Bol. Soc. Paran. Mat. (3s.) **v. 33** 2 (2015): 203–217.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v33i2.24307

Existence and multiplicity of solutions for a $p(x)$-Kirchhoff type problems

El Miloud Hssini, Mohammed Massar and Najib Tsouli

ABSTRACT: This paper is concerned with the existence and multiplicity of solutions for a class of $p(x)$-Kirchhoff type equations with Neumann boundary condition. Our technical approach is based on variational methods.

Key Words: Variational methods, p(x)-Kirchhoff type equation, nonlocal problems.

Contents

1. Introduction

In this work, we study the existence and multiplicity of solutions for the nonlocal elliptic problem under Neumann boundary condition:

$$\begin{cases} -M(t)\left(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - a(x)|u|^{p(x)-2}u\right) = \lambda f(x,u) & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \geq 2)$, with smooth boundary, $\dfrac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $a \in L^\infty(\Omega)$, with $\operatorname{ess\,inf}_\Omega a > 0$, $\lambda > 0$ and $p(x) \in C_+(\overline{\Omega})$ with

$$N < p^- := \inf_{\overline{\Omega}} p(x) \leq p^+ := \sup_{\overline{\Omega}} p(x) < +\infty.$$

In the statement of problem (1.1), $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an Carathéodory function and $M(t)$ is a continuous function with $t := \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\right) dx$.

The $p(x)$-Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator, mainly due to the fact that it is not homogeneous. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years, we can for example refer to [1,4,17,24,29,35]. This great interest may be justified by their various physical

2000 *Mathematics Subject Classification:* 35D05, 35D10, 35J60

applications. In fact, there are applications concerning elastic mechanics [41], electrorheological fluids [38,39], image restoration [13], dielectric breakdown, electrical resistivity and polycrystal plasticity [7,8] and continuum mechanics [5].

As it is well know, problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff [36]. More precisely, Kirchhoff introduced a model given by the following equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where ρ, ρ_0, h, E, L are constants,which extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. A distinguishing feature of the Kirchhoff equation is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ which depends on the average $\frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ of the kinetic energy $\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2$ on $[0, L]$, and hence the equation is no longer a point wise identity. On the othere hand, stationary counterpart of (1.2) is given as

$$\begin{cases} \left(a + b \int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x, u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

which is has attracted much attention after Lions's paper [31], where a functional analysis frame work for the problem was proposed; see, e.g., [6,12,16] for some interesting results. Moreover, nonlocal problems like

$$- M \left(\int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x, u) \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega, \tag{1.4}$$

can be used for modeling several physical and biological systems where u describes a process which depends on the average of it self, such as the population density, see [3]. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian

$$-M \left(\int_\Omega |\nabla u|^p dx \right) \Delta_p u = f(x, u) \text{ in } \Omega,$$

see, e.g., [11,26,30]. In [11], the authors present several sufficient conditions for the existence of positive solutions to a class of nonlocal boundary value problems of the p-Kirchhoff type equation. However, to our knowledge, there is not a great number of papers which have dealt with nonlocal $p(x)$-Laplacian equations. We refer the reader to [14,18,19,20,34] and the references therein for an overview on this subject.

Our aim is to establish the existence and multiplicity results for problem (1.1) through variational methods. First we will exploit a critical point theorem by Bonanno ([9], Theorem 5.1) which provides for the existence of a local minima for a parameterized abstract functional, and a classical theorem of Ambrosetti-Rabinowitz, to guarantee that (1.1) has at least two distinct nontrivial weak solutions (Theorem 3.1). Next, we will get the existence of a nontrivial solution of the problem (1.1) where the nonlinearity $f(x,u)$ does not satisfy Ambrosetti-Rabinowitz condition (Theorem 3.2), by employing a local minimum theorem ([9], Theorem 5.3). These results can be viewed as generalizations to the nonlocal and variable exponent space setting of some results obtained in [10,33].

