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Multiple solutions for p-Laplacian eigenproblem with nonlinear boundary conditions

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abstract: In this paper we study the existence of at least two nontrivial solutions for the nonlinear p-Laplacian problem, with nonlinear boundary conditions. We establish that there exist at least two solutions, which are opposite signs. For this reason, we characterize the first eigenvalue of an intermediary eigenvalue problem by the minimization method. In fact, in some sense, we establish the non-resonance below the first eigenvalues of nonlinear Steklov-Robin problem.

Key Words: Eigenproblem, p-Laplacian operator, variational method, non-resonance problems.

Contents

1. Introduction

Let us consider the following nonlinear boundary problem

$$(S_1)\begin{cases} -\triangle_p u + c(x)|u|^{p-2}u = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$, the functions $c : \Omega \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(C_1) $c \in L^p(\Omega)$ and $c \geq 0$ a.e $x \in \Omega$ with $\int_\Omega c(x)dx > 0$;

(C_2) f and g are Carathéodory functions;

(C_3) $|f(x,t)| \leq c(1+|t|^{q-1})$, a.e $x \in \Omega, t \in \mathbb{R}$, where $q \in [p, p^*[$, with $p^* = \frac{pN}{N-p}$, if $1 < p < N, p^* = \infty$, if $1 < N < p$.

(C_4) $|g(x,t)| \leq c(1+|t|^{q-1})$, a.e $x \in \partial\Omega, t \in \mathbb{R}$;

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The nonlinear p-Laplacian problem with Dirichlet, Neumann or Stecklov conditions has been studied by severals authors, for example, we cite the papers [1,7] and [2] in which the authors established the existence of positive and multiple solutions for the following quasilinear problem

$$\begin{cases} \triangle_p u &= |u|^{p-2}u \quad \text{in} \quad \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= f(x,u) \quad \text{on} \quad \partial\Omega, \end{cases}$$

under appropriate assumptions on f. In this sense we have extended this work to a nonlinear problem which has a strongly nonlinear second member. The resolution of the problem (S_1) appears in several works, we cite for example [6,8], but they are only in the case of quasilinear problems. This paper is organized as follows. In Section 2, we give a characterization of the first eigenvalue of the problem (S_2) bellow. In Section 3, we give some preliminary results and notations, that will be useful to prove the principal results of this article. In Section 4, we establish the main result.

2. Characterization of the first eigenvalues

Firstly, we consider the following eigenvalue problem

$$(S_2)\begin{cases} -\triangle_p u + c(x)|u|^{p-2}u &= \lambda m(x)|u|^{p-2} \quad \text{in} \quad \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda n(x)|u|^{p-2} \quad \text{on} \quad \partial\Omega, \end{cases}$$

$(m,n) \in L^\infty(\Omega) \times \in L^\infty(\partial\Omega)$, $mes\{x \in \Omega |\ m(x) > 0\} \neq 0$ and $mes\{x \in \partial\Omega |\ n(x) > 0\} \neq 0$.
The variational problem associated to the problem (S_2), see [5], is given by

$$\begin{cases} \text{Find } (\lambda, u) \in \mathbb{R}^+ \times W^{1,p}(\Omega) \setminus \{0\}, \text{ such that for all } w \in W^{1,p}(\Omega), \\ \int |\nabla u|^{p-2}\nabla u.\nabla w + \int c(x)|u|^{p-2}u.w = \lambda \left[\int m(x)|u|^{p-2}u.w + \oint n(x)|u|^{p-2}u.w\right]. \end{cases} \tag{2.1}$$

The existence of the eigenvalues sequence $(\lambda_k)_{k\geq 1}$ of the problem (S_2), see [4] and [5], is given as follows

$$\lambda_k^{-1}(m,n) = \sup_{B\in A_k} \min_{u\in B} \left(\int m(x)|u|^p + \oint n(x)|u|^p\right), \tag{2.2}$$

where $\mathcal{A}_k = \{B \in W^{1,p}(\Omega) \mid B \subset S_c,$ compact, symmetric and $\gamma(B) \geq k\}$, γ is the genus's function and $S_c = \{u \in W^{1,p}(\Omega) |\ \|u\|_c^p = \int |\nabla u|^p + \int c(x)|u|^p = 1\}$.
The proof of (2.2) is given by the L.Ljusternik-L. Schnirelmann critical point theory on C^1 manifolds using the genus γ. The first eigenvalue is characterized by

$$\lambda_1^{-1}(m,n) = \sup_{u\in S_c} \left(\int m(x)|u|^p + \oint n(x)|u|^p\right).$$

We have proved in [5] that the first eigenvalue λ_1 is simple and an eigenfunction u_1 associated with it, does not change sign in Ω and $|u_1| > 0$ in Ω.

