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On semiderivations of ∗−prime rings

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abstract: Let R be a ∗-prime ring with involution ∗ and center $Z(R)$. An additive mapping $F : R \to R$ is called a semiderivation if there exists a function $g : R \to R$ such that (i) $F(xy) = F(x)g(y) + xF(y) = F(x)y + g(x)F(y)$ and (ii) $F(g(x)) = g(F(x))$ hold for all $x, y \in R$. In the present paper, some well known results concerning derivations of prime rings are extended to semiderivations of ∗-prime rings.

Key Words: ∗−prime rings, derivations, semiderivations.

Contents

1. Introduction

Let R will be an associative ring with center Z. For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $* : R \to R$ is called an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or ∗−ring. A ring with an involution is said to ∗−prime if $xRy = xRy^* = 0$ or $xRy = x^*Ry = 0$ implies that $x = 0$ or $y = 0$. Every prime ring with an involution is ∗−prime but the converse need not hold general. An example due to Oukhtite [7] justifies the above statement that is, R be a prime ring, $S = R \times R^o$ where R^o is the opposite ring of R. Define involution ∗ on S as $*(x, y) = (y, x)$. S is ∗−prime, but not prime. This example shows that every prime ring can be injected in a ∗−prime ring and from this point of view ∗−prime rings constitute a more general class of prime rings. In all that follows the symbol $S_{a_*}(R)$, first introduced by Oukhtite, will denote the set of symmetric and skew symmetric elements of R, i.e. $S_{a_*}(R) = \{x \in R \mid x^* = \pm x\}$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. The study of derivations in prime rings was initiated by E. C. Posner in [11]. Recently, Bresar defined the following notation in [1]: An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$F(xy) = F(x)y + xd(y), \quad \text{for all } x, y \in R.$$

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Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \to ax + xb$ for some $a, b \in R$). Several authors consider the structure of a prime ring in the case that the derivation d is replaced by a generalized derivation. Generalized derivations have been primarily studied on operator algebras.

In [2] J. Bergen has introduced the notion of semiderivations of a ring R which extends the notion of derivations of a ring R. An additive mapping $F : R \to R$ is called a semiderivation if there exists a function $g : R \to R$ such that (i) $F(xy) = F(x)g(y) + xF(y) = F(x)y + g(x)F(y)$ and (ii) $F(g(x)) = g(F(x))$ hold for all $x, y \in R$. In case g is an identity map of R, then all semiderivations associated with g are merely ordinary derivations. On the other hand, if g is a homomorphism of R such that $g \neq 1$, then $f = g - 1$ is a semiderivation which is not a derivation. In case R is prime and $F \neq 0$, it has been shown by Chang [3] that g must necessarily be a ring endomorphism.

Let S be a nonempty subset of R. A mapping F from R to R is called centralizing on S if $[F(x), x] \in Z$ for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$ for all $x \in S$. The study of such mappings was initiated by E. C. Posner in [11]. A famous result due to Herstein [5] states that if R is a prime ring of characteristic not 2 which admits a nonzero derivation d such that $[d(x), a] = 0$ for all $x \in R$, then $a \in Z$. Also, Herstein showed that if $d(R) \subset Z$, then R must be commutative. On the other hand, in [4], Daif and Bell proved that if a semiprime ring R has a derivation d satisfying the following condition, then I is a central ideal;

$$\text{there exists a nonzero ideal } I \text{ of } R \text{ such that}$$
$$\text{either } d([x, y]) = [x, y] \text{ for all } x, y \in I, \text{ or } d([x, y]) = -[x, y] \text{ for all } x, y \in I.$$

Many authors have studied commutativity of prime and semiprime rings admitting derivations, generalized derivations and semiderivations which satisfy appropriate algebraic conditions on suitable subsets of the rings. Recently, some well-known results concerning prime rings have been proved for $*-$prime ring by Oukhtite et al. (see, [6-10], where further references can be found). In the present paper our objective is to generalize above results for semiderivations of a $*-$prime ring.

Throughout the paper, R will be a $*-$prime ring and F be a semiderivation of R associated with a surjective function g of R such that $*F = F*$. Also, we will make some extensive use of the basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z$$
$$[xy, z] = [x, z]y + x[y, z].$$

2. Results

Lemma 2.1. *Let R be a $*-$prime ring and $a \in R$. If R admits a semiderivation F of R such that $aF(x) = 0$ (or $F(x)a = 0$) for all $x \in R$, then $a = 0$ or $F = 0$.*

Proof: For all $x, y \in R$, we get $aF(xy) = 0$, and hence

$$aF(x)g(y) + axF(y) = 0,$$

and so

$$aRF(y) = 0, \text{ for all } y \in R.$$

Replacing y by y^* in this equation and using $*F = F*$, we find that

$$aRF(y)^* = 0, \text{ for all } y \in R.$$

Since R is a $*-$prime ring, we have $a = 0$ or $F = 0$. Similarly holds case $F(x)a = 0$. □

The following theorem is be obtained using the same methods in [3, Theorem 1].

