Existence and multiplicity results for elliptic problems with Nonlinear Boundary Conditions and variable exponents

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ABSTRACT: By applying the Ricceri's three critical points theorem, we show the existence of at least three solutions to the following elleptic problem:

$$-div[a(x, \nabla u)] + |u|^{p(x)-2}u = \lambda f(x, u), \quad \text{in } \Omega,$$
$$a(x, \nabla u).\nu = \mu g(x, u), \quad \text{on } \partial\Omega,$$

where λ, $\mu \in \mathbb{R}^+$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain of smooth boundary $\partial\Omega$ and ν is the outward normal vector on $\partial\Omega$. $p : \overline{\Omega} \mapsto \mathbb{R}$, $a : \overline{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}^N$, $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ are fulfilling appropriate conditions.

Key Words: Variable exponents; Elliptic problem; Nonlinear boundary conditions;Multiple solutions; Three critical points theorem; Variational methods

Contents

1. Introduction and main result

In this article, we consider the elliptic problem with nonlinear boundary conditions and variable exponents

$$-div[a(x, \nabla u)] + |u|^{p(x)-2}u = \lambda f(x, u), \quad \text{in } \Omega,$$
$$a(x, \nabla u).\nu = \mu g(x, u), \quad \text{on } \partial\Omega, \tag{1.1}$$

where λ, $\mu \in [0, \infty)$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$ and ν is the outward normal vector on $\partial\Omega$. $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ are two Carathéodory functions. $p \in C(\overline{\Omega})$ is the variable exponent.
Throughout this paper, we denote

$$p^- = \min_{x \in \overline{\Omega}} p(x); \ p^+ = \max_{x \in \overline{\Omega}} p(x);$$

$$p^*(x) = \begin{cases} Np(x)/[N - p(x)] & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

$$p^\partial(x) = \begin{cases} (N-1)p(x)/[N-p(x)] & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

and

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : 1 < p^- < p^+ < \infty\}.$$

Our variable exponent p fulfills $p \in C_+(\overline{\Omega})$ and for this p we introduce a characterization of the Carathéodory function $a : \overline{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}^N$.

(H_0) $a(x,-s) = -a(x,s)$ for a.e. $x \in \overline{\Omega}$ and all $s \in \mathbb{R}^N$.

(H_1) There exists a Carathéodory function $A : \overline{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}$ continuously differentiable with respect to its second argument, such that $a(x,s) = \nabla_s A(x,s)$ all $s \in \mathbb{R}^N$ and a.e. $x \in \overline{\Omega}$.

(H_2) $A(x,0) = 0$ for a.e. $x \in \overline{\Omega}$.

(H_3) There exists $c > 0$ such that the function a satisfies the growth condition $|a(x,s)| \leq c(1 + |s|^{p(x)-1})$ for a.e. $x \in \overline{\Omega}$ and all $s \in \mathbb{R}^N$, where $|.|$ denotes the Euclidean norm.

(H_4) The monotonicity condition $0 \leq [a(x,s_1) - a(x,s_2)](s_1 - s_2)$ holds for a.e. $x \in \overline{\Omega}$ and all $s_1, s_2 \in \mathbb{R}^N$. With equality if and only if $s_1 = s_2$.

(H_5) The inequalities $|s|^{p(x)} \leq a(x,s)s \leq p(x)A(x,s)$ hold for a.e. $x \in \overline{\Omega}$ and all $s \in \mathbb{R}^N$.

A first remark is that hypothesis (H_0) is only needed to obtain the multiplicity of solutions. As in [6], we have decided to use this kind of function a satisfying (H_0)–(H_5) because we want to assure a high degree of generality to our work. Here we invoke the fact that, with appropriate choices of a, we can obtain many types of operators. We give, in the following, two examples of well known operators which are present in lots of papers.

Examples:

1. If $a(x,s) = |s|^{p(x)-2}s$, we have $A(x,s) = \frac{1}{p(x)}|s|^{p(x)}$.
   (H_0)–(H_5) are verified, and we arrive to the $p(x)$-Laplace operator
   $div(a(x,\nabla u)) = div(|\nabla u|^{p(x)-2}\nabla u) = \triangle_{p(x)}u.$

2. If $a(x,s) = (1 + |s|^2)^{(p(x)-2)/2}s$, we have $A(x,s) = \frac{1}{p(x)}[(1 + |s|^2)^{p(x)/^2} - 1]$.
   (H_0)–(H_5) are verified, and we find a generalized mean curvature operator
   $div(a(x,\nabla u)) = div((1 + |\nabla u|^2)^{(p(x)-2)/2}\nabla u).$