2. Preliminaries

Our main tools are two consequences of a local minimum theorem [9, Theorem 3.1] which are recalled below. Given X a set and two functionals $\Phi, \Psi : X \to \mathbb{R}$, put

$$\beta(r_1,r_2) = \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u)-\Psi(v)}{r_2-\Phi(v)}, \tag{2.1}$$

$$\rho_1(r_1,r_2) = \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v)-\sup_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u)}{\Phi(v)-r_1}, \tag{2.2}$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, and

$$\rho_2(r) = \sup_{v\in\Phi^{-1}(]r,+\infty[)} \frac{\Psi(v)-\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{\Phi(v)-r}, \tag{2.3}$$

for all $r \in \mathbb{R}$.

Theorem 2.1 ([9], Theorem 5.1)**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1,r_2) < \rho_1(r_1,r_2), \tag{2.4}$$

where β and ρ_1 are given by (2.1) and (2.2). Then, for each $\lambda \in \left]\frac{1}{\rho_1(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1,r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1,r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 ([9], Theorem 5.3)**.** *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that*

$$\rho_2(r) > 0, \tag{2.5}$$

where ρ_2 is given by (2.3), and for each $\lambda > \frac{1}{\rho_2(r)}$ the function $I_\lambda = \Phi - \lambda\Psi$ is coercive. Then, for each $\lambda > \frac{1}{\rho_2(r)}$ there is $u_{0,\lambda} \in \Phi^{-1}(]r,+\infty[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r,+\infty[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

In the sequel, let $p(x) \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \left\{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\right\}.$$

The variable exponent Lebesgue space is defined by

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable and} \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\sigma > 0 : \int_\sigma |\frac{u(x)}{\sigma}|^{p(x)}\, dx \leq 1\},$$

and the variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.3 ([27,28]). *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, uniformly convex, reflexive Banach spaces. The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $q(x)$ is the conjugate function of $p(x)$; i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}.$$

Proposition 2.4 ([27,28]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p^*(x)$ ($r(x) < p^*(x)$) for all $x \in \overline{\Omega}$, there is a continuous (compact) embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega),$$

where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Now, for any $u \in X := W^{1,p(x)}(\Omega)$ define

$$\|u\|_a := \inf\Big\{\sigma > 0 : \int_\Omega \big(|\frac{\nabla u(x)}{\sigma}|^{p(x)} + a(x)|\frac{u(x)}{\sigma}|^{p(x)}\big)dx \leq 1\Big\}.$$

Since $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a > 0$, we see that $\|.\|_a$ is a norm on X equivalent to $\|.\|_{W^{1,p(x)}(\Omega)}$. Now, we introduce the modular $\rho : X \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega (|\nabla u|^{p(x)} + a(x)|u|^{p(x)})dx$$

for all $u \in X$. Here, we give some relations between the norm $||.||_a$ and the modular ρ.

Proposition 2.5 ([27]). *For $u \in X$ we have*

(i) $\|u\|_a < 1(= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$;

(ii) If $\|u\|_a < 1 \Rightarrow \|u\|_a^{p^+} \leq \rho(u) \leq \|u\|_a^{p^-}$;

(iii) If $\|u\|_a > 1 \Rightarrow \|u\|_a^{p^-} \leq \rho(u) \leq \|u\|_a^{p^+}$.

Now, let

$$k := \max\left\{ \sup_{u \in X\setminus\{0\}} \frac{\max\limits_{x\in\overline{\Omega}}|u(x)|}{||u||_a} \right\}. \tag{2.6}$$

It is well know that $X \hookrightarrow W^{1,p^-}(\Omega)$ is a continuous embedding, and the embedding $W^{1,p^-}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact when $N < p^-$. So we obtain the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $N < p^-$, and hence $k < \infty$. If Ω is convex, an explicit upper bound for the constant k is

$$k \leq 2^{\frac{p^- - 1}{p^-}} \max\left\{ \left(\frac{1}{\|a\|_1}\right)^{\frac{1}{p^-}}, \frac{d}{N^{\frac{1}{p^-}}} \left(\frac{p^- - 1}{p^- - N}|\Omega|\right)^{\frac{p^- - 1}{p^-}} \frac{\|a\|_\infty}{\|a\|_1} \right\} (1 + |\Omega|),$$

where $\|a\|_1 := \int_\Omega a(x)dx$, $\|a\|_\infty := \sup_{x\in\Omega} a(x)$, $d := \operatorname{diam}(\Omega)$ and $|\Omega|$ is the Lebesgue measure of Ω (see [23]).