Definition 2.1. *A weak solution of problem* (S_1) *is a function* $u \in W^{1,p}(\Omega)$ *satisfied*

$$\int |\nabla u|^{p-2}\nabla u.\nabla v + \int c(x)|u|^{p-2}u.v = \int f(x,u).v + \oint g(x,u).v, \ \forall v \in W^{1,p}(\Omega)$$

3. Preliminary results

We consider the following truncated problem

$$(S_{1+})\left\{\begin{array}{rcll} -\triangle_p u + c(x)|u|^{p-2}u &=& f_+(x,u) & \text{in} \quad \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &=& g_+(x,u) & \text{on} \quad \partial\Omega, \end{array}\right.$$

and

$$(S_{1-})\left\{\begin{array}{rcll} -\triangle_p u + c(x)|u|^{p-2}u &=& f_-(x,u) & \text{in} \quad \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &=& g_-(x,u) & \text{on} \quad \partial\Omega, \end{array}\right.$$

where

$$f_\pm(x,t) = \left\{\begin{array}{ll} f(x,t), & \text{if} \pm t \geq 0, \\ 0, & \text{if } \pm t < 0, \end{array}\right. \quad \text{and } g_\pm(x,t) = \left\{\begin{array}{ll} g(x,t), & \text{if } \pm t \geq 0, \\ 0, & \text{if } \pm t < 0. \end{array}\right.$$

We denote by $u^+ = \max(u,0)$ and $u^- = \max(-u,0)$. We also consider the functions $F_\pm$ and $G_\pm$ are defined as follows $F_\pm(x,t) = \int_0^t f_\pm(x,s)ds$ and $G_\pm(x,t) = \oint_0^t g_\pm(x,s)ds$. We assume the following assumptions

$$|F_+(x,t)| \leq c|t|^q + c \ p < q < p^* \ a.e. \ x \in \Omega, \tag{3.1}$$

and

$$|G_+(x,t)| \leq c|t|^q + c \ p < q < p^* \ a.e. \ x \in \partial\Omega, \tag{3.2}$$

Lemma 3.1. *All solutions for* (S_{1+}) *(respectively* (S_{1-})*) are the positive (respectively negative) solutions for* (S_1)*.*

Proof: Let ϕ_+ and ϕ_- be the energies functionals associated with problems (S_{1+}) and (S_{1-}) respectively, defined from $W^{1,p}(\Omega) \to \mathbb{R}$ such as

$$\phi_+(u) = \frac{1}{p}\left[\int |\nabla u|^p + \int c(x)|u|^p\right] - \int F_+(x,u) - \oint G_+(x,u), \tag{3.3}$$

and

$$\phi_-(u) = \frac{1}{p}\left[\int |\nabla u|^p + \int c(x)|u|^p\right] - \int F_-(x,u) - \oint G_-(x,u), \tag{3.4}$$

Using the assumptions $(C_1)-(C_4)$, $\phi_\pm$ are well defined and of class C^1 in $W^{1,p}(\Omega)$. Let u be a solution of (S_{1+}), then it is clear that u is a critical point of ϕ_+. We have for all $v \in W^{1,p}(\Omega)$,

$$0 =< \phi_+'(u), v >= \int |\nabla u|^{p-2}\nabla u.\nabla v + \int c(x)|u|^{p-2}u.v - \int f_+(x,u).v - \oint g_+(x,u).v. \tag{3.5}$$

We take $v = u^-$ in (3.5), we have

$$\int |\nabla u|^{p-2}\nabla u.\nabla u^- + \int c(x)|u|^{p-2}u.u^- - \int f_+(x,u)u^- - \oint g_+(x,u)u^- = 0.$$