The energy functional corresponding to problem (1.1) is defined on $W^{1,p(x)}(\Omega)$ as

$$H(u) = \Phi(u) + \lambda\Psi(u) + \mu J(u), \tag{1.2}$$

where

$$\Phi(u) = \int_\Omega A(x, \nabla u)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx, \tag{1.3}$$

$$\Psi(u) = -\int_\Omega F(x, u)dx, \tag{1.4}$$

$$J(u) = -\int_{\partial\Omega} G(x, u)d\sigma, \tag{1.5}$$

where $F(x, u) = \int_0^u f(x, s)ds$, $G(x, u) = \int_0^u g(x, s)ds$, and $d\sigma$ is the $N - 1$ dimensional Hausdorff measure. Let us recall that a weak solution of (1.1) is any $u \in W^{1,p(x)}(\Omega)$ such that

$$\int_\Omega a(x, \nabla u)\nabla v dx + \int_\Omega |u|^{p(x)-2}uv dx$$
$$= \lambda \int_\Omega f(x, u)v dx + \mu \int_{\partial\Omega} g(x, u)v d\sigma \quad \text{for all } v \in W^{1,p(x)}(\Omega).$$

The study of differential and partial differential equation with variable exponent has been received considerable attention in recent years. This importance reflects directly into a various range of applications. There are applications concerning elastic materials [25], image restoration [7], thermorheological and electrorheological fluids [2,21] and mathematical biology [10].

Ricceri's three critical points theorem is a powerful tool to study boundary problem of differential equation (see, for example, [1,3,4,5]). Particularly, Mihailescu [17] use three critical points theorem of Ricceri [19] study a particular $p(x)$-Laplacian equation. He proved existence of three solutions for the problem. Liu [16] study the solutions of the general $p(x)$-Laplacian equations with Neumann or Dirichlet boundary condition on a bounded domain, and obtain three solutions under appropriate hypotheses. Shi [22] generalizes the corresponding result of [17]. The multiple solutions of $p(x)$-biharmonic equation under sublinear condition has been studied in [15] by L. Li, L. Ding and W.W. Pan. To our knowledge, there is no result of multiple solutions of elliptic problems with nonlinear boundary conditions and variable exponents.

We enumerate the hypotheses concerning the functions f, F and g.

(I1) For $t \in C(\overline{\Omega})$ and $t(x) < p^*(x)$ for all $x \in \overline{\Omega}$, we have

$$\sup_{(x,s)\in\Omega\times\mathbb{R}} \frac{|f(x,s)|}{1+|s|^{t(x)-1}} < +\infty;$$

(I2) There exist positive constant c_1 such that $F(x, s) > 0$ for a.e. $x \in \Omega$ and all $s \in]0, c_1]$;

(I3) there exist $p_1(x) \in C(\overline{\Omega})$ and $p^+ < p_1^- \leq p_1(x) < p^*(x)$, such that

$$\limsup_{s\to 0} \sup_{x\in\Omega} \frac{F(x,s)}{|s|^{p_1(x)}} < +\infty;$$

(I4) There exist positive constant c_2 and a function $\gamma(x) \in C(\overline{\Omega})$ with $1 < \gamma^- \leq \gamma^+ < p^-$, such that $|F(x,s)| \leq c_2(1 + |s|^{\gamma(x)})$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$;

(I5) For $p_2 \in C(\overline{\Omega})$ and $p_2(x) < p^\partial(x)$ for all $x \in \overline{\Omega}$, we have

$$\sup_{(x,s)\in\partial\Omega\times\mathbb{R}} \frac{|g(x,s)|}{1 + |s|^{p_2(x)-1}} < +\infty,$$

.

This article is divided into three sections. In Section 2, we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces at first and we recall B. Ricceri's three critical points theorem (Theorem 2.3). In the third section, we prove the following theorem which is the main result of this paper.