Hereafter, we state the assumptions on $M(t)$ and $f(x,t)$:

(M_0) $M(t) : \mathbb{R} \to (m_0, +\infty)$ is a continuous and increasing function, with $m_0 > 0$.

(M_1) there exists $0 < \theta < 1$ such that

$$\widehat{M}(t) \geq (1-\theta)M(t)t \quad \text{for all } t \geq 0.$$

(f_0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition and there exists $c > 0$ such that

$$|f(x,t)| \leq c\left(1 + |t|^{\alpha(x)-1}\right) \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

where $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$ for all $x \in \Omega$.

(f_1) there exist two constants $\mu > \frac{p^+}{1-\theta}$ and $R > 0$ such that

$$0 < \mu F(x,s) \leq s f(x,s) \text{ for all } x \in \Omega \text{ and for all } |s| \geq R,$$

where θ is given in (M_1).

Definition 2.6. *We say that $u \in X$ is a weak solution of problem (1.1) if*

$$M\left(\int_\Omega \frac{|\nabla u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)} dx\right) \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + a(x)|u|^{p(x)-2}uv\right) dx$$
$$-\lambda \int_\Omega f(x,u) v dx = 0,$$

for all $v \in X$.

We introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$, defined by

$$\Phi(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)} dx\right), \quad \Psi(u) = \int_\Omega F(x,u) dx, \tag{2.7}$$

for all $u \in X$, where

$$\widehat{M}(t) = \int_0^t M(s) ds, \text{ for all } t \geq 0,$$
$$F(x,t) = \int_0^t f(x,\xi) d\xi, \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

It is well known that Φ and Ψ are well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u \in X$ are given by

$$\langle \Phi'(u), v\rangle = M\left(\int_\Omega \frac{|\nabla u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)} dx\right)\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + a(x)|u|^{p(x)-2}uv\right) dx$$
$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u) v\, dx,$$

for all $v \in X$.

We need the following theorem in the proofs of our main results.

Theorem 2.7 ([21], Theorem 2.1)**.** *If (M_0) holds, then*

(i) *Φ is weakly lower semicontinuous.*

(ii) *Φ' is strictly monotone*

(iii) *Φ' is of (S_+) type, namely*

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle \leq 0 \text{ implies } u_n \to u.$$

(iv) *Φ' admits a continuous inverse on X^*.*

3. Main results

In order to introduce our result, given two positive constants c and d with

$$\frac{p^-}{p^+}\left(\frac{c}{k}\right)^{p^-} \neq ||a||_1 d^{p^+}.$$

Set

$$\mathcal{A}_d(c) := \frac{\int_\Omega \max_{|\xi|\leq\sigma(c)} F(x,\xi)dx - \int_\Omega F(x,d)dx}{\widehat{M}\left(\frac{1}{p^+}\left(\frac{c}{k}\right)^{p^-}\right) - \widehat{M}\left(\frac{1}{p^-}\|a\|_1 d^{p^+}\right)},$$

where

$$\sigma(c) := k\left[\frac{p^+}{m_0}\widehat{M}\left(\frac{1}{p^+}\left(\frac{c}{k}\right)^{p^-}\right)\right]^{\frac{1}{p^-}}$$

and k is given by (2.6).

Theorem 3.1. *If (f_0), (f_1), (M_0) and (M_1) hold, and there exist three constants $c_1 \geq k$, $c_2 \geq k$ and $d \geq 1$ with*

$$\left(\frac{c_1}{k}\right)^{p^-} < ||a||_1 d^{p^-} \leq ||a||_1 d^{p^+} < \frac{p^-}{p^+}\left(\frac{c_2}{k}\right)^{p^-} \tag{3.1}$$

such that

$$\mathcal{A}_d(c_2) < \mathcal{A}_d(c_1).$$

Then, for each $\lambda \in \left]\frac{1}{\mathcal{A}_d(c_1)}, \frac{1}{\mathcal{A}_d(c_2)}\right[$, problem (1.1) admits at least two nontrivial weak solutions $\overline{u}_1$ and $\overline{u}_2$ such that $\frac{p^-}{p^+}\left(\frac{c_1}{k}\right)^{p^-} < \rho(\overline{u}_1) < \left(\frac{c_2}{k}\right)^{p^-}$.

Proof: Let Φ, Ψ be the functionals defined in (2.7). Since $p^- > 1$, for each $u \in X$ such that $\|u\|_a \geq 1$ we have

$$\frac{\langle\Phi(u),u\rangle}{\|u\|_a} \geq \frac{m_0}{p^+}\frac{\rho(u)}{\|u\|_a} \geq \frac{m_0}{p^+}\|u\|_a^{p^- -1} \to \infty \quad \text{as } \|u\|_a \to \infty.$$

So, Φ is a coercive. From Theorem 2.7, of course, Φ' admits a continuous inverse on X^*, moreover, Ψ has a compact derivative, it results sequentially weakly continuous. Hence Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.1 and that the critical points of the functional $\Phi - \lambda\Psi$ in X are exactly the weak solutions of problem (1.1). So, our aim is to verify condition (2.4) of Theorem 2.1. To this end, let $u_0(x) = d$ for all $x \in \overline{\Omega}$, and put

$$r_1 = \widehat{M}\left(\frac{1}{p^+}\left(\frac{c_1}{k}\right)^{p^-}\right) \quad \text{and} \quad r_2 = \widehat{M}\left(\frac{1}{p^+}\left(\frac{c_2}{k}\right)^{p^-}\right).$$

Clearly $u_0 \in X$, and

$$\Psi(u_0) = \int_\Omega F(x,u_0)dx = \int_\Omega F(x,d)dx, \tag{3.2}$$

$$\Phi(u_0) = \widehat{M}\left(\int_\Omega \frac{a(x)|u_0|^{p(x)}}{p(x)}dx\right).$$

Then, in virtu of the strict monotonicity of $\widehat{M}$, we get

$$\widehat{M}\left(\frac{\|a\|_1 d^{p^-}}{p^+}\right) \le \Phi(u_0) \le \widehat{M}\left(\frac{\|a\|_1 d^{p^+}}{p^-}\right).$$

Hence, it follows from (3.1) that

$$r_1 < \Phi(u_0) < r_2. \tag{3.3}$$

Now, let $u \in X$ such that $u \in \Phi^{-1}]-\infty, r_2[$. By (M_0) and Proposition 2.5, we obtain

$$\min\left\{\|u\|_a^{p^+}, \|u\|_a^{p^-}\right\} < \frac{r_2 p^+}{m_0}.$$

Then

$$\|u\|_a < \max\left\{\left(\frac{r_2 p^+}{m_0}\right)^{\frac{1}{p^+}}, \left(\frac{r_2 p^+}{m_0}\right)^{\frac{1}{p^-}}\right\},$$

the fact that $c_2 \ge k$, we get $\frac{r_2 p^+}{m_0} \ge 1$ and

$$\|u\|_a < \left(\frac{r_2 p^+}{m_0}\right)^{\frac{1}{p^-}}.$$

This together with (2.6), yields

$$|u(x)| \le k\|u\|_a < k\left(\frac{r_2 p^+}{m_0}\right)^{\frac{1}{p^-}} = \sigma(c_2) \text{ for all } x \in \Omega. \tag{3.4}$$

So

$$\Psi(u) = \int_\Omega F(x,u)dx \le \int_\Omega \max_{|\xi| \le \sigma(c_2)} F(x,\xi)dx,$$

for all $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r_2[)$. Thus

$$\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) \le \int_\Omega \max_{|\xi| \le \sigma(c_2)} F(x,\xi)dx. \tag{3.5}$$

On the other hand, arguing as before we obtain

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \le \int_\Omega \max_{|\xi| \le \sigma(c_1)} F(x,\xi)dx. \tag{3.6}$$

In view of (3.2)-(3.3) and (3.5)-(3.6), one has

$$\begin{aligned}
\beta(r_1, r_2) &\le \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) - \Psi(u_0)}{r_2 - \Phi(u_0)} \\
&\le \frac{\int_\Omega \max_{|\xi| \le \sigma(c_2)} F(x,\xi)dx - \int_\Omega F(x,d)dx}{\widehat{M}\left(\frac{1}{p^+}\left(\frac{c_2}{k}\right)^{p^-}\right) - \widehat{M}\left(\frac{\|a\|_1}{p^-}d^{p^+}\right)} \\
&= \mathcal{A}_d(c_2)
\end{aligned}$$

and

$$
\begin{aligned}
\rho_1(r_1,r_2) &\geq \frac{\Psi(u_0)-\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{\Phi(u_0)-r_1} \\
&\geq \frac{\int_\Omega \max_{|\xi|\leq\sigma(c_1)} F(x,\xi)dx-\int_\Omega F(x,d)dx}{\widehat{M}\left(\frac{1}{p^+}\left(\frac{c_1}{k}\right)^{p^-}\right)-\widehat{M}\left(\frac{\|a\|_1}{p^-}d^{p^+}\right)} \\
&= \mathcal{A}_d(c_1).
\end{aligned}
$$

So, by our assumption it follows that

$$\beta(r_1,r_2)<\rho_1(r_1,r_2).$$

Hence, from Theorem 2.1 for each $\lambda\in\left]\frac{1}{\mathcal{A}_d(c_1)},\frac{1}{\mathcal{A}_d(c_2)}\right[\subset\left]\frac{1}{\rho_1(r_1,r_2)},\frac{1}{\beta(r_1,r_2)}\right[$, the functional $I_\lambda:=\Phi-\lambda\Psi$ admits at least one critical point $\overline{u}_1$ such that $r_1<\Phi(\overline{u}_1)<r_2$. Therefore

$$\frac{p^-}{p^+}\left(\frac{c_1}{k}\right)^{p^-}<\rho(\overline{u}_1)<\left(\frac{c_2}{k}\right)^{p^-}.$$

Now we prove the existence of the second local minimum distinct from the first one. To this purpose, we verify the hypotheses of the mountain pass theorem for the functional I_λ. Clearly I_λ is of class C^1 and $I_\lambda(0)=0$. The first part of proof guarantees that $\overline{u}_1\in X$ is a local nontrivial local minimum for I_λ in X. Therefore there is $\varrho>0$ such that

$$\inf_{\|u-\overline{u}_1\|_a=\varrho} I_\lambda(u)\geq I_\lambda(\overline{u}_1),$$

so condition [37, (I_1), Theorem 2.2] is verified. From condition (f_1), by standard computations, there is a positive constant c_1 such that

$$F(x,s)\geq c_1|s|^\mu. \tag{3.7}$$

By integrating (M_1), we get

$$\widehat{M}(t)\leq\frac{\widehat{M}(t_0)}{t_0^{\frac{1}{1-\theta}}}t^{\frac{1}{1-\theta}}=c_2t^{\frac{1}{1-\theta}}\quad\text{for all } t\geq t_0>0. \tag{3.8}$$

Hence, from (3.7) and (3.8), for $u\in X\backslash\{0\}$ and $t>1$, we obtain

$$
\begin{aligned}
I_\lambda(tu) &\leq \widehat{M}\left(\int_\Omega\frac{|t\nabla u|^{p(x)}+a(x)|tu|^{p(x)}}{p(x)}dx\right)-\lambda\int_\Omega F(x,tu(x))dx \\
&\leq c_3\left(\int_\Omega\left(|t\nabla u|^{p(x)}+a(x)|tu|^{p(x)}\right)dx\right)^{\frac{1}{1-\theta}}-c_1\lambda t^\mu\int_\Omega|u(x)|^\mu dx \\
&\leq c_3t^{\frac{p^+}{1-\theta}}\left(\int_\Omega\left(|\nabla u|^{p(x)}+a(x)|u|^{p(x)}\right)dx\right)^{\frac{1}{1-\theta}} \\
&\quad -c_1\lambda t^\mu\int_\Omega|u(x)|^\mu dx\to-\infty
\end{aligned}
$$

as $t \to \infty$, since $\mu > \frac{p^+}{1-\theta}$. So the condition [37, (I_2), Theorem 2.2] is verified.

Now, we verify that I_λ satisfies the (PS)-condition. To this end, suppose that $(u_n) \subset X$ is a (PS)-sequence; i.e., there is $M > 0$ such that

$$\sup |I_\lambda(u_n)| \leq M, \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to +\infty.$$

Let us show that (u_n) is bounded in X. Using hypothesis (f_1) and (M_1), for n large enough, we have

$$\begin{aligned}
M + \|u_n\|_a &\geq I_\lambda(u_n) - \frac{1}{\mu}\langle I'_\lambda(u_n), u_n\rangle \\
&= \widehat{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}}{p(x)} dx\right) - \lambda \int_\Omega F(x, u_n) dx \\
&\quad - \frac{1}{\mu} M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}}{p(x)} dx\right) \\
&\quad \int_\Omega \left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right) dx + \lambda \int_\Omega \frac{1}{\mu} f(x, u_n) u_n dx \\
&\geq \left(\frac{1-\theta}{p^+} - \frac{1}{\mu}\right) M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}}{p(x)} dx\right) \\
&\quad \int_\Omega \left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right) dx \\
&\quad + \lambda \int_\Omega \left(\frac{1}{\mu} f(x, u_n) u_n - F(x, u_n)\right) dx \\
&\geq m_0 \left(\frac{1-\theta}{p^+} - \frac{1}{\mu}\right) \|u_n\|_a^{p^-} + \lambda \int_\Omega \left(\frac{1}{\mu} f(x, u_n) u_n - F(x, u_n)\right) dx \\
&\geq m_0 \left(\frac{1-\theta}{p^+} - \frac{1}{\mu}\right) \|u_n\|_a^{p^-} - c_4.
\end{aligned}$$

Since $\mu > \frac{p^+}{1-\theta}$, (u_n) is bounded, for a subsequence still denoted (u_n), we can assume that $u_n \rightharpoonup u$ in X, then $\langle I'_\lambda(u_n), u_n - u\rangle \to 0$. Thus, we have

$$\begin{aligned}
&M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}}{p(x)} dx\right) \\
&\int_\Omega \left(|\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) + a(x)|u_n|^{p(x)-2} u_n (u_n - u)\right) dx \\
&- \int_\Omega f(x, u_n)(u_n - u)\, dx \to 0.
\end{aligned}$$

From (f_0) and Proposition 2.3, we get that $\int_\Omega f(x, u_n)(u_n - u)\, dx \to 0$. therefore, one has

$$M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}+a(x)|u_n|^{p(x)}}{p(x)}dx\right)$$
$$\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)+a(x)|u_n|^{p(x)-2}u_n(u_n-u)\right)dx \to 0.$$

In view of condition (M_0), we obtain

$$\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)+a(x)|u_n|^{p(x)-2}u_n(u_n-u)\right)dx \to 0.$$

We write

$$J(u) := \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)}+a(x)|u|^{p(x)}\right)dx.$$

Using Theorem 2.7, the mapping $J' : X \to X^*$ is of (S_+) type. Then we have $u_n \to u$. Consequently, the classical theorem of Ambrosetti and Rabinowitz ensures a critical point $\overline{u}_2$ such that $I_\lambda(\overline{u}_2) > I_\lambda(\overline{u}_1)$. So $\overline{u}_1$ and $\overline{u}_2$ are distinct weak solutions of the problem, and the proof of Theorem 3.1 is achieved. □

Corollary 3.2. *Assume that $f(x,s)=\alpha(x)g(s)$ for all $(x,s)\in\Omega\times\mathbb{R}$, where $\alpha\in L^1(\Omega)$ such that $\alpha\geq 0$ a.e. $x\in\Omega$, $\alpha\not\equiv 0$, and $g:\mathbb{R}\to\mathbb{R}$ be a nonnegative continuous function. If (f_0), (f_1), (M_0) and (M_1) hold, and there exist three constants $c_1\geq k, c_2\geq k$ and $d\geq 1$ such that (3.1) and*

$$\frac{G(c_2)-G(d)}{\widehat{M}\left(\frac{1}{p^+}\left(\frac{c_2}{k}\right)^{p^-}\right)-\widehat{M}\left(\frac{1}{p^-}\|a\|_1 d^{p^+}\right)} < \frac{G(c_1)-G(d)}{\widehat{M}\left(\frac{1}{p^+}\left(\frac{c_1}{k}\right)^{p^-}\right)-\widehat{M}\left(\frac{1}{p^-}\|a\|_1 d^{p^+}\right)}$$

Then, for each

$$\lambda \in \left]\frac{\widehat{M}\left(\frac{1}{p^+}\left(\frac{c_1}{k}\right)^{p^-}\right)-\widehat{M}\left(\frac{1}{p^-}\|a\|_1 d^{p^+}\right)}{G(c_1)-G(d)}, \frac{\widehat{M}\left(\frac{1}{p^+}\left(\frac{c_2}{k}\right)^{p^-}\right)-\widehat{M}\left(\frac{1}{p^-}\|a\|_1 d^{p^+}\right)}{G(c_2)-G(d)}\right[,$$

the problem

$$-M\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)}+a(x)|u|^{p(x)}\right)dx\right)\left(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)-a(x)|u|^{p(x)-2}u\right)$$
$$=\lambda\alpha(x)g(u) \text{ in } \Omega$$
$$\frac{\partial u}{\partial \nu}=0 \text{ on } \partial\Omega,$$

admits at least two nonnegative weak solutions.

Proof: Clearly, one has $F(x,s)=\alpha(x)G(s)$ for all $(x,s)\in\Omega\times\mathbb{R}$. Therefore, taking into account that G is a nondecreasing function, one has

$$\begin{aligned}\mathcal{A}_d(c_2) &= \|\alpha\|_1 \frac{G(c_2)-G(d)}{\widehat{M}\left(\frac{1}{p^+}\left(\frac{c_2}{k}\right)^{p^-}\right)-\widehat{M}\left(\frac{1}{p^-}\|a\|_1 d^{p^+}\right)} \\ &< \|\alpha\|_1 \frac{G(c_1)-G(d)}{\widehat{M}\left(\frac{1}{p^+}\left(\frac{c_1}{k}\right)^{p^-}\right)-\widehat{M}\left(\frac{1}{p^-}\|a\|_1 d^{p^+}\right)} = \mathcal{A}_d(c_1).\end{aligned}$$

Therefore, Theorem 3.1 ensure the existence of at last two solutions, and by standard argument we see that they are nonnegative. □

Finally, we give an application of Theorem 2.2.

Theorem 3.3. *If (M_0) and (f_0) hold, and there exist two constants $\overline{c}$ and $\overline{d}\geq 1$ with*

$$1\leq\left(\frac{\overline{c}}{k}\right)^{p^-}<||a||_1\overline{d}^{p^-}$$

such that

$$\int_\Omega \max_{|\xi|\leq\sigma(\overline{c})} F(x,\xi)dx<\int_\Omega F(x,\overline{d})dx \tag{3.9}$$

and

$$\limsup_{|\xi|\to+\infty}\frac{F(x,\xi)}{|\xi|^{p^-}}\leq 0 \text{ uniformly in } x. \tag{3.10}$$

Then, for each

$$\lambda\in\overline{\Lambda}:=\left]\frac{\widehat{M}\left(\frac{1}{p^+}\left(\frac{\overline{c}}{k}\right)^{p^-}\right)-\widehat{M}\left(\frac{1}{p^-}\|a\|_1 d^{p^+}\right)}{\int_\Omega \max_{|\xi|\leq\sigma(\overline{c})}F(x,\xi)dx-\int_\Omega F(x,\overline{d})dx},+\infty\right[,$$

problem (1.1) admits at least one nontrivial weak solution $\overline{u}$ such that $\rho(\overline{u})>\frac{p^-}{p^+}\left(\frac{\overline{c}}{k}\right)^{p^-}$.

Proof: The functionals Φ and Ψ given by (2.7) satisfy all regularity assