



Majorization Problems and Integral Transforms for a Class of Univalent Functions with Missing Coefficients

Som P. Goyal, Rakesh Kumar and Teodor Bulboacă

ABSTRACT: In 2005, Ponnusamy and Sahoo have introduced a special subclass of univalent functions $\mathcal{U}_n(\lambda)$ ($n \in \mathbb{N}$, $\lambda > 0$) and obtained some geometrical properties, including strongly starlikeness and convexity, for the functions of this subclass $\mathcal{U}_n(\lambda)$. Moreover, they have studied some important properties of an integral transform connected with these subclasses. The aim of the present paper is to investigate another important concept of majorization for the functions belonging to the class $\mathcal{U}_n(\lambda)$ ($0 < \lambda \leq 1$). We shall also discuss a majorization problem for some special integral transforms.

Key Words: Univalent functions, quasi-subordination, starlike functions, majorization property, integral transforms.

Contents

1	Introduction and Preliminaries	219
2	Majorization problem for the class $\mathcal{U}_n(\lambda)$	222
3	Integral Transforms	225

1. Introduction and Preliminaries

Let \mathcal{H} denote the class of functions which are analytic in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For a fixed $n \in \mathbb{N} = \{1, 2, \dots\}$, let \mathcal{A}_n be the class of functions $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad z \in \Delta. \quad (1.1)$$

We denote $\mathcal{A} := \mathcal{A}_1$, while the subclass of \mathcal{A} consisting of all univalent functions in Δ is denoted by \mathcal{S} .

Definition 1.1. [11, p. 226] *If $f, g \in \mathcal{H}$, then f is said to be subordinate to g , if there exists a function $w \in \mathcal{H}$ satisfying $w(0) = 0$ and $|w(z)| < 1$, $z \in \Delta$, such that*

$$f(z) = g(w(z)), \quad z \in \Delta.$$

The subordination relation is denoted by

$$f(z) \prec g(z). \quad (1.2)$$

2000 *Mathematics Subject Classification:* 30C45, 30C80.

Moreover, if g is univalent in Δ , then this subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$ (cf. Duren [3], Goodman [4]).

Definition 1.2. [17] If $f, g \in \mathcal{H}$, then f is said to be quasi-subordinate to g , if there exists a function $\varphi \in \mathcal{H}$ satisfying $\frac{f}{\varphi} \in \mathcal{H}$ and $|\varphi(z)| \leq 1$, $z \in \Delta$, such that

$$\frac{f(z)}{\varphi(z)} \prec g(z).$$

The quasi-subordination relation is denoted by

$$f(z) \prec_q g(z). \quad (1.3)$$

Note that, the quasi-subordination (1.3) is equivalent to

$$f(z) = \varphi(z)g(w(z)), \quad z \in \Delta, \quad (1.4)$$

where $w \in \mathcal{H}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$.

Remark that, for the special case when if $\varphi(z) \equiv 1$, the quasi-subordination (1.3) becomes the subordination (1.2).

Definition 1.3. [10] If $f, g \in \mathcal{H}$, we say that f is majorized by g , if there exists a function $\varphi \in \mathcal{H}$, satisfying $|\varphi(z)| \leq 1$, $z \in \Delta$, such that

$$f(z) = \varphi(z)g(z), \quad z \in \Delta.$$

The majorization relation is denoted by

$$f(z) \ll g(z). \quad (1.5)$$

If we take $w(z) \equiv z$ in (1.4), then the quasi-subordination (1.3) becomes the majorization (1.5).

Recently, Ponnusamy and Sahoo [16] defined and studied the class $\mathcal{U}_n(\lambda)$ of functions $f \in \mathcal{A}_n$ which satisfy the inequality

$$\left| \left(\frac{z}{f(z)} \right)^{n+1} f'(z) - 1 \right| < \lambda, \quad z \in \Delta, \quad (1.6)$$

for some real number $\lambda > 0$. Several important properties of the class $\mathcal{U}(\lambda) := \mathcal{U}_1(\lambda)$ has been studied by many authors in [8], [12], [13], [14], [15].

Some interesting majorization problems were investigated earlier by Altınas et al. [2] for the class of starlike functions of complex order, and in the recent years many authors investigate majorization problems for various subclasses of univalent functions, like in [1], [5], [6], [7], [9].

The main aim of this paper is to investigate the majorization problem for the function $f \in \mathcal{A}_n$ which contains the well-known subclass $\mathcal{U}_n(\lambda)$ ($0 < \lambda \leq 1$) of univalent functions. Throughout the paper, the coefficient a_{n+1} from the power series expansion (1.1) is meant for $\frac{f^{(n+1)}(0)}{(n+1)!}$.

Remark 1.4. 2. If $f \in \mathcal{U}_n(\lambda)$, with $\lambda > 0$, then from [16] we have

$$\left| \left(\frac{z}{f(z)} \right)^n - 1 \right| \leq n|z|^n (|a_{n+1}| + \lambda|z|), \quad z \in \Delta. \tag{1.7}$$

2. In particular, if a_{n+1} and λ satisfy the inequality $n(|a_{n+1}| + \lambda) < 1$, then (1.7) is equivalent to

$$\left| \left(\frac{f(z)}{z} \right)^n - \frac{1}{1 - n^2|z|^{2n} (|a_{n+1}| + \lambda|z|)^2} \right| \leq \frac{n|z|^n (|a_{n+1}| + \lambda|z|)}{1 - n^2|z|^{2n} (|a_{n+1}| + \lambda|z|)^2}, \quad z \in \Delta.$$

Thus, if $f \in \mathcal{U}_n(\lambda)$, then we have

$$\operatorname{Re} \left(\frac{f(z)}{z} \right)^n \geq \frac{1}{1 + n|z|^n (|a_{n+1}| + \lambda|z|)} > \frac{1}{1 + n(|a_{n+1}| + \lambda)}, \quad z \in \Delta.$$

To prove these last two results, it is easy to see that $\left| \frac{1}{w} - 1 \right| \leq R$ is equivalent to

$$(1 - R^2) |w|^2 - 2 \operatorname{Re} w + 1 \leq 0,$$

that is

$$\left| w - \frac{1}{1 - R^2} \right| \leq \frac{R}{1 - R^2}, \quad \text{if } R < 1. \tag{1.8}$$

Moreover, this inequality implies that

$$\frac{1}{1 + R} \leq \operatorname{Re} w \leq \frac{1}{1 - R}, \quad \text{if } R < 1. \tag{1.9}$$

Since $n(|a_{n+1}| + \lambda) < 1$, we deduce that $n|z|^n (|a_{n+1}| + \lambda|z|) < 1$ for all $z \in \Delta$. Substituting the values $w := \left(\frac{f(z)}{z} \right)^n$ and $R := n|z|^n (|a_{n+1}| + \lambda|z|) < 1$ in (1.8) and (1.9), we obtain, respectively, the above inequalities.

In order to obtain our main results, we need the following well-known Schwarz's lemma:

Lemma 1.5. [4] If $w \in \mathcal{B}_n$, where

$$\mathcal{B}_n := \left\{ w \in \mathcal{H} : w(0) = w'(0) = \dots = w^{(n)}(0) = 0, |w(z)| < 1, z \in \Delta \right\},$$

then

$$|w(z)| \leq |z|^{n+1}, \quad z \in \Delta. \tag{1.10}$$

The equality in (1.10) is attained if and only if $w(z) = e^{i\theta} z^{n+1}$, with $\theta \in \mathbb{R}$.

From this lemma we deduce the following lemma that will be used in the proof of our first main result:

Lemma 1.6. *Let $p \in \mathcal{P}_n$, where*

$$\mathcal{P}_n := \left\{ p \in \mathcal{H} : p(0) = p'(0) = \dots = p^{(n)}(0) = 0 \right\}.$$

If the function p satisfies the condition

$$\left| p(z) - \frac{1}{n} z p'(z) \right| < \lambda, \quad z \in \Delta, \tag{1.11}$$

where $\lambda > 0$, then there exists a function $w \in \mathcal{B}_n$ of the form

$$w(z) = \sum_{k=n+1}^{\infty} w_k z^k, \quad z \in \Delta, \tag{1.12}$$

such that

$$p(z) = -n\lambda \sum_{k=n+1}^{\infty} \frac{w_k}{k-n} z^k = -n\lambda \int_0^1 \frac{w(tz)}{t^{n+1}} dt, \quad z \in \Delta.$$

Proof: Since $p \in \mathcal{P}_n$, therefore by the assumption (1.11) it follows that there exists a function $w \in \mathcal{B}_n$ of the form (1.12), such that

$$p(z) - \frac{1}{n} z p'(z) = \lambda w(z), \quad z \in \Delta. \tag{1.13}$$

If we let

$$p(z) = \sum_{k=n+1}^{\infty} p_k z^k, \quad z \in \Delta,$$

then (1.13) gives

$$\left(1 - \frac{k}{n} \right) p_k = w_k, \quad k \geq n + 1,$$

that is

$$p_k = -n\lambda \frac{w_k}{k-n}, \quad k \geq n + 1,$$

and from this last relation immediately arrive at our conclusion. □

2. Majorization problem for the class $\mathcal{U}_n(\lambda)$

We begin by proving the following result contained in:

Theorem 2.1. *Let $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$, such that $g \in \mathcal{U}_n(\lambda)$, where $0 < \lambda \leq 1$.*

If the function $f \in \mathcal{A}_n$ is majorized by g , i.e. $f(z) \ll g(z)$, then

$$|f'(z)| \leq |g'(z)|, \quad \text{for } |z| \leq r_1(\lambda, n), \tag{2.1}$$

where $r_1(\lambda, n)$ is the smallest positive real root of the equation

$$\lambda r^{n+3} - 2n\lambda r^{n+2} - (2n|b_{n+1}| + \lambda) r^{n+1} - r^2 - 2r + 1 = 0. \tag{2.2}$$

Proof: For $g \in \mathcal{U}_n(\lambda)$, with $0 < \lambda \leq 1$, let define the function $w \in \mathcal{B}_n$ by

$$w(z) = \frac{1}{\lambda} \left[\left(\frac{z}{g(z)} \right)^{n+1} g'(z) - 1 \right], \quad z \in \Delta.$$

Since

$$\left(\frac{z}{g(z)} \right)^{n+1} g'(z) = \left(\frac{z}{g(z)} \right)^n - \frac{1}{n} z \left[\left(\frac{z}{g(z)} \right)^n \right]',$$

therefore, we have

$$\left(\frac{z}{g(z)} \right)^n - \frac{1}{n} z \left[\left(\frac{z}{g(z)} \right)^n \right]' - 1 = \lambda w(z). \tag{2.3}$$

Using the fact that

$$\left(\frac{z}{g(z)} \right)^n = 1 - nb_{n+1}z^n - \dots, \quad z \in \Delta, \tag{2.4}$$

from (2.3) and (2.4), according to Lemma 1.6, we obtain

$$\left(\frac{z}{g(z)} \right)^n - 1 + nb_{n+1}z^n = -n\lambda \sum_{k=n+1}^{\infty} \frac{w_k}{k-n} z^k,$$

and therefore

$$\left(\frac{z}{g(z)} \right)^n = 1 - nb_{n+1}z^n - n\lambda \int_0^1 \frac{w(tz)}{t^{n+1}} dt.$$

If we denote

$$W(z) = \int_0^1 \frac{w(tz)}{t^{n+1}} dt, \quad z \in \Delta,$$

then

$$\left(\frac{z}{g(z)} \right)^n = 1 - nb_{n+1}z^n - n\lambda W(z),$$

and thus, we have

$$\frac{zg'(z)}{g(z)} = \frac{1 + \lambda w(z)}{1 - nb_{n+1}z^n - n\lambda W(z)}, \quad z \in \Delta. \tag{2.5}$$

In addition, a simple calculation combined with Lemma 1.5 yields that $w \in \mathcal{B}_n$ implies that

$$|w(z)| \leq |z|^{n+1}, \quad z \in \Delta, \tag{2.6}$$

and

$$|W(z)| = \left| \int_0^1 \frac{w(tz)}{t^{n+1}} dt \right| \leq \int_0^1 \frac{|w(tz)|}{t^{n+1}} dt \leq |z|^{n+1} \int_0^1 dt = |z|^{n+1}, \quad z \in \Delta. \tag{2.7}$$

Now, from (2.5) we get

$$|g(z)| \leq \frac{(1+n|b_{n+1}||z|^n + n\lambda|W(z)|)|z|}{1-\lambda|w(z)|} |g'(z)|, \quad z \in \Delta.$$

Since $f(z) \ll g(z)$, from (1.5) and (1.6) we have that

$$f(z) = \varphi(z)g(z), \quad z \in \Delta, \quad (2.8)$$

and differentiating (2.8) we get

$$f'(z) = \varphi'(z)g(z) + \varphi(z)g'(z), \quad z \in \Delta.$$

Noting that $\varphi \in \mathcal{B}_n$ satisfies the inequality (see, e.g. Nehari [11])

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2}, \quad z \in \Delta. \quad (2.9)$$

Therefore, we have

$$|f'(z)| \leq \left[\frac{1-|\varphi(z)|^2}{1-|z|^2} \frac{(1+n|b_{n+1}||z|^n + n\lambda|W(z)|)|z|}{1-\lambda|w(z)|} + |\varphi(z)| \right] |g'(z)|, \quad z \in \Delta,$$

and using (2.6) and (2.7), after some simple calculations we get

$$|f'(z)| \leq \left[\frac{1-|\varphi(z)|^2}{1-|z|^2} \frac{(1+n|b_{n+1}||z|^n + n\lambda|z|^{n+1})|z|}{1-\lambda|z|^{n+1}} + |\varphi(z)| \right] |g'(z)|, \quad z \in \Delta,$$

which gives

$$|f'(z)| \leq \frac{(1-|z|^2)(1-\lambda|z|^{n+1})|\varphi(z)| + (1-|\varphi(z)|^2)(1+n|b_{n+1}||z|^n + n\lambda|z|^{n+1})|z|}{(1-|z|^2)(1-\lambda|z|^{n+1})} |g'(z)|,$$

for all $z \in \Delta$.

Upon setting $|z| = r$, $0 \leq r < 1$, and $|\varphi(z)| = \rho$, $0 \leq \rho \leq 1$, this leads to the inequality

$$|f'(z)| \leq \frac{\Theta(r, \rho)}{(1-r^2)(1-\lambda r^{n+1})} |g'(z)|, \quad z \in \Delta, \quad (2.10)$$

where

$$\begin{aligned} \Theta(r, \rho) := & -(1+n|b_{n+1}|r^n + n\lambda r^{n+1})r\rho^2 \\ & + (1-r^2)(1-\lambda r^{n+1})\rho + (1+n|b_{n+1}|r^n + n\lambda r^{n+1})r. \end{aligned}$$

If we denote

$$\Psi(r, \rho) := \frac{\Theta(r, \rho)}{(1-r^2)(1-\lambda r^{n+1})},$$

then (2.10) becomes

$$|f'(z)| \leq \Psi(r, \rho)|g'(z)|, \quad z \in \Delta.$$

From the above relation, in order to prove our result, we need to determine $r_1 = \max \{r \in [0, 1] : \Psi(r, \rho) \leq 1, \forall \rho \in [0, 1]\} = \max \{r \in [0, 1] : \varphi(r, \rho) \geq 0, \forall \rho \in [0, 1]\}$, where

$$\varphi(r, \rho) := (1 - r^2) (1 - \lambda r^{n+1}) - \Theta(r, \rho) = (1 - r^2) (1 - \lambda r^{n+1}) (1 - \rho) - (1 - \rho^2) (1 + n |b_{n+1}| r^n + n \lambda r^{n+1}) r.$$

A simple calculation shows that the inequality $\varphi(r, \rho) \geq 0$ is equivalent to

$$u(r, \rho) := (1 - r^2) (1 - \lambda r^{n+1}) - (1 + n |b_{n+1}| r^n + n \lambda r^{n+1}) r (1 + \rho) \geq 0.$$

Obviously the function $u(r, \rho)$ takes its minimum value at $\rho = 1$, that is

$$\min \{u(r, \rho) : \rho \in [0, 1]\} = u(r, 1) =: v(r),$$

where

$$v(r) = \lambda r^{n+3} - 2n \lambda r^{n+2} - (2n |b_{n+1}| + \lambda) r^{n+1} - r^2 - 2r + 1.$$

Since $v(0) = 1 > 0$ and $v(1) < 0$, it follows that $v(r) \geq 0$ for all $r \in [0, r_1]$, where $r_1 := r_1(\lambda, n)$, is the smallest positive root of the equation (2.2), which completes our proof. \square

For $n = 1$, Theorem 2.1 gives the following special case:

Corollary 2.2. *Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, such that $g \in \mathcal{U}(\lambda)$, where $0 < \lambda \leq 1$. If the function $f \in \mathcal{A}$ is majorized by g , i.e. $f(z) \ll g(z)$, then*

$$|f'(z)| \leq |g'(z)|, \text{ for } |z| \leq \rho_1(\lambda),$$

where $\rho_1(\lambda) := r_1(\lambda, 1)$ is the smallest positive real root of the equation

$$\lambda r^4 - 2\lambda r^3 - (2|b_2| + \lambda + 1) r^2 - 2r + 1 = 0.$$

3. Integral Transforms

In this section we consider the following integral transform $I_{c,n} : \mathcal{A}_n \rightarrow \mathcal{A}_n$ defined by

$$I_{c,n}(f)(z) = z \left[\frac{c+1-n}{z^{c+1-n}} \int_0^z \left(\frac{t}{f(t)} \right)^n t^{c-n} dt \right]^{1/n}, \quad c+1-n > 0, \quad (3.1)$$

(see also [16]). For $c = n = 1$ the transform (3.1) reduces to

$$I_{1,1}(f)(z) = \int_0^z \frac{t}{f(t)} dt,$$

which is similar to the *Alexander transform*. Also, the operator $I_{c,n}$ is similar to the *Bernardi transformation*, for $n = 1$ and $c > 0$.

Theorem 3.1. For $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$, such that $g \in \mathcal{U}_n(\lambda)$, where $0 < \lambda \leq 1$, let

$$G = I_{c,n}(g), \quad (3.2)$$

where $I_{c,n}$ is defined by (3.1). Let the function $f \in \mathcal{A}_n$, such that $F = I_{c,n}(f)$ is majorized by G , i.e. $F(z) \ll G(z)$, and

$$2|c_{n+1}| + \frac{\lambda(c+1-n)(2n+1)}{c+2} \leq 1, \quad (3.3)$$

where

$$c_{n+1} = -\frac{n(c+1-n)}{c+1} b_{n+1}. \quad (3.4)$$

Then,

$$|F'(z)| \leq |G'(z)|, \text{ for } |z| \leq r_2(\lambda, n),$$

where $r_2(\lambda, n)$ is the smallest positive real root of the equation

$$\begin{aligned} & \frac{\lambda(c+1-n)}{c+2} (2n+1)r^{n+3} + 2n(c+1-n) \left(\frac{|c_{n+1}|}{c+1} - \frac{\lambda}{c+2} \right) r^{n+2} \\ & - (c+1-n) \left(\frac{(2n+1)\lambda}{c+2} + \frac{2n|c_{n+1}|}{c+1} \right) r^{n+1} \\ & - \frac{2n(c+1-n)}{c+1} |c_{n+1}| r^n - r^2 - 2r + 1 = 0. \end{aligned} \quad (3.5)$$

Proof: For $g \in \mathcal{U}_n(\lambda)$, from (3.2) we easily obtain that

$$(c+1-n) \left(\frac{G(z)}{z} \right)^n + z \frac{d}{dz} \left(\frac{G(z)}{z} \right)^n = (c+1-n) \left(\frac{z}{g(z)} \right)^n.$$

Differentiating the above relation we deduce that

$$\begin{aligned} & \frac{1}{n(c+1-n)} \left[(c-n)(n+1) \left(\frac{G(z)}{z} \right)^n - (c-2n) \frac{d}{dz} \left(z \left(\frac{G(z)}{z} \right)^n \right) \right. \\ & \left. - z \frac{d^2}{dz^2} \left(z \left(\frac{G(z)}{z} \right)^n \right) \right] = \left(\frac{z}{g(z)} \right)^{n+1} g'(z). \end{aligned} \quad (3.6)$$

Letting

$$P(z) = z \left(\frac{G(z)}{z} \right)^n, \quad (3.7)$$

from (3.6) and the assumption $g \in \mathcal{U}_n(\lambda)$ it follows that P satisfies the second order differential equation

$$\frac{(c-n)(n+1)}{n(c+1-n)} \frac{P(z)}{z} - \frac{(c-2n)P'(z)}{n(c+1-n)} - \frac{zP''(z)}{n(c+1-n)} = 1 + \lambda w(z), \quad (3.8)$$

where $w \in \mathcal{B}_n$. If we set

$$P(z) = z + \sum_{k=n+1}^{\infty} c_k z^k \quad \text{and} \quad w(z) = \sum_{k=n+1}^{\infty} w_k z^k,$$

then (3.8) could be written like

$$\begin{aligned} & \frac{(c-n)(n+1)}{n(c+1-n)} \left[\frac{P(z)}{z} - 1 - c_{n+1}z^n \right] - \frac{(c-2n)}{n(c+1-n)} [P'(z) - 1 - (n+1)c_{n+1}z^n] \\ & - \frac{1}{n(c+1-n)} [zP''(z) - n(n+1)c_{n+1}z^n] = \lambda w(z), \quad z \in \Delta. \end{aligned} \tag{3.9}$$

Denoting

$$H(z) := \frac{P(z)}{z} - 1 - c_{n+1}z^n = \sum_{k=n+1}^{\infty} c_{k+1}z^k, \quad z \in \Delta,$$

then

$$\begin{aligned} zH'(z) + H(z) &= P'(z) - 1 - (n+1)c_{n+1}z^n, \\ z^2H''(z) + 2zH'(z) &= zP''(z) - n(n+1)c_{n+1}z^n, \end{aligned}$$

and replacing these values in (3.9) we have

$$\frac{-n^2 + n + cn}{n(c+1-n)} H(z) + \frac{-c + 2n - 2}{n(c+1-n)} zH'(z) - \frac{1}{n(c+1-n)} z^2H''(z) = \lambda w(z), \quad z \in \Delta. \tag{3.10}$$

Equating the coefficients of z^n in (3.10) we get the relations

$$c_{k+1} = -\frac{\lambda n(c+1-n)}{c+1} \left(\frac{1}{k-n} - \frac{1}{k-n+c+1} \right) w_k, \quad k \geq n+1,$$

then

$$\begin{aligned} H(z) &= \sum_{k=n+1}^{\infty} c_{k+1}z^k \\ &= -\frac{\lambda n(c+1-n)}{c+1} \left(\sum_{k=n+1}^{\infty} \frac{w_k}{k-n} z^k - \sum_{k=n+1}^{\infty} \frac{w_k}{k-n+c+1} z^k \right) \\ &= -\frac{\lambda n(c+1-n)}{c+1} \left(\int_0^1 \frac{w(tz)}{t^{n+1}} dt - \int_0^1 \frac{t^{c+1}w(tz)}{t^{n+1}} dt \right), \end{aligned} \tag{3.11}$$

or

$$\frac{P(z)}{z} = 1 + c_{n+1}z^n - \frac{\lambda n(c+1-n)}{c+1} W_1(z),$$

where

$$W_1(z) = \int_0^1 \frac{w(tz)}{t^{n+1}} (1 - t^{c+1}) dt,$$

and c_{n+1} is given by (3.4).

Also, differentiating the first part of the formula (3.11) we have

$$\begin{aligned} zH'(z) + H(z) &= -\frac{\lambda n(c+1-n)}{c+1} \left[\sum_{k=n+1}^{\infty} \frac{(k+1)w_k}{k-n} z^k - \sum_{k=n+1}^{\infty} \frac{(k+1)w_k}{k-n+c+1} z^k \right] = \\ &= -\frac{\lambda n(c+1-n)}{c+1} \left[(n+1) \sum_{k=n+1}^{\infty} \frac{w_k}{k-n} z^k + (c-n) \sum_{k=n+1}^{\infty} \frac{w_k}{k-n+c+1} z^k \right] = \\ &= -\frac{\lambda n(c+1-n)}{c+1} \left[(n+1) \int_0^1 \frac{w(tz)}{t^{n+1}} dt + (c-n) \int_0^1 \frac{t^{c+1}w(tz)}{t^{n+1}} dt \right]. \end{aligned}$$

Thus, we obtain

$$P'(z) = 1 + (n+1)c_{n+1}z^n - \frac{\lambda n(c+1-n)}{c+1}W_2(z),$$

where

$$W_2(z) = \int_0^1 \frac{w(tz)}{t^{n+1}} [n+1 + (c-n)t^{c+1}] dt,$$

and c_{n+1} is given by (3.4).

From (3.7) we obtain

$$\begin{aligned} \frac{zG'(z)}{G(z)} &= \frac{1}{n} \left[\frac{zP'(z)}{P(z)} - 1 \right] + 1 \\ &= \frac{1}{n} \left[\frac{1 + (n+1)c_{n+1}z^n - \frac{\lambda n(c+1-n)}{c+1}W_2(z)}{1 + c_{n+1}z^n - \frac{\lambda n(c+1-n)}{c+1}W_1(z)} - 1 \right] + 1, \end{aligned}$$

which easily gives

$$\frac{zG'(z)}{G(z)} = \frac{1 + 2c_{n+1}z^n - \frac{\lambda(c+1-n)}{c+1} [(n-1)W_1(z) + W_2(z)]}{1 + c_{n+1}z^n - \frac{\lambda n(c+1-n)}{c+1}W_1(z)},$$

and from this equality we get

$$|G(z)| \leq \frac{\left[1 + |c_{n+1}| |z|^n + \frac{\lambda n(c+1-n)}{c+1} |W_1(z)| \right] |z|}{1 - 2|c_{n+1}| |z|^n - \frac{\lambda(c+1-n)}{c+1} [(n-1)|W_1(z)| + |W_2(z)|]} |G'(z)|. \tag{3.12}$$

Using the fact that $w \in \mathcal{B}_n$, a simple calculation combined with Lemma 1.5 implies that

$$\begin{aligned} |W_1(z)| &= \left| \int_0^1 \frac{w(tz)}{t^{n+1}} (1-t^{c+1}) dt \right| \\ &\leq \int_0^1 \frac{|w(tz)|}{t^{n+1}} (1-t^{c+1}) dt \leq \frac{c+1}{c+2} |z|^{n+1}, \quad z \in \Delta, \end{aligned} \tag{3.13}$$

and

$$|W_2(z)| = \left| \int_0^1 \frac{w(tz)}{t^{n+1}} [(n+1) + (c-n)t^{c+1}] dt \right| \leq \int_0^1 \frac{|w(tz)|}{t^{n+1}} [(n+1) + (c-n)t^{c+1}] dt \leq \frac{(n+2)(c+1)}{c+2} |z|^{n+1}, \quad z \in \Delta. \quad (3.14)$$

Since $F(z) \ll G(z)$, from (1.5) and (1.6) we obtain that

$$F(z) = \varphi(z)G(z), \quad z \in \Delta,$$

and differentiating the above relation we have

$$F'(z) = \varphi'(z)G(z) + \varphi(z)G'(z), \quad z \in \Delta. \quad (3.15)$$

Thus, noting that $\varphi \in \mathcal{B}_n$ satisfies the inequality (2.9), by (3.12) and (3.15) we deduce that

$$|F'(z)| \leq \left[\frac{1 - |\varphi(z)|^2}{1 - |z|^2} \frac{\left[1 + |c_{n+1}| |z|^n + \frac{\lambda n(c+1-n)}{c+1} |W_1(z)| \right] |z|}{1 - 2|c_{n+1}| |z|^n - \frac{\lambda(c+1-n)}{c+1} [(n-1)|W_1(z)| + |W_2(z)|]} + |\varphi(z)| \right] |G'(z)|,$$

for all $z \in \Delta$. Using (3.13) and (3.14), the above inequality gives that

$$|F'(z)| \leq \left[\frac{1 - |\varphi(z)|^2}{1 - |z|^2} \frac{\left[1 + |c_{n+1}| |z|^n + \frac{\lambda n(c+1-n)}{c+2} |z|^{n+1} \right] |z|}{1 - 2|c_{n+1}| |z|^n - \frac{\lambda(c+1-n)}{c+2} (2n+1)|z|^{n+1}} + |\varphi(z)| \right] |G'(z)|,$$

that is

$$|F'(z)| \leq \frac{\Theta(|z|, |\varphi(z)|)}{(1 - |z|^2) \left[1 - 2|c_{n+1}| |z|^n - \frac{\lambda(c+1-n)}{c+2} (2n+1)|z|^{n+1} \right]} |G'(z)|, \quad z \in \Delta,$$

where

$$\begin{aligned} \Theta(|z|, |\varphi(z)|) := & (1 - |z|^2) \left[1 - 2|c_{n+1}| |z|^n - \frac{\lambda n(c+1-n)}{c+2} (2n+1)|z|^{n+1} \right] |\varphi(z)| \\ & + (1 - |\varphi(z)|^2) \left[1 + |c_{n+1}| |z|^n + \frac{\lambda n(c+1-n)}{c+2} |z|^{n+1} \right] |z|. \end{aligned}$$

Upon setting $|z| = r$, $0 \leq r < 1$, and $|\varphi(z)| = \rho$, $0 \leq \rho \leq 1$, this inequality implies that

$$|F'(z)| \leq \frac{\Theta(r, \rho)}{(1 - r^2) \left[1 - 2|c_{n+1}| r^n - \frac{\lambda(c+1-n)}{c+2} (2n+1)r^{n+1} \right]} |G'(z)|, \quad z \in \Delta. \quad (3.16)$$

According to the assumption (3.3), we remark that the denominator of the right-hand side fraction is positive, for all $r \in [0, 1)$.

If we denote

$$\Psi(r, \rho) := \frac{\Theta(r, \rho)}{(1-r^2) \left[1 - 2|c_{n+1}|r^n - \frac{\lambda(c+1-n)}{c+2}(2n+1)r^{n+1} \right]},$$

then (3.16) becomes

$$|F'(z)| \leq \Psi(r, \rho) |G'(z)|, \quad z \in \Delta.$$

From the above relation, in order to prove our result, we need to determine

$$r_2 = \max \{r \in [0, 1] : \Psi(r, \rho) \leq 1, \forall \rho \in [0, 1]\} = \max \{r \in [0, 1] : \varphi(r, \rho) \geq 0,$$

$\forall \rho \in [0, 1]\}$, where

$$\begin{aligned} \varphi(r, \rho) := & (1-r^2) \left[1 - 2|c_{n+1}|r^n - \frac{\lambda(c+1-n)}{c+2}(2n+1)r^{n+1} \right] (1-\rho) \\ & - (1-\rho^2) \left[1 + |c_{n+1}|r^n + \frac{\lambda n(c+1-n)}{c+2}r^{n+1} \right] r. \end{aligned}$$

A simple calculation shows that the inequality $\varphi(r, \rho) \geq 0$ is equivalent to

$$\begin{aligned} u(r, \rho) := & (1-r^2) \left[1 - 2|c_{n+1}|r^n - \frac{\lambda(c+1-n)}{c+2}(2n+1)r^{n+1} \right] \\ & - \left[1 + |c_{n+1}|r^n + \frac{\lambda n(c+1-n)}{c+2}r^{n+1} \right] r(1+\rho) \geq 0. \end{aligned}$$

Obviously, the function $u(r, \rho)$ takes its minimum value at $\rho = 1$, i.e.

$$\min \{u(r, \rho) : \rho \in [0, 1]\} = u(r, 1) =: v(r),$$

where

$$\begin{aligned} v(r) = & \frac{\lambda(c+1-n)}{c+2}(2n+1)r^{n+3} + 2n(c+1-n) \left(\frac{|c_{n+1}|}{c+1} - \frac{\lambda}{c+2} \right) r^{n+2} \\ & - (c+1-n) \left(\frac{(2n+1)\lambda}{c+2} + \frac{2n|c_{n+1}|}{c+1} \right) r^{n+1} \\ & - \frac{2n(c+1-n)}{c+1} |c_{n+1}|r^n - r^2 - 2r + 1. \end{aligned}$$

Since $v(0) = 1 > 0$ and $v(1) < 0$, it follows that $v(r) \geq 0$ for all $r \in [0, r_2]$, where $r_2 := r_2(\lambda, n)$, is the smallest positive root of the equation (3.1), which completes our proof. \square

For $n = 1$, Theorem 3.1 reduces to the following result:

Corollary 3.2. For $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, such that $g \in \mathcal{U}(\lambda)$, where $0 < \lambda \leq 1$, let $G = I_{c,1}(g)$, where $I_{c,1}$ is defined by

$$I_{c,1}(f)(z) = \frac{c}{z^{c-1}} \int_0^z \frac{t^c}{f(t)} dt, \quad c > 0.$$

Let the function $f \in \mathcal{A}$, such that $F = I_{c,1}(f)$ is majorized by G , i.e. $F(z) \ll G(z)$, and

$$\frac{2c|b_2|}{c+1} + \frac{3\lambda c}{c+2} \leq 1.$$

Then,

$$|F'(z)| \leq |G'(z)|, \text{ for } |z| \leq \rho_2(\lambda),$$

where $\rho_2(\lambda) := r_2(\lambda, 1)$ is the smallest positive real root of the equation

$$\begin{aligned} \frac{3\lambda c}{c+2} r^4 + 2c \left[\frac{|b_2|c}{(c+1)^2} - \frac{\lambda}{c+2} \right] r^3 - \left[\frac{3\lambda c}{c+2} + 2 \left(\frac{c}{c+1} \right)^2 |b_2| + 1 \right] r^2 \\ - 2 \left[\left(\frac{c}{c+1} \right)^2 |b_2| + 1 \right] r + 1 = 0. \end{aligned}$$

Conflict of Interests. The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgments

The first author (S. P. G) is thankful to CSIR, New Delhi, India, for awarding Emeritus Scientistship, under scheme number 21(084)/10/EMR-II.

References

1. O. Altinas, Majorization for certain analytic functions, *Eur. J. Pure Appl. Math.*, **5**(1)(2012), 16–23.
2. O. Altinas, Ö. Özkan and H. M. Srivastava, Majorization by starlike functions of complex order, *Complex Var.*, **46**(2001), 207–218.
3. P. L. Duren, *Univalent functions*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
4. A. W. Goodman, *Univalent functions*, Vol. I and Vol. II, Mariner Publishing Company, Tampa, Florida, 1983.
5. S. P. Goyal and P. Goswami, Majorization for certain classes of analytic functions defined by fractional derivatives, *Appl. Math. Lett.*, **22**(12)(2009), 1855–1858.
6. S. P. Goyal and P. Goswami, Majorization for certain classes of meromorphic functions defined by integral operator, *Annales. Univ. Mariae Curie-Skłodowska*, **LXVI**(2)(2012), 57–62.
7. J. K. Prajapat and M. K. Aouf, Majorization problem for certain class of p -valently analytic function defined by generalized fractional differintegral operator, *Comput. Math. Appl.*, **63**(1)(2012), 42–47.

8. K. Kuroki, T. Hayami, N. Uyanik and S. Owa, Some properties for a certain class concerned with univalent functions, *Comput. Math. Appl.*, **63**(2012), 1425–1432.
9. S.-H. Li, H. Tang and E. Ao, Majorization properties for certain new classes of analytic functions using the Salagean operator, *J. Ineq. Appl.* 2013, **2013**:86.
10. T. H. MacGregor, Majorization by univalent functions, *Duke Math. J.*, **34**(1967), 95–102.
11. Z. Nehari, *Conformal Mappings*, Mc Graw-Hill, New York (1952).
12. M. Obradović, A class of univalent functions, *Hokkaido Math. J.*, **27**(1998), 329–335.
13. M. Obradović and S. Ponnusamy, New criteria and distortion theorems for univalent functions, *Complex Variables: Theory and Appl.*, **44**(2001), 173–191.
14. M. Obradović and S. Ponnusamy, Univalence and starlikeness of certain integral transforms defined by convolution of analytic functions, *J. Math. Anal. Appl.*, **336**(2007), 758–767.
15. S. Ponnusamy, Starlikeness criteria for a certain class of analytic functions, *Appl. Math. Lett.*, **24**(2011), 381–386.
16. S. Ponnusamy and P. Sahoo, Special classes of univalent functions with missing coefficients and integral transforms, *Bull. Malays. Math. Sci. Soc.*, **28**(2)(2005), 141–156.
17. M. S. Robertson, Quasi-subordination and coefficient conjectures, *Bull. Amer. Math. Soc.*, **76**(1970), 1–9.

Som P. Goyal
Department of Mathematics,
University of Rajasthan, Jaipur-302055, India.
E-mail address: somprg@gmail.com

and

Rakesh Kumar (Corresponding author)
Department of Mathematics,
Amity University Rajasthan, NH-11C, Jaipur-302002, India.
E-mail address: rkyadav11@gmail.com

and

Teodor Bulboacă
Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 400084 Cluj-Napoca, Romania.
E-mail address: bulboaca@math.ubbcluj.ro