Theorem 2.2. *Let R be a $*-$prime ring, F a nonzero semiderivation of R associated with a function g (not necessarily surjective). Then g is a homomorphism of R.*

Proof: For any $x, y, z \in R$, we get

$$\begin{aligned} F(z(x+y)) &= F(z)g(x+y) + zF(x+y) \\ &= F(z)g(x+y) + zF(x) + zF(y). \end{aligned}$$

On the other hand,

$$\begin{aligned} F(z(x+y)) &= F(zx+zy) \\ &= F(z)g(x) + zF(x) + F(z)g(y) + zF(y). \end{aligned}$$

Comparing these two equations, we arrive at $F(z)(g(x+y) - g(x) - g(y)) = 0$, for all $x, y, z \in R$. Using Lemma 2.1 and $F \neq 0$, we obtain that

$$g(x+y) = g(x) + g(y), \text{ for all } x, y \in R.$$

Now, let $x, y, z \in R$. Then

$$\begin{aligned} F((xy)z) &= g(xy)F(z) + F(xy)z \\ &= g(xy)F(z) + g(x)F(y)z + F(x)yz. \end{aligned}$$

On the other hand,

$$\begin{aligned} F((xy)z) = F(x(yz)) &= g(x)F(yz) + F(x)yz \\ &= g(x)g(y)F(z) + g(x)F(y)z + F(x)yz. \end{aligned}$$

Hence we get $(g(xy) - g(x)g(y))F(z) = 0$, for all $x, y, z \in R$. Again using Lemma 2.1 and $F \neq 0$, we have

$$g(xy) = g(x)g(y), \text{ for all } x, y \in R.$$

□

Theorem 2.3. *Let R be a $*-$prime ring, F a semiderivation of R such that $F(R) \subseteq Z$, then $F = 0$ or R is commutative.*

Proof: By the hypothesis, we have

$$F(xy) \in Z, \text{ for all } x, y \in R.$$

That is

$$F(x)g(y) + xF(y) \in Z, \text{ for all } x, y \in R.$$

Commuting this term with x and using the hypothesis, we get

$$\begin{aligned} 0 &= [F(x)g(y) + xF(y), x] \\ &= F(x)[g(y), x] \end{aligned}$$

Since $F(x) \in Z$ and g is surjective function of R, we arrive at

$$F(x)R[y, x] = 0, \text{ for all } x, y \in R.$$

Using $*F = F*$, for any $x \in S_{a_*}(R)$, we have

$$F(x)^* R[y, x] = 0, \text{ for all } x \in S_{a_*}(R), y \in R.$$

Since R is a $*-$prime ring, we arrive at

$$F(x) = 0 \text{ or } [y, x] = 0, \text{ for all } x \in S_{a_*}(R), y \in R.$$

Using the fact that $x + x^* \in S_{a_*}(R)$, $x - x^* \in S_{a_*}(R)$ for all $x \in R$, we easily deduce $F(x \pm x^*) = 0$ or $[y, x \pm x^*] = 0$. Hence we obtain R is union of its two additive subgroups such that

$$K = \{x \in R \mid F(x) = 0\}$$

and

$$L = \{x \in R \mid x \in Z\}.$$

Clearly each of K and L is additive subgroup of R. Morever, R is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $K = R$ or $L = R$. In the former case, we have $F = 0$ and the second case, R is commutative. □

Theorem 2.4. *Let R be a $2-$torsion free $*-$prime ring, F a semiderivation of R such that $F^2(x) = 0$, for all $x \in R$, then $F = 0$.*

Proof: Assume that

$$F^2(x) = 0, \text{ for all } x \in R.$$

Replacing x by xy in this equation, we get

$$\begin{aligned} 0 &= F^2(xy) = F(F(x)g(y) + xF(y)) \\ &= F^2(x)g^2(y) + F(x)F(g(y)) + F(x)g(F(y)) + xF^2(y) \end{aligned}$$

and so

$$2F(x)F(g(y)) = 0, \text{ for all } x, y \in R.$$

Using R is a $2-$torsion free and g is surjective function of R, we have

$$F(x)F(y) = 0, \text{ for all } x, y \in R.$$

By Lemma 2.1, we complete the proof. □

Theorem 2.5. *Let R be a $2-$torsion free $*-$prime ring and $a \in R$. If R admits a semiderivation Fsuch that $[F(x), a] = 0$, for all $x \in R$, then $F = 0$ or $a \in Z$.*

Proof: Replacing x by xy and using the hypothesis, we have

$$\begin{aligned} 0 &= [a, F(xy)] = [a, F(x)y + g(x)F(y)] \\ &= F(x)[a, y] + [a, g(x)]F(y) \end{aligned} \tag{2.1}$$

Writing y for $F(y)$ in this equation and again using the hypothesis, we obtain that

$$[a, g(x)]F^2(y) = 0, \text{ for all } x, y \in R.$$

Since g is surjective function of R, we have

$$[a, x]F^2(y) = 0, \text{ for all } x, y \in R.$$

Substituting xz for x in this equation, we get

$$[a, x]RF^2(y) = 0, \text{ for all } x, y \in R.$$

Since $*F = F*$, it reduces

$$[a, x]RF^2(y)^* = 0, \text{ for all } x, y \in R.$$

By the $*-$primeness of R, we find that

$$a \in Z \text{ or } F^2(y) = 0, \text{ for all } y \in R.$$

If $F^2(y) = 0$, for all $y \in R$, then $F = 0$ by Theorem 2.4. □

Theorem 2.6. *Let R be a $2-$torsion free $*-$prime ring and F a semiderivation of R such that $[F(R), F(R)] = 0$, then $F = 0$ or R is commutative.*

Proof: By Theorem 2.5, we have $F = 0$ or $F(R) \subseteq Z$. If $F(R) \subseteq Z$, then $F = 0$ or R is commutative by Theorem 2.3. □

Theorem 2.7. *Let R be a $*-$prime ring, F a semiderivation of R such that $[F(x), x] = 0$, for all $x \in R$, then $F = 0$ or R is commutative.*

Proof: Linearizing the hypothesis, we have

$$[F(x), y] + [F(y), x] = 0, \text{ for all } x, y \in R.$$

Replacing y by yx in this equation and using the hypothesis, we get

$$\begin{aligned} 0 &= [F(x), yx] + [F(yx), x] \\ &= [F(x), y]x + [F(y)x + g(y)F(x), x], \end{aligned}$$

and so

$$[g(y), x]F(x) = 0, \text{ for all } x, y \in R.$$

Since g is surjective function of R, we have

$$[y, x]F(x) = 0, \text{ for all } x, y \in R.$$

Writing yz for y and using this equation, we obtain that

$$[y, x]RF(x) = 0, \text{ for all } x, y \in R.$$

Using the same arguments as we used in the last part of proof of the Theorem 2.3, we get the required result. □

Theorem 2.8. *Let R be a $*-$prime ring, F a nonzero semiderivation of R such that $F([x, y]) = 0$, for all $x, y \in R$, then R is commutative.*

Proof: Replacing y by xy in the hypothesis, we get

$$\begin{aligned} 0 &= F(x[x, y]) = F(x)g([x, y]) + xF([x, y]) \\ &= F(x)g([x, y]). \end{aligned}$$

We know that g is homomorphism of R by Theorem 1. Hence we have

$$F(x)[g(x), g(y)] = 0, \text{ for all } x, y \in R.$$

Since g is surjective function of R, we get

$$F(x)[g(x), y] = 0, \text{ for all } x, y \in R.$$

Writing yz for y and using this equation, we obtain that

$$F(x)R[g(x), z] = 0, \text{ for all } x, z \in R.$$

Using $*F = F*$, for any $x \in S_{a_*}(R)$, we have

$$F(x)^* R[g(x), z] = 0, \text{ for all } x \in S_{a_*}(R), z \in R.$$

Since R is a $*-$prime ring, we arrive at

$$F(x) = 0 \text{ or } [g(x), y] = 0, \text{ for all } x \in S_{a_*}(R), y \in R.$$

Using the fact that $x+x^* \in S_{a_*}(R), x-x^* \in S_{a_*}(R)$ for all $x \in R$, we easily deduce $F(x \pm x^*) = 0$ or $[g(x \pm x^*), y] = 0$. Hence we obtain that R is union of its two additive subgroups such that

$$K = \{x \in R \mid F(x) = 0\}$$

and

$$L = \{x \in R \mid [g(x), y] = 0, \text{for all } y \in R\}.$$

Clearly each of K and L is additive subgroup of R. Morever, R is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $K = R$ or $L = R$. In the former case, we have $F = 0$, a contradiction. So, we must have $L = R$. Hence R is commutative. □

Theorem 2.9. *Let R be a $*-$prime ring, F a nonzero semiderivation of R such that $F([x, y]) = \pm[x, y]$, for all $x, y \in R$, then R is commutative.*

Proof: Replacing y by xy in the hypothesis, we get

$$\begin{aligned} F(x[x, y]) &= \pm x[x, y] \\ F(x)g([x, y]) + xF([x, y]) &= \pm x[x, y], \end{aligned}$$

and so

$$F(x)g([x, y]) = 0.$$

Using the same arguments as we used in the last part of proof of the Theorem 2.8, we get the required result. □

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