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The Order of Minimal Realization of Jordan Canonical Form Systems

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ABSTRACT: This paper presents a new method based on controllability and observability of Jordan canonical form systems useful in determining the order of minimal realization. Since any standard system is equivalent to a Jordan canonical form system, this method is applicable to any standard system.

Key Words: Standard systems, Minimal realization, Jordan canonical form.

Contents

1. Introduction

Consider the standard time invariant linear system:

$$\begin{cases} \dot{x}\,(t) = Ax\,(t) + Bu(t) \\ y\,(t) = Cx\,(t) + Du\,(t) \end{cases}, \tag{1.1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$ and $D \in R^{p\times m}$ are the coefficient matrices of the system and $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^p$ are the state, input and output vectors respectively. The dimention of $x(t)$ is called the order of the system. The order of system (1.1) is equal to n.

The matrix $G(s) = C(sI - A)^{-1}B + D$ is called the transfer matrix of system (1.1). The system (1.1) is controllable if and only if $rank\ [B\ \ AB\ \ \ldots\ \ A^{n-1}B] = n$, and is observable if and only if $rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$. For the sake of brevity, we

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show system (1.1) as $[A, B, C, D]$. The system $[A, B, C, D]$ is called a realization for transfer matrix $G(s)$ if $G(s) = C(sI - A)^{-1}B + D$. The realization $[A, B, C, D]$ is minimal if it has the smallest possible order [4, 7]. The realization of $[A, B, C, D]$ is minimal if and only if it is controllable and observable [1, 2, 3, 5, 8].

Two systems $[A, B, C, D]$ and $[\bar{A}, \bar{B}, \bar{C}, \bar{D}]$ are equivalent if they have the same order and the number of inputs and outputs is equal and there exists nonsingular matrices P and Q such that:

$$\bar{A} = QAP, \quad \bar{B} = QB, \quad \bar{C} = CP, \quad \bar{D} = D.$$

Equivalent systems have the same minimal order, since their transfer matrices are the same [2, 3].
Selecting matrices P and Q properly, each standard system $[A, B, C, D]$ will be equivalent to a system $[J, \bar{B}, \bar{C}, \bar{D}]$ in which matrix J is in Jordan canonical form [6].

2. Recognizing the controllability and observability of a system using Jordan canonical form

Consider the system $[J, \bar{B}, \bar{C}, \bar{D}]$ in which J is in the Jordan canonical form and suppose:

$$J = \begin{bmatrix} J_{P_1}(\lambda_1) & & & & & \\ & \ddots & & & & \\ & & J_{P_\alpha}(\lambda_1) & & & \\ & & & \lambda_{k+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} B_{P_1} \\ \vdots \\ B_{P_\alpha} \\ B_{k+1} \\ \vdots \\ B_n \end{bmatrix}, \tag{2.1}$$

$$\bar{C} = \begin{bmatrix} C_{P_1} & \cdots & C_{P_\alpha} & C_{k+1} & \cdots & C_n \end{bmatrix},$$

where eigenvalue λ_1 has multiplicity k and other eigenvalues $\lambda_{k+1}, ..., \lambda_n$ are distinct. The matrices $B_{P_1}, ..., B_{P_\alpha}, B_{k+1}, ..., B_n$ and matrices $C_{P_1}, ..., C_{P_\alpha}, C_{k+1}, ..., C_n$ are classified corresponding to Jordan blocks in J. The following theorems are associated with the controllability and observability of system $[J, \bar{B}, \bar{C}, \bar{D}]$.

Theorem 2.1. *System (2.1) is controllable if and only if*
1. The last rows of submatrices $\bar{B}$ corresponding to Jordan blocks which correspond to the same eigenvalues are linearly independent.
2. If a frequent eigenvalue has only one Jordan block, the last row of its corresponding submatrix in $\bar{B}$ is nonzero.
3. The rows of $\bar{B}$ corresponding to distinct eigenvalues are nonzero.

Proof: See [2] and [3]. □

Theorem 2.2. *System (2.1) is observable if and only if*
1. The first columns of submatrices $\bar{C}$ corresponding to Jordan blocks which correspond to the same eigenvalues are linearly independent.
2. If a frequent eigenvalue has only one Jordan block, the first column of the corresponding submatrix in $\bar{C}$, is a nonzero vector.
3. The columns in $\bar{C}$ corresponding to distinct eigenvalues are nonzero.

Proof: See [2] and [3]. □

Consider system (2.1), the linearly dependent number of uncontrollability (unobservability), for two Jordan blocks with the same eigenvalues in which the last rows (the first columns) of submatrices in $\bar{B}$ ($\bar{C}$) corresponding to the two Jordan blocks are linearly dependent is defined as follows:

Definition 2.1. *The number of linearly dependent consecutive rows with the last rows of submatrices $\bar{B}$, corresponding to Jordan blocks having the same eigenvalues is called linearly dependent number of uncontrollability in that block.*

Definition 2.2. *The number of linearly dependent consecutive columns with the columns of submatrices $\bar{C}$, corresponding to Jordan blocks having the same eigenvalues is called linearly dependent number of unobservability in that block.*

3. Determining the order of minimal realization of standard systems

In this section, the system is considered in the form of $[J, \bar{B}, \bar{C}, \bar{D}]$ in which J is in Jordan canonical form. The following two states are considered:

I) The Jordan matrix J includes Jordan blocks with distinct eigenvalues

If the system $[J, \bar{B}, \bar{C}, \bar{D}]$ satisfies the conditions of theorems 2.1 and 2.2, the dimension of matrix J is equal to the order of minimal realization. Otherwise steps 1 and 2 are applied in order to determine the order of minimal realization:

Step 1 (elimination of uncontrollable factors in system $[J, \bar{B}, \bar{C}, \bar{D}]$)
Considering the last rows of submatrices in $\bar{B}$ corresponding to Jordan block for each eigenvalue. If they are equal to zero, then the rows in submatrices $\bar{B}$ are eliminated together with their corresponding rows and columns in matrix J and their corresponding columns in submatrices $\bar{C}$. The process is continued as long as the last rows of submatrices in $\bar{B}$ corresponding to Jordan block of that eigenvalue are nonzero. This process is repeated for all the eigenvalues. Finally the resultant matrices from the elimination of the rows in $\bar{B}$ and the rows and columns in J and the columns in $\bar{C}$ are called $\bar{B}_1$, J_1, $\bar{C}_1$ respectively.

Step 2 (elimination of unobservability factors in system $[J_1, \bar{B}_1, \bar{C}_1, \bar{D}]$)
Considering the first columns of submatrices $\bar{C}_1$ which correspond to the jordan block for each eigenvalue. If the columns are zero then the columns in submatrices

$\bar{C}_1$, together with their corresponding rows and columns in matrix J_1 and their corresponding rows in submatrices $\bar{B}_1$ are eliminated. The process is continued as long as the first columns of submatrices in $\bar{C}_1$ corresponding to Jordan block of that eigenvalue are nonzero. The process is repeated for all the eigenvalues. Finally, the resultant matrix from the elimination of the rows and columns in J_1 is called J_2. The dimension of matrix J_2 is termed the order of minimal realization of $[J, \bar{B}, \bar{C}, \bar{D}]$.

II) The Jordan matrix J includes Jordan blocks with repetitive eigenvalues

Consider the case in which at most two blocks of submatrices in $\bar{B}$ ($\bar{C}$) have the last rows (the first columns) which are linearly dependent.

Step 1 (elimination of uncontrollable factors in $[J, \bar{B}, \bar{C}, \bar{D}]$)
If one Jordan block with the size of n_1 has linearly dependent number of uncontrollability and linearly dependent number of unobservability n_1, then the block is eliminated. If one other block with the size of n_2 possesses the same property, then the smaller sized block is eliminated and Stage (I) adopted, otherwise the following step is adopted:
First two blocks with the same eigenvalues containing the last rows of submatrices $\bar{B}$ which correspond to the blocks which are linearly dependent, are considered. If at least one of the rows is a zero vector then (1-1) is applied, otherwise (2-1) is applied.

(1-1): The zero row (two rows) in submatrix $\bar{B}$, its corresponding row and column in J and its corresponding column in $\bar{C}$ are eliminated. The process continues as long as the last rows of submatrices $\bar{B}$ corresponding to two Jordan blocks are not zero vectors. (The names of the matrices $\bar{B}$, J, $\bar{C}$ in which elimination operation is performed are not changed). Now, if with the elimination of zero vectors in the submatrices of $\bar{B}$ corresponding to the two Jordan blocks, the last rows of submatrices $\bar{B}$ corresponding to the two Jordan blocks are linearly dependent, then (2-1) is applied, otherwise step 2 is applied.

(2-1): Between two Jordan blocks, the block which has the less linearly dependent number of uncontrollability (If the numbers are equal, the choice is arbitrary) is selected. Being equal to the linearly dependent number of uncontrollability, elimination is made from the last rows of submatrices $\bar{B}$ corresponding to that Jordan block, together with its corresponding rows and columns in the block of Jordan matrix J and the corresponding columns in submatrices $\bar{C}$. The aforementioned process is performed for both the Jordan blocks with the same eigenvalues in which the last rows of submatrices $\bar{B}$ corresponding to the two Jordan blocks are linearly dependent. Stage (I) is performed for Jordan blocks with distinct eigenvalues. Finally, the resultant matrices from the eliminations are called $\bar{B}_1$, J_1, $\bar{C}_1$.

Step 2 (elimination of unobservable factors $[J_1, \bar{B}_1, \bar{C}_1, \bar{D}]$)
Considering the two Jordan blocks with the same eigenvalues containing linearly dependent first columns of submatrices in $\bar{C}_1$ which correspond to the two Jordan blocks, if at least one of the columns is zero then (1-2) is applied, otherwise (2-2) is applied.

(1-2): The column (two columns) in submatrix $\bar{C}_1$, together with its corresponding row and column in J_1 and its corresponding column in $\bar{B}_1$ is eliminated. The process is continued as long as the first columns of submatices in $\bar{C}_1$ corresponding to two Jordan block are not zero vectors. (The names of the matrices $\bar{B}_1$, J_1, $\bar{C}_1$ in which elimination operation is performed, are not changed). Now, if with the elimination of zero vectors in the submatrices of $\bar{C}_1$ corresponding to the two Jordan blocks, the first columns of submatrices $\bar{C}_1$ corresponding to the two Jordan blocks are linearly dependent, then step (2-2) is applied, otherwise the process is finished.

(2-2): Between two Jordan blocks, the block containing the less linearly dependent number of unobservability is selected (If the numbers are equal, the choice is arbitrary). Being equal to the linearly dependent number of unobservability, elimination is made from the first column of submatrix $\bar{C}_1$ corresponding to that of Jordan block, together with its corresponding rows and columns in J_1 and the corresponding rows in submatrices $\bar{B}_1$. The aforementioned process is performed for both Jordan blocks containing the same eigenvalues, in which the first columns of submatrices $\bar{C}_1$ corresponding to the two Jordan blocks are linearly dependent. For Jordan blocks with distinct eigenvalues, step 2 of (I) is applied. Finally, the resultant matrix from the elimination of rows and columns in J_1 is called J_2. The dimension of matrix J_2 is the order of minimal realization of the system $[J, \bar{B}, \bar{C}, \bar{D}]$.

Example 3.1. *We consider system* $[J, \bar{B}, \bar{C}, \bar{D}]$ *in which*

$$J = \begin{bmatrix} 2 & 1 & 0 & & & \\ 0 & 2 & 1 & & & \\ 0 & 0 & 2 & & & \\ & & & 2 & 1 & 0 \\ & & & 0 & 2 & 1 \\ & & & 0 & 0 & 2 \end{bmatrix}, \quad \bar{B} = \begin{array}{cc} \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 0 & 0 \\ \cdots & \cdots \\ 2 & 7 \\ 1 & 5 \\ 3 & 6 \end{bmatrix} & \begin{array}{c} \bar{B}_1 \\ \\ \\ \\ \bar{B}_2 \end{array} \end{array},$$

$$\bar{C} = \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 3 & 4 \\ 6 & 12 & 5 & \vdots & 6 & 3 & 8 \end{bmatrix} \\ \bar{C}_1 \qquad\qquad\qquad \bar{C}_2 \end{array}.$$

Eliminating zero vector in submatrix $\bar{B}_1$*, its corresponding row and column in* J *and its corresponding column in* C_1 *leads to:*

$$J = \begin{bmatrix} 2 & 1 & & & \\ 0 & 2 & & & \\ & & 2 & 1 & 0 \\ & & 0 & 2 & 1 \\ & & 0 & 0 & 2 \end{bmatrix}, \quad \begin{array}{c} \\ \bar{B} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ \cdots & \cdots \\ 2 & 7 \\ 1 & 5 \\ 3 & 6 \end{bmatrix}, \end{array} \begin{array}{c} \bar{B}_1 \\ \\ \\ \bar{B}_2 \end{array}$$

$$\begin{array}{c} \bar{C} = \begin{bmatrix} 1 & 2 & \vdots & 1 & 3 & 4 \\ 6 & 12 & \vdots & 6 & 3 & 8 \end{bmatrix}. \\ \quad\quad \bar{C}_1 \quad\quad\quad\quad \bar{C}_2 \end{array}$$

Linearly dependent number of uncontrollability of $\bar{B}_2$ *is less than* $\bar{B}_1$*, consequently the last row of submatrix* $\bar{B}_2$ *is eliminated, also its corresponding row and column in* J *and its corresponding column in* $\bar{C}$ *which leads to:*

$$J = \begin{bmatrix} 2 & 1 & & \\ 0 & 2 & & \\ & & 2 & 1 \\ & & 0 & 2 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ \cdots & \cdots \\ 2 & 7 \\ 1 & 5 \end{bmatrix}, \begin{array}{c} \bar{B}_1 \\ \\ \\ \bar{B}_2 \end{array}$$

$$\begin{array}{c} \bar{C} = \begin{bmatrix} 1 & 2 & \vdots & 1 & 3 \\ 6 & 12 & \vdots & 6 & 3 \end{bmatrix}. \\ \quad\quad \bar{C}_1 \quad\quad\quad \bar{C}_2 \end{array}$$

The linear dependent number of unobservability of $\bar{C}_2$ *is equal to 1 and the linear dependent number of unobservability of* $\bar{C}_1$ *is equal to 2, consequently the first column of* $\bar{C}_2$ *and its corresponding row and column in* J_1 *and its corresponding row in* $\bar{B}_1$ *are eliminated:*

$$J = \begin{bmatrix} 2 & 1 & \\ 0 & 2 & \\ & & 2 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ \cdots & \cdots \\ 1 & 5 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 2 & \vdots & 3 \\ 6 & 12 & \vdots & 3 \end{bmatrix}.$$

The aforementioned system is of the 3^{rd} *order and, the minimal realization of the first system is also of the* 3^{rd} *order.*

4. Algorithm

Algorithm: Determining the order of minimal realization of standard systems using Jordan canonical form

Input: A standard system $[A, B, C, D]$.
Output: The order of minimal realization of $[A, B, C, D]$.
Step 1: Converting the system $[A, B, C, D]$ to the system$[J, \bar{B}, \bar{C}, \bar{D}]$ in which J is in Jordan canonical form.
Step 2: Elimination of zero vector from the last rows in submatrices $\bar{B}$ and also elimination of the corresponding columns in submatrices $\bar{C}$ and the corresponding rows and columns in J.
Step 3: Elimination of zero vector from first columns in submatrices $\bar{C}$ and also elimination of the corresponding rows in submatrices $\bar{B}$ and the corresponding rows and columns in J.
Step 4: If the resultant matrix J from previous step lacks Jordan blocks with the same eigenvalues, then the dimension of J is the order of minimal realization, otherwise step 5 is applied.
Step 5: If one Jordan block with the dimension of n_1 has linearly dependent number of uncontrollability and linearly dependent number of unobservability n_1, the block is eliminated. (If there is another block with the dimension of n_2 which has similar property, the block with fewer dimension is eliminated).
Step 6: If the last rows of submatrices $\bar{B}$ corresponding to Jordan blocks with the same eigenvalues are linearly independent then step 8 is applied, otherwise step 7 is applied.
Step 7: Between two Jordan blocks containing the same eigenvalues, the block which has the less linear dependent number of uncontrollability is selected and equal to the number, the rows of $\bar{B}$ from the last row in $\bar{B}$ corresponding to that Jordan block and the corresponding columns in $\bar{C}$ and the corresponding rows and columns in J are eliminated.
Step 8: If the first column of submatrix $\bar{C}$ corresponding to Jordan blocks with the same eigenvalues are linearly dependent, then the dimension of J is the order of minimal realization, otherwise step 9 is applied.
Step 9: Between two Jordan blocks containing the same eigenvalues, the block which has the less linear dependent number of unobservability is selected and equal to the number, the columns of $\bar{C}$ from the first column in $\bar{C}$ corresponding to that Jordan block and the corresponding rows in $\bar{B}$ and the corresponding rows and columns in J are eliminated.

The dimension of matrix J which resulted from these steps is the order of minimal realization of the initial system.

5. Conclusion

This paper, by applying a new method and defining two new concepts, has demonstrated that with the elimination of unobservable and uncontrollable factors of a system in which the state matrix is in Jordan form, the system order can be decreased and this decrease can continue in order to remove all unobservable and uncontrollable components and the minimal order of the system can be determined. Since each system is equivalent to a system in which the state matrix is in Jordan form, then this method is efficient to achieve the minimal order of each standard system.

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