Thus

$$\int |\nabla u^-|^p + \int c(x)|u^-|^p = 0.$$

So

$$\|u^-\| = 0,$$

which proves that $u^- = 0$, so $u = u^+$, is also a critical point of ϕ, with the critical value is $\phi(u) = \phi(u^+) = \phi_+(u)$. Similarly for the problem (S_{1-}). □

We pose

$$k_\pm(x) = \liminf_{t\to\pm\infty} \frac{f(x,t)}{|t|^{p-2}t} \text{ and } K_\pm(x) = \limsup_{t\to\pm 0} \frac{pF(x,t)}{|t|^p} \text{ uniformly } x \in \Omega, \quad (3.6)$$

$$l_\pm(x) = \liminf_{t\to\pm\infty} \frac{g(x,t)}{|t|^{p-2}t} \text{ and } L_\pm(x) = \limsup_{t\to\pm 0} \frac{pG(x,t)}{|t|^p} \text{ uniformly } x \in \partial\Omega, \quad (3.7)$$

4. Main result

Theorem 4.1. *Suppose the conditions $(C_2) - (C_4)$ and*

$$\lambda_1(k_+, l_+) < 1 < \lambda_1(K_+, L_+), \quad (4.1)$$

or

$$\lambda_1(k_-, l_-) < 1 < \lambda_1(K_-, L_-). \quad (4.2)$$

Then, the nonlinear problem (S_1), has at least two (non trivial) solutions in $W^{1,p}(\Omega)$. where in the first case is positive and the second is negative.

Remark 4.2. *If the assumptions (4.1) and (4.2) are satisfied simultaneously , then the nonlinear problem (S_1), has at least two (non trivial) solutions in $W^{1,p}(\Omega)$, which are opposite signs.*

For proof of theorem, we need the following lemma

Lemma 4.3. *under the assumptions of Theorem 4.1, the functional ϕ_+ and ϕ_- satisfied the Palais-Smale (P.S) conditions.*

Proof:

$$\phi_+(u) = \frac{1}{p}\left[\int |\nabla u|^p + \int c(x)|u|^p\right] - \int F_+(x,u) - \oint G_+(x,u). \quad (4.3)$$

Let $(u_n)_n$ a sequence satisfying the (P.S) conditions, $(\phi(u_n)_n)_n$ is bounded and $\|\phi'(u_n)\| \to 0$. We show that $(u_n)_n$ has a convergent subsequence. As $W^{1,p}(\Omega) \hookrightarrow$

$L^p(\Omega)$, it suffices to show that $(u_n)_n$ is a bounded sequence. By absurd, pose $v_n = \frac{u_n}{\|u_n\|}$, with $\|u_n\| \to +\infty$, so $\|v_n\| = 1$. As $(v_n)_n$ is bounded in $W^{1,p}(\Omega)$, then there exist a subsequence also noted $(v_n)_n$ such that,

$$\begin{cases} v_n & \rightharpoonup & v & \text{in} & W^{1,p}(\Omega), \\ v_n & \to & v & \text{in} & L^p(\Omega), \\ v_n & \to & v & \text{in} & L^p(\partial\Omega), \\ v_n(x) & \to & v(x) & \text{a.e in} & \Omega. \end{cases}$$

Firstly, we show that $v \neq 0$. By absurd, we suppose that $v = 0$, so $v_n \to 0$ in $L^p(\Omega)$.
As $\|\phi'_+(u_n)\| \to 0$ and $p > 1$, then $\frac{<\phi'_+(u_n),(u_n)}{\|u_n\|^p} \to 0$, with $\forall w \in W^{1,p}(\Omega)$

$$\phi'_+(u)w = \int |\nabla u|^{p-2}\nabla u.\nabla w + \int c(x)|u|^{p-2}u.w - \int f_+(x,u).w - \oint g_+(x,u).w, \tag{4.4}$$

so

$$\phi'_+(u_n)u_n = \int |\nabla u_n|^p + \int c(x)|u_n|^p - \int f_+(x,u_n).u_n - \oint g_+(x,u_n).u_n \ \forall n \in \mathbb{N}. \tag{4.5}$$

$$\begin{aligned} \frac{\phi'_+(u_n)u_n}{\|u_n\|^p} &= \int |\nabla v_n|^p + \int c(x)|v_n|^p - \int f_+(x,u_n).\frac{u_n}{\|u_n\|^p} - \oint g_+(x,u_n).\frac{u_n}{\|u_n\|^p} \\ &= \int |\nabla v_n|^p + \int c(x)|v_n|^p \\ &- \int \frac{f_+(x,u_n)}{|u_n|^{p-2}u_n}.\frac{|u_n|^{p-2}u_n^2}{\|u_n\|^p} - \oint \frac{g_+(x,u_n)}{|u_n|^{p-2}u_n}.\frac{|u_n|^{p-2}u_n^2}{\|u_n\|^p} \\ &= \|v_n\|^p - \int \frac{f_+(x,u_n)}{|u_n|^{p-2}u_n}.|v_n|^p - \oint \frac{g_+(x,u_n)}{|u_n|^{p-2}u_n}.|v_n|^p. \end{aligned}$$

As $\frac{f_+(x,u_n)}{|u_n|^{p-2}u_n}$ and $\frac{g_+(x,u_n)}{|u_n|^{p-2}u_n}$ are bounded in $L^{p'}(\Omega)$ and $L^{p'}(\partial\Omega)$ respectively, $v_n \to 0$ in $L^p(\Omega)$ and $v_n \to 0$ in $L^p(\partial\Omega)$, we deduce that

$$\int \frac{f_+(x,u_n)}{|u_n|^{p-2}u_n}.|v_n|^p \to 0 \text{ and } \oint \frac{g_+(x,u_n)}{|u_n|^{p-2}u_n}.|v_n|^p \to 0.$$

Therefore, we have $0 = 1 - 0 - 0$, it's absurd. Consequently $v \neq 0$.
Now we prove that (u_n) is bounded.

We have $\forall w \in W^{1,p}(\Omega)$, $\frac{\phi'_+(u_n)w}{\|u_n\|^{p-1}} \to 0$ and $f(x,s)=0$, for $s \leq 0$, so

$$\begin{aligned}\frac{\phi'_+(u_n)w}{\|u_n\|^{p-1}} &= \int \frac{|\nabla u_n|^{p-2}\nabla u_n.\nabla w}{\|u_n\|^{p-1}} + \int c(x)\frac{|u_n|^{p-2}.u_n}{\|u_n\|^{p-1}}.w - \int \frac{f_+(x,u_n)}{\|u_n\|^{p-1}}.w \\ &\quad - \oint \frac{g_+(x,u_n)}{\|u_n\|^{p-1}}.w \\ &= \int |\nabla v_n|^{p-2}\nabla v_n.\nabla w + \int c(x)|v_n|^{p-2}v_n.w - \int \frac{f(x,u_n^+)}{(u_n^+)^{p-1}}.\frac{(u_n^+)^{p-1}}{\|u_n\|^{p-1}}.w \\ &\quad - \oint \frac{g(x,u_n^+)}{(u_n^+)^{p-1}}.\frac{(u_n^+)^{p-1}}{\|u_n\|^{p-1}}.w,\end{aligned}$$

because $f_+(x,u_n) = f(x,u_n^+)$ and $g_+(x,u_n) = g(x,u_n^+)$.
Since $\frac{f(x,u_n^+)}{(u_n^+)^{p-1}}$ and $\frac{g(x,u_n^+)}{(u_n^+)^{p-1}}$ are bounded in $L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$ respectively, then $\frac{f(x,u_n^+)}{(u_n^+)^{p-1}}$ and $\frac{g(x,u_n^+)}{(u_n^+)^{p-1}}$ convergent to h_1 in $L^\infty(\Omega)$ and to h_2 in $L^\infty(\partial\Omega)$ respectively.
So we obtain $\forall w \in W^{1,p}(\Omega)$,

$$\int |\nabla v|^{p-2}.\nabla v.\nabla w + \int c(x)|v|^{p-2}v.w - \int h_1(x)(v^+)^{p-1}.w - \oint h_2(x)(v^+)^{p-1}.w = 0.$$

In particular, for $w = v^-$, we obtain $\int |\nabla v^-|^p + \int c(x)|v^-|^p = 0$, so $\|v^-\| = 0$, which proves that $v = v^+ \geq 0$, and satisfies the problem

$$\begin{cases} -\triangle_p v + c(x)|v|^{p-2}.v = h_1(x)|v|^{p-2}.v & \text{in} \quad \Omega, \\ |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = h_2(x)|v|^{p-2}.v & \text{on} \quad \partial\Omega. \end{cases}$$

Since the regularity of solutions at this problem are $C^{1,\alpha}(\bar{\Omega})$, (see [3]). We can apply Harnack inequality, we have $v > 0$ in Ω, so $\lambda_1(h_1(x), h_2(x)) = 1$. Since $v > 0$ in Ω, then $u_n(x) = \|u_n\|v_n \to +\infty$ *a.e.* Thus, we have

$$\liminf_{n\to+\infty} \frac{f(x,u_n)}{|u_n|^{p-2}.u_n} = k_+(x) \text{ and } \liminf_{n\to+\infty} \frac{g(x,u_n)}{|u_n|^{p-2}.u_n} = l_+(x), \tag{4.6}$$

and

$$\lim_{n\to+\infty} \frac{f(x,u_n^+)}{(u_n^+)^{p-1}} = h_1(x) \text{ and } \lim_{n\to+\infty} \frac{g(x,u_n^+)}{(u_n^+)^{p-1}} = h_2(x). \tag{4.7}$$

Using Fatou's lemma, we obtain

$$\begin{aligned}\int k_+(x)|u|^p &= \int \liminf_n \frac{f(x,u_n)}{|u_n|^{p-2}.u_n}|u|^p \\ &\leq \liminf_n \int \frac{f(x,u_n^+)}{|u_n^+|^{p-2}.u_n^+}|u|^p \\ &= \int h_1(x)|u|^p.\end{aligned}$$

The same, we prove that

$$\oint l_+(x)|u|^p \leq \oint h_2(x)|u|^p.$$

As

$$\begin{aligned}\frac{1}{\lambda_1(h_1,h_2)} &= \sup_{x\in W^{1,p}(\Omega),\ u\neq 0} \frac{\int h_1(x)|u|^p + \oint h_2(x)|u|^p}{\|u\|_c^p} \\ &\geq \sup_{x\in W^{1,p}(\Omega),\ u\neq 0} \frac{\int h_+(x)|u|^p + \oint l_+(x)|u|^p}{\|u\|_c^p} \\ &= \frac{1}{\lambda_1(k_+,l_+)}.\end{aligned}$$

thus $1 = \lambda_1(h_1,h_2) \leq \lambda_1(k_+,l_+) < 1$, absurd. So $(u_n)_n$ is bounded.
Now, we prove that ϕ_+ satisfies the geometric conditions of the mountain pass, in order to prove that ϕ_+ admits a critical point. We have

$$\frac{1}{\lambda_1(k_+,l_+)} \geq \frac{\int K_+|u|^p + \oint L_+|u|^p}{\int |\nabla u|^p + \int c(x)|u|^p}\ \forall u \in W^{1,p}(\Omega)\setminus\{0\}. \tag{4.8}$$

Using (3.1) and (3.2), we have

$$\forall \epsilon > 0, \exists \delta > 0, \forall 0 \leq t < \delta\ pF(x,t) \leq K_+(x)t^p + \epsilon t^p\ \forall x \in \Omega,$$

and

$$\exists \delta > 0, \forall 0 \leq t < \delta\ pG(x,t) \leq L_+(x)t^p + \epsilon t^p\ \forall x \in \partial\Omega.$$

Thus

$$F(x,t) \leq \frac{1}{p}K_+(x)t^p + \frac{\epsilon}{p}t^p + c|t|^p,\ \forall t \in \mathbb{R}, \forall x \in \Omega, \tag{4.9}$$

and

$$G(x,t) \leq \frac{1}{p}L_+(x)t^p + \frac{\epsilon}{p}t^p + c|t|^p,\ \forall t \in \mathbb{R}, \forall x \in \partial\Omega.\text{ Hence} \tag{4.10}$$

$$\begin{aligned}
\phi_+(u) &= \frac{1}{p}\left[\int |\nabla u|^p + \int c(x)|u|^p - \int F_+(x,u) - \oint G_+(x,u)\right] \\
&\geq \frac{1}{p}\|u\|^p - \frac{1}{p}\int K_+(x)|u|^p - \frac{\epsilon}{p}\int |u|^p - c\int |u|^q - \frac{1}{p}\oint L_+(u)|u|^p \\
&- \frac{c}{p}\oint |u|^p - c\oint |u|^q \\
&\geq \frac{1}{p}\|u\|^p - \frac{1}{p}\left[\int K_+(x)|u|^p + \oint L_+(u)|u|^p\right] - \frac{\epsilon}{p}\left[\int |u|^p + \oint |u|^p\right] \\
&- c\left[\int |u|^q + \oint |u|^q\right] \\
&\geq \frac{1}{p}\|u\|^p - \frac{1}{p\lambda_1(K_+,L_+)}\|u\|^p - \frac{\epsilon}{p\lambda_1(1,1)}\|u\|^p - \frac{c}{\lambda_1(1,1)}\|u\|^q \\
&\geq \frac{1}{p}\left(1 - \frac{1}{\lambda_1(K_+,L_+)} - \frac{\epsilon}{\lambda_1(1,1)}\right)\|u\|^p - \frac{c}{\lambda_1(1,1)}\|u\|^q.
\end{aligned}$$

For sufficiently small ϵ, we have $1 - \frac{1}{\lambda_1(K_+,L_+)} - \frac{\epsilon}{\lambda_1(1,1)} > 0$. Since $p < q$, then there exist $a > 0$, $\rho > 0$ and if $\|u\| = \rho$ then $\phi_+(u) \geq a > 0$. □

Proof: [Proof of Theorem 4.1] Let $\varphi(k_+,l_+)$ the first eigenfunction associated, such that $\varphi(k_+,l_+) > 0$, thus

$$\begin{aligned}
\frac{\phi_+(t\varphi(k_+,l_+))}{t^p} &= \frac{1}{p}\left(\int |\nabla\varphi(k_+,l_+)|^p + \int c(x)|\varphi(k_+,l_+)|\right) \\
&- \int \frac{F_+(x,t\varphi(k_+,l_+))}{|t\varphi(k_+,l_+)|^p} - \oint \frac{G_+(x,t\varphi(k_+,l_+))}{|t\varphi(k_+,l_+)|^p}.
\end{aligned}$$

Using Fatou's lemma, we obtain

$$\begin{aligned}
\liminf_{n\to+\infty} \frac{\phi_+(t\varphi(k_+,l_+))}{t^p} &\leq \frac{1}{p}\|\varphi(k_+,l_+)\|^p - \int \liminf_{n\to+\infty} \frac{F_+(x,t\varphi(k_+,l_+))}{|t\varphi(k_+,l_+)|^p} \\
&- \oint \liminf_{n\to+\infty} \frac{G_+(x,t\varphi(k_+,l_+))}{|t\varphi(k_+,l_+)|^p} \\
&\leq \frac{1}{p}\|\varphi(k_+,l_+)\|^p - \frac{1}{p}\int k_+(x)|\varphi(k_+,l_+)|^p \\
&- \frac{1}{p}\oint l_+(x)|\varphi(k_+,l_+)|^p \\
&\leq \frac{1}{p}\|\varphi(k_+,l_+)\|^p \\
&- \frac{1}{p}\left(\int k_+(x)|\varphi(k_+,l_+)|^p + \oint l_+(x)|\varphi(k_+,l_+)|^p\right) \\
&\leq \frac{1}{p}\left(1 - \frac{1}{\lambda_1(k_+,l_+)}\right)\|\varphi(k_+,l_+)\|^p \\
&\leq 0, \ (\text{ because } \lambda_1(k_+,l_+) < 1).
\end{aligned}$$

Thus $\exists t_0 > 0$ such that $\phi_+(t\varphi(k_+, l_+)) < 0$, therefore ϕ_+ satisfies the assumptions of Mountain pass theorem, thus $\exists u \in W^{1,p}(\Omega) \setminus \{0\}$

$$\int |\nabla u|^{p-2}\nabla u \nabla w + \int c(x)|u|^{p-2}u.w = \int f_+(x,u).w + \oint g_+(x,u).w, \ \forall w \in W^{1,p}(\Omega).$$

For $w = u^-$ and as for all $t \leq 0$, $f_+(x,t) = 0$ and $g_+(x,t) = 0$, we obtain

$$\int |\nabla u^-|^p + \int c(x)|u^-|^p = 0.$$

Thus $\|u^-\| = 0$, so $u^- = 0$. Which proves that $u > 0$ is the solution of the problem (S_1).
The same manner, using the condition (4.2), we prove that the solution of the problem (S_1) is negative. □

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