x_{i}t_{i},x_{i+1}\in U_{i}$*, for* $i=0,...,n-1$*.*

Definition 3.8. *Let* $\mathcal{O}$ *be an admissible family of open coverings of* M*. The* Ω-chain limit set *of* X *is defined as*

$$\Omega\left(X\right)=\bigcap_{\mathcal{U}\in\mathcal{O},t\in S}\Omega\left(X,\mathcal{U},t\right),$$

where $\Omega\left(X,\mathcal{U},t\right)=\{y\in M$*: there is a point* $x\in X$ *and a* $\left(\mathcal{U},t\right)$*-chain from* x *to* $y\}$*, and the* Ω^{*}-chain limit set *of* X *is defined as*

$$\Omega^{*}\left(X\right)=\bigcap_{\mathcal{U}\in\mathcal{O},t\in S}\Omega^{*}\left(X,\mathcal{U},t\right),$$

where $\Omega^{*}(X,\mathcal{U},t)=\{y\in M$*: there is a point* $x\in X$ *and a* $\left(\mathcal{U},t\right)$*-chain from* y *to* $x\}$*. A point* $x\in M$ *is* chain recurrent *if* $x\in\Omega\left(x\right)$*. A subset* $Y\subset M$ *is* chain recurrent *if all the points in* Y *are chain recurrent. A subset* $Y\subset M$ *is* chain transitive *if* $Y\subset\Omega\left(x\right)$ *for all* $x\in Y$*.*

Let $\mathfrak{R}$ be the *chain recurrence set*, that is, the set of all chain recurrent points of (M, G). If M is a compact Hausdorff space, the Conley theorem for semigroup actions proved in [2, Theorem 4.1] says that $\mathfrak{R} = \bigcap \{\mathcal{A} \cup \mathcal{A}^* : \mathcal{A} \text{ is an attractor}\}$. If M is a (noncompact) Tychonoff space, there is an admissible family of open coverings of M such that $\mathfrak{R} = \bigcap \{\mathcal{A} \cup \mathcal{A}^* : \mathcal{A} \text{ is a } \beta\text{-attractor}\}$ ([17, Theorem 2]). If $x \in \mathfrak{R}$, then the maximal chain transitive set containing x is the set $M_x = \Omega(x) \cap \Omega^*(x)$. The chain transitivity is an equivalence relation in $\mathfrak{R}$ whose the equivalence classes are the maximal chain transitivity sets. In general, the limit sets $\omega(x)$ and $\omega^*(x)$ are chain transitive for every $x \in M$. Hence, if $y \in \omega(x)$ and $z \in \omega^*(x)$, then $y, z \in \mathfrak{R}$, $\omega(x) \subset M_y$, and $\omega^*(x) \subset M_z$, although M_y may be different from M_z. Maximal chain transitive sets are G-invariant (see [15]).

We now introduce the important concept of isomorphism.

Definition 3.9. *A* homomorphism *of the group action* (M, G) *into the group action* (N, G) *is a continuous map* $\phi : M \to N$ *such that* $\phi(xg) = \phi(x) g$ *for all* $x \in M$ *and* $g \in G$. *If a homomorphism* ϕ *is also a homeomorphism of* M *onto* N, *then it is an* isomorphism *of* (M, G) *onto* (N, G).

The limits sets, the attractor-repeller pairs, and the maximal chain transitive sets are dynamical invariant, that is, they are invariant under isomorphisms.

3.1. The β-compactification of groups

We now recall the main result of [15] on limit behavior in the β-compactification of a topological group.

Let G be a noncompact T_4 topological group with identity e. Then the β-compactification βG is described as the set of all closed ultrafilters on G. For each $g \in G$, we have the ultrafilter $u_g = \{A \subset G : g \in A\}$. The mapping $g \in G \longrightarrow u_g \in \beta G$ is an embedding of G as a dense subset of βG. Thus, we may consider $G \subset \beta G$. The group G acts on the right on βG with the mapping $\sigma : \beta G \times G \to \beta G$: $(u, g) \to ug$, where $ug = \{Ag : A \in u\}$.

Let $S \subsetneq G$ be a generating centric subsemigroup and assume that it is closed and has nonempty interior in G. Consider the direction as stated in Definition 3.3.

Definition 3.10. *A subsemigroup* H *of* G *is called* total *if* $H \cup H^{-1} = G$; *it is called* semitotal *if there is an element* $h \in H$ *such that* $h^{-1}H \cup hH^{-1} = G$.

Note that a total subsemigroup is semitotal. The main theorem in [15, Theorem 3.9] relates an algebraic property of S to a dynamical property of S, and describes completely the limit behavior of $(\beta G, G)$, as follows.

Theorem 3.11. *The semigroup* S *is semitotal if and only if* $\omega(e)$ *and* $\omega^*(e)$ *are the maximal chain transitive sets in* βG. *Equivalently,* S *is semitotal if and only if* $(\omega(e), \omega^*(e))$ *is the only nontrivial attractor-repeller pair in* βG.

In particular, if S is total, then $(\omega(e), \omega^*(e))$ is the only nontrivial attractor-repeller pair in βG. Since G is an invariant dense set in βG, it follows that $G \cap (\omega(e) \cup \omega^*(e)) = \varnothing$, that is, there is no chain recurrent point in G.

4. Compactification of space-time

In this section we present the main result of the paper. By assuming the galilean space-time structure, we consider the extension of the four-dimensional euclidean group action to the Stone-Čech compactification of the space-time. We refer to [1] for the formulation of galilean space-time structure.

The special principle of relativity was first explicitly enunciated by Galileo Galilei in 1632 in his *Dialogue Concerning the Two Chief World Systems.* The space is three-dimensional and euclidean, and time is one-dimensional. The laws of nature at all moments of time are the same in all inertial coordinate systems. Every coordinate system in uniform rectilinear motion with respect to an inertial one is itself inertial. This principle is a basic experimental fact that lie at the foundation of mechanics.

Let $\mathbb{A}^n$ denote the affine n-dimensional space. The abelian group $\mathbb{R}^n$ acts on the right on $\mathbb{A}^n$ as the group of parallel displacements:

$$A \to A + u, \quad A \in \mathbb{A}^n, u \in \mathbb{R}^n, A + u \in \mathbb{A}^n.$$

If $A + u = B$, then we denote $u = B - A$. The distance between two points $X, Y \in \mathbb{A}^n$ is defined as $\mathrm{d}(X, Y) = \|X - Y\|$, where $\|\cdot\|$ is a norm in $\mathbb{R}^n$. The affine space with this distance is well-known as the euclidean space $\mathbb{E}^n$.

The *galilean space-time structure* consists of the following three elements:

1. The universe – a four-dimensional affine space $\mathbb{A}^4$. The points of $\mathbb{A}^4$ are called *world points* or *events*. The parallel displacements of the universe $\mathbb{A}^4$ constitute the abelian group $\mathbb{R}^4$.

2. Time – a non-trivial linear functional $t : \mathbb{R}^4 \to \mathbb{R}$ from the group of parallel displacements of the universe to the real "time axis". The *time interval* from the event $A \in \mathbb{A}^4$ to the event $B \in \mathbb{A}^4$ is the number $t(B - A)$. If $t(B - A) = 0$, then the events A and B are called *simultaneous*. Since the kernel of the linear functional t is a three-dimensional linear vector subspace $\mathbb{R}^3$ of the vector space $\mathbb{R}^4$, the set of events simultaneous with a given event forms a three-dimensional affine subspace in $\mathbb{A}^4$. It is called a *space of simultaneous events* $\mathbb{A}^3$.

3. The *distance between simultaneous events* $\mathrm{d}(A, B) = \|A - B\|$, $A, B \in \mathbb{A}^3$, is given by a norm $\|\cdot\|$ in $\mathbb{R}^3$. This distance makes every space of simultaneous events into a three-dimensional euclidean space $\mathbb{E}^3$.

A space $\mathbb{A}^4$ equipped with a galilean space-time structure is called a *galilean space*.

By considering the self-action of $\mathbb{R}^4$, the group actions $(\mathbb{R}^4, \mathbb{R}^4)$ and $(\mathbb{A}^4, \mathbb{R}^4)$ are isomorphic. An isometric isomorphism between $(\mathbb{R}^4, \mathbb{R}^4)$ and $(\mathbb{A}^4, \mathbb{R}^4)$ may be obtained by fixing a world point $O \in \mathbb{A}^4$ and then defining the mapping $\phi : \mathbb{R}^4 \to \mathbb{A}^4$ as $\phi(u) = O + u$. This isomorphism transferees the galilean structure of $\mathbb{A}^4$ to $\mathbb{R}^4$. In particular, the group action $(\mathbb{A}^4, \mathbb{R}^4)$ has the same limit behavior as

$(\mathbb{R}^4, \mathbb{R}^4)$, and the β-commpactification of the galilean space $\mathbb{A}^4$ coincides with the β-compactification of the abelian group $\mathbb{R}^4$.

The limit behavior in the time direction is evidently the interest in the galilean space-time structure. Let $S \subset \mathbb{R}^4$ be the subsemigroup defined as

$$S = \left\{ v \in \mathbb{R}^4 : t(v) \geq 0 \right\}.$$

This semigroup establishes the time direction to the action of $\mathbb{R}^4$. It is easily seen that S is total in $\mathbb{R}^4$. Furthermore, we have $S + u = \left\{ v \in \mathbb{R}^4 : t(v) \geq t(u) \right\}$, for every $u \in \mathbb{R}^4$. Thus, the limit sets of $X \subset \beta\mathbb{R}^4$ are described by

$$\omega(X) = \left\{ \begin{array}{c} x \in \beta\mathbb{R}^4 : \text{there are sequences } (u_n) \text{ in } S \text{ and } (x_n) \text{ in } X \\ \text{such that } t(u_n) \to +\infty \text{ and } x_n u_n \to x \end{array} \right\},$$

$$\omega^*(X) = \left\{ \begin{array}{c} x \in \beta\mathbb{R}^4 : \text{there are sequences } (u_n) \text{ in } S \text{ and } (x_n) \text{ in } X \\ \text{such that } t(u_n) \to -\infty \text{ and } x_n u_n \to x \end{array} \right\}.$$

If 0 is the origin of $\mathbb{R}^4$, then we have the limit sets

$$\omega(0) = \left\{ x \in \beta\mathbb{R}^4 : \text{there is } t(u_n) \to +\infty \text{ with } x_n u_n \to x \right\},$$

$$\omega^*(0) = \left\{ x \in \beta\mathbb{R}^4 : \text{there is } t(u_n) \to -\infty \text{ with } x_n u_n \to x \right\}.$$

By Theorem 3.11, $\omega(0)$ is an attractor, $\omega^*(0)$ is its complementary repeller, and $(\omega(0), \omega^*(0))$ is the only nontrivial attactor-repeller pair in $\beta\mathbb{R}^4$. This fact means that the dynamics in $\beta\mathbb{R}^4$ are totally described by $\omega(0)$ and $\omega^*(0)$, since $\omega(0)$ contains all the ω-limit sets and $\omega^*(0)$ contains all the ω^*-limit sets. Moreover, $\omega(0)$ and $\omega^*(0)$ are the only maximal chain transitive sets, or in other words, they are the only two equivalence class of the chain recurrence relation in $\beta\mathbb{R}^4$.

By fixing a world point $O \in \mathbb{A}^4$, it follows that the limit sets $\omega(O)$ and $\omega^*(O)$ describe the global dynamical behavior in the β-compactification of the galilean space $\mathbb{A}^4$. The trajectories which traverse the space-time are repelled from $\omega^*(O)$ and attracted to $\omega(O)$. As $(\omega(O), \omega^*(O))$ is the only nontrivial attactor-repeller pair in $\beta\mathbb{A}^4$, it does not depend on the world point O. Thus we may denote $\mathrm{A} = \omega^*(O)$ and $\Omega = \omega(O)$. The sets A and Ω represents respectively the beginning times and the end times.

4.1. Proceeding models

Notice that the sets A and Ω are regions of the compactification $\beta\mathbb{A}^4$ outside the space-time, that is, they do not intersect $\mathbb{A}^4$. In other words, the regions A and Ω transcend space-time. Although we can not illustrate these regions which are outside the space-time, we may describe their projections on compact sets of $\mathbb{A}^4$ by using the extension property stated in Theorem 2.2.

Let O be a fixed world point in $\mathbb{A}^4$. Since $\mathbb{A}^4$ is homeomorphic with the space $\mathrm{B}_1(O) \times (0,1)$, where $(0,1) \subset \mathbb{R}$ and $\mathrm{B}_1(O)$ is the open 1-ball centered at O in the space of simultaneous events $\mathbb{A}^3$, then there is a projection $\beta\mathbb{A}^4 \to \mathrm{B}_1[O] \times [0,1]$ of $\beta\mathbb{A}^4$ onto $\mathrm{B}_1[O] \times [0,1]$, where $\mathrm{B}_1[O]$ is the closed 1-ball centered at O. The projections of A and Ω coincide respectively with the slices $\mathrm{B}_1[O] \times \{0\}$ and

$B_1[O] \times \{1\}$ (see Figure 1-(a)). This model comes near to the Penrose model of closed universe dominated by radiation, where the cosmological singularity is a c-boundary with the same spherical topology as the closed universe ([9], [10]).

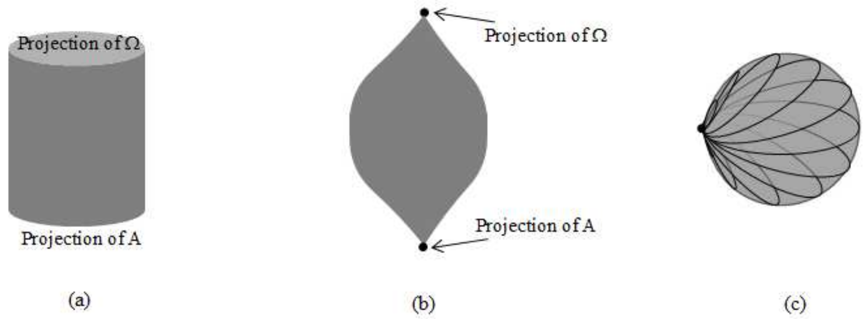

Figure 1: (a) Projection of $\beta\mathbb{A}^4$ onto $B_1[O] \times [0,1]$. The time evolution of the universe occurs inside the cylinder, on the vertical direction, from the projection of the repeller A within the projection of the attractor Ω. (b) Collapse of the projections of A and Ω into the quotient space of $\beta\mathbb{A}^4$ by the chain recurrence relation, forming the suspension $\sum B_1[O]$. The time evolution of the universe occurs inside the suspension, from the initial point within the end point. (c) Identification of A and Ω to a single point. The circles represent the expansion of the universe, which collapses back on itself.

Another way of interpreting A and Ω is based on a quotient map. Since A and Ω are the only distinct chain recurrence classes in $\beta\mathbb{A}^4$, they identify with two distinct single points in the quotient space of $\beta\mathbb{A}^4$ by the chain recurrence relation. The quotient space of $B_1[O] \times [0,1]$ by the induced chain recurrence relation is the suspension $\sum B_1[O]$ obtained by identifying the slice $B_1[O] \times \{0\}$ to a single point, and the slice $B_1[O] \times \{1\}$ to another point (Figure 1-(b)). This model approaches the Big Bang–Big Crunch model of space-time with point singularities.

We may also view A and Ω by means of the projection of $\beta\mathbb{A}^4$ onto the closed 1-ball $B_1[O]$ in $\mathbb{A}^4$. The projections of A and Ω on $B_1[O]$ form the two hemispheres of the sphere $S_1[O]$ with intersection in the equator. Because of this intersection, the quotient space of $B_1[O]$ by the induced chain recurrence relation is obtained by identifying the whole sphere $S_1[O]$ to a single point (Figure 1-(c)). This model suggests that the expansion of the universe reverses and it collapses ultimately ending as a single point or restarting the expansion. One may include the idea of a cyclic universe, in what it could collapse to the state where it began and then initiate another expansion. In this way the universe would last forever, but would pass through phases of expansion (Big Bang) and contraction (Big Crunch). The peculiar aspect of this model is that the collapsed point is a limit point of the space during all the phases of its evolution.

5. Conclusion

The present paper contributes to the formal discussion about the limits of the universe. The Stone-Čech compactification of the galilean space could be used as a mathematical tool of studying the infinite quantities and the singularities in the space-time. The theoretical existence of a universal attractor-repeller pair yields a proceeding model of cosmology that approaches the hypothetical occurrence of the Big Bang and the Big Crunch. This attractor-repeller pair describes the global dynamics in such a way that all events in the space-time begin at the repeller and terminate at the attractor.

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