Theorem 1.1. *Assume (H_0)–(H_5) and $(I1)$–$(I4)$. Then there exist an open interval $\Lambda \subseteq (0,+\infty)$ and a positive real number ω such that, for each $\lambda \in \Lambda$ and each function $g\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfying $(I5)$, there exists $\delta > 0$ which satisfies, for each $\mu \in [0,\delta]$, the problem (1.1) has at least three weak solutions whose norms in $W^{1,p(x)}(\Omega)$ are less than ω.*

2. Preliminaries

We first recall some basic facts about the variable exponent Lebesgue-Sobolev.
For $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{ u; u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u|^{p(x)}dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{p(x)} := \inf \left\{ \alpha > 0; \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\},$$

which is separable and reflexive Banach space (see [13]).
Let us define the space

$$W^{1,p(x)}(\Omega) := \{ u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega) \},$$

equipped with the norm

$$\|u\| = \inf \left\{ \alpha > 0; \int_\Omega \left(\left| \frac{\nabla u(x)}{\alpha} \right|^{p(x)} + \left| \frac{u(x)}{\alpha} \right|^{p(x)} \right) dx \leq 1 \right\}; \ \forall u \in W^{1,p(x)}(\Omega).$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1. *[8,11,12,13]*

(1) $W_0^{1,p(x)}(\Omega)$ is separable reflexive Banach space;

(2) If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \bar{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous;

(3) If $q \in C_+(\bar{\Omega})$ and $q(x) < p^{\partial}(x)$ for any $x \in \bar{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous;

(4) (Poincaré) There is a constant $C > 0$, such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)} \ \forall u \in W_0^{1,p(x)}(\Omega).$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping ρ defined by

$$\rho(u) := \int_{\Omega} \left[|\nabla u|^{p(x)} + |u|^{p(x)} \right] dx, \ \forall u \in W^{1,p(x)}(\Omega).$$

Proposition 2.2. [9] For $u, u_k \in W^{1,p(x)}(\Omega); k = 1, 2, ...,$ we have

(1) $\|u\| \geq 1$ if and only if $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

(2) $\|u\| \leq 1$ if and only if $\|u\|^{p^-} \geq \rho(u) \geq \|u\|^{p^+}$;

(3) $\|u_k\| \to 0$ as $k \to +\infty$ if and only if $\rho(u_k) \to 0$ as $k \to +\infty$;

(4) $\|u_k\| \to +\infty$ as $k \to +\infty$ if and only if $\rho(u_k) \to +\infty$ as $k \to +\infty$.

In the sequel, we recall the revised form of Ricceri's three critical points theorem [20, Theorem 1] and [18, Proposition 3.1].

Theorem 2.3 ([20, Theorem 1]). *Let X be a reflexive real Banach space. $\Phi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X', where X' is the dual of X, and Φ is bounded on each bounded subset of X; $\Psi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbb{R}$ is an interval. Assume that*

$$\lim_{\|x\|\to+\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty \tag{2.1}$$

for all $\lambda \in I$, and that there exists $h \in \mathbb{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) + h)) < \inf_{x \in X} \sup_{\lambda \in I}(\Phi(x) + \lambda(\Psi(x) + h)). \tag{2.2}$$

Then, there exists an open interval $\Lambda \subseteq I$ and a positive real number ρ with the following property: for every $\lambda \in \Lambda$ and every C^1 functional $J\colon X \mapsto \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the equation

$$\Phi'(x) + \lambda\Psi'(x) + \mu J'(x) = 0$$

has at least three solutions in X whose norms are less than ω.

Proposition 2.4 ([18, Proposition 3.1]). *Let X be a non-empty set and Φ, Ψ two real functions on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = -\Psi(x_0) = 0, \quad \Phi(x_1) > r, \quad \sup_{x \in \Phi^{-1}(]-\infty,r])} -\Psi(x) < r\frac{-\Psi(x_1)}{\Phi(x_1)}.$$

Then, for each h satisfying

$$\sup_{x \in \Phi^{-1}(]-\infty,r])} -\Psi(x) < h < r\frac{-\Psi(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(h + \Psi(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(h + \Psi(x))).$$

In our work, we designate by X the Sobolev space with variable exponent $W^{1,p(x)}(\Omega)$.

3. Proof of main result

The operator Φ is well defined and of class C^1 (see [6]). The Fréchet derivative of Φ is the operator $\Phi' \colon X \to X'$ defined as

$$\langle \Phi'(u), v \rangle = \int_\Omega a(x, \nabla u)\nabla v dx + \int_\Omega |u|^{p(x)-2}uv dx \text{ for any } u, v \in X.$$

We start by proving some properties of the operator Φ'.

Theorem 3.1. *Suppose that the mapping a satisfies (H_0)–(H_5). Then the following statements holds.*

(1) Φ' is continuous and strictly monotone.

(2) Φ' is of (S_+) type.

(3) Φ' is a homeomorphism.

Proof: (1) Since Φ' is the Fréchet derivative of Φ, it follows that Φ' is continuous. Using (H_4) and the elementary inequalities [23]

$$|x - y|^\gamma \leq 2^\gamma(|x|^{\gamma-2}x - |y|^{\gamma-2}y)(x - y) \quad \text{if } \gamma \geq 2,$$

$$|x - y|^2 \leq \frac{1}{(\gamma - 1)}(|x| + |y|)^{2-\gamma}(|x|^{\gamma-2}x - |y|^{\gamma-2}y)(x - y) \quad \text{if } 1 < \gamma < 2,$$

for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, we obtain for all $u, v \in X$ such that $u \neq v$,

$$\langle \Phi'(u) - \Phi'(v), u - v \rangle > 0,$$

which means that Φ' is strictly monotone.

(2) Let $(u_n)_n$ be a sequence of X such that

$$u_n \rightharpoonup u \text{ weakly in } X \text{ as } n \to +\infty \quad \text{and} \quad \limsup_{n \to +\infty} \langle \Phi'(u_n), u_n - u \rangle \leq 0.$$

Using the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, we have

$$\lim_{n \to +\infty} \int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx = 0.$$

Thus

$$\limsup_{n \to +\infty} \int_\Omega a(x, \nabla u_n)(\nabla u_n - \nabla u)dx \leq 0.$$

The following theorem assure that $u_n \to u$ strongly in $W^{1,p(x)}(\Omega)$ as $n \to +\infty$.

Theorem 3.2 ([14], Theorem 4.1). *The Carathéodory function $a \colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ described by (H_0)–(H_5) is an operator of type (S_+) that is, if $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$ as $n \to +\infty$ and*

$$\limsup_{n \to +\infty} \int_\Omega a(x, \nabla u_n)(\nabla u_n - \nabla u)dx \leq 0,$$

then $u_n \to u$ strongly in $W^{1,p(x)}(\Omega)$ as $n \to +\infty$.

(3) Note that the strict monotonicity of Φ' implies its injectivity. Moreover, Φ' is a coercive operator. Indeed, using (H_5), Proposition 2.2 and since $p^- - 1 > 0$, for each $u \in X$ such that $\|u\| \geq 1$ we have

$$\frac{\langle \Phi'(u), u \rangle}{\|u\|} = \frac{\int_\Omega [a(x, \nabla u)\nabla u + |u|^{p(x)}]dx}{\|u\|} \geq \|u\|^{p^- - 1} \to \infty \quad \text{as } \|u\| \to \infty.$$

Consequently, the operator Φ' is a surjection and admits an inverse mapping. It suffices then to show the continuity of Φ'^{-1}. Let $(f_n)_n$ be a sequence of X' such that $f_n \to f$ in X' as $n \to +\infty$. Let u_n and u in X such that

$$\Phi'^{-1}(f_n) = u_n \quad \text{and} \quad \Phi'^{-1}(f) = u.$$

By the coercivity of Φ', one deducts that the sequence (u_n) is bounded in the reflexive space X. For a subsequence, we have $u_n \rightharpoonup \widehat{u}$ weakly in X as $n \to +\infty$, which implies

$$\lim_{n \to +\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - \widehat{u} \rangle = \lim_{n \to +\infty} \langle f_n - f, u_n - \widehat{u} \rangle = 0.$$

It follows by the property (S_+) and the continuity of Φ' that

$$u_n \to \widehat{u} \quad \text{strongly in } X \quad \text{and} \quad \Phi'(u_n) \to \Phi'(\widehat{u}) = \Phi'(u) \quad \text{in } X' \text{ as } n \to +\infty.$$

Moreover, since Φ' is an injection, we conclude that $u = \widehat{u}$. $\square$

Now we can give the proof of our main result.

Proof: [Proof of Theorem 1.1] Set $\Phi(u)$, $\Psi(u)$ and $J(u)$ as (1.3), (1.4) and (1.5). So, for each u, $v \in X$, one has

$$\langle \Phi'(u), v \rangle = \int_\Omega [a(x, \nabla u)\nabla v + |u|^{p(x)-2}uv]dx,$$

$$\langle \Psi'(u), v \rangle = -\int_\Omega f(x, u)v \, dx,$$

$$\langle J'(u), v \rangle = -\int_{\partial\Omega} g(x, u)v \, d\sigma.$$

From Theorem 3.1 and [[6], Proposition 4] the functional Φ is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X'. By (I1) and (I5), Ψ and J are continuously Gâteaux differentiable functionals. Moreover, using the compacity of embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ and the trace embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\partial\Omega)$ (Proposition 2.1), we deduce that Ψ' and J' are compact. Obviously, Φ is bounded on each bounded subset of X under our assumptions.
From (H_5), if $\|u\| \geq 1$ then

$$\begin{aligned}
\Phi(u) &= \int_\Omega A(x, \nabla u)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx \\
&\geq \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx \\
&\geq \frac{1}{p^+}\rho(u) \\
&\geq \frac{1}{p^+}\|u\|^{p^-},
\end{aligned}$$

Meanwhile, for each $\lambda \in \Lambda$,

$$\begin{aligned}
\lambda\Psi(u) &= -\lambda \int_\Omega F(x, u)dx \\
&\geq -\lambda \int_\Omega c_2(1 + |u|^{\gamma(x)})dx \\
&\geq -\lambda c_2(|\Omega| + \|u\|_{\gamma(x)}^{\gamma^+}) \\
&\geq -c_2'(1 + \|u\|_{\gamma(x)}^{\gamma^+}) \\
&\geq -c_2''(1 + \|u\|^{\gamma^+})
\end{aligned}$$

for any $u \in X$, where c_2' and c_2'' are positive constants and $\|.\|_{\gamma(x)}$ is the usual norm of $W^{1,\gamma(x)}(\Omega)$. Combining the two inequalities above, we obtain

$$\Phi(u) + \lambda\Psi(u) \geq \frac{1}{p^+}\|u\|^{p^-} - c_2''(1 + \|u\|^{\gamma^+}),$$

since $\gamma^+ < p^-$, it follows that

$$\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty \quad \forall u \in X, \quad \lambda \in [0, +\infty).$$

Then assumption (2.1) of Theorem 2.3 is satisfied.

Next, we will prove that assumption (2.2) is also satisfied. It suffices to verify the conditions of Proposition 2.4. Let $u_0 = 0$. By (H_2) and the definition of F, we can easily have

$$\Phi(u_0) = -\Psi(u_0) = 0.$$

Now we claim that (2.2) is satisfied.

From (I3), there exist $\eta \in [0, 1]$, $c_3 > 0$, such that

$$F(x, s) < c_3 |s|^{p_1(x)} < c_3 |s|^{p_1^-} \quad \forall s \in [-\eta, \eta], \text{ a.e. } x \in \Omega.$$

Then, from (I4), we can find a constant M such that

$$F(x, s) < M|s|^{p_1^-}$$

for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Consequently, by the Sobolev embedding theorem, $W^{1,p(x)}(\Omega) \hookrightarrow L^{p_1^-}(\Omega)$ is continuous. And for suitable positive constant c_4, c_5), we have

$$-\Psi(u) = \int_\Omega F(x, u)dx < M \int_\Omega |u|^{p_1^-} dx \le c_4 \|u\|^{p_1^-} \le c_5 r^{p_1^-/p^+},$$

when $\|u\|^{p^+}/p^+ \le r$. Hence, being $p_1^- > p^+$, it follows that

$$\lim_{r \to 0^+} \frac{\sup_{\|u\|^{p^+}/p^+ \le r} -\Psi(u)}{r} = 0. \tag{3.1}$$

Let $u_1 \in C^1(\Omega)$ be a function positive in Ω, with $\max_{\overline{\Omega}} u_1 \le c_1$. Then, $u_1 \in X$ and $\Phi(u_1) > 0$. In view of (I2) we also have $-\Psi(u_1) = \int_\Omega F(x, u_1(x))dx > 0$. Therefore, from (3.1), we can find $r \in \left(0, \min\{\Phi(u_1), \frac{1}{p^+}\}\right)$ such that

$$\sup_{\|u\|^{p^+}/p^+ \le r} (-\Psi(u)) < r\frac{-\Psi(u_1)}{\Phi(u_1)}.$$

Now, let $u \in \Phi^{-1}((-\infty, r])$. Then, $\int_\Omega (p(x)A(x, \nabla u) + |u|^{p(x)})dx \le rp^+ < 1$ which, by Proposition 2.2, implies $\|u\| < 1$. Consequently,

$$\frac{1}{p^+}\|u\|^{p^+} \le \frac{1}{p^+}\rho(u) \le \int_\Omega (p(x)A(x, \nabla u) + |u|^{p(x)})dx < r.$$

Therefore, we infer that $\Phi^{-1}((-\infty, r]) \subset \left\{u \in X : \frac{1}{p^+}\|u\|^{p^+} < r\right\}$, and so

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} (-\Psi(u)) < r\frac{-\Psi(u_1)}{\Phi(u_1)}.$$

At this point, conclusion follows from Proposition 2.4 and Theorem 2.3. $\qquad \square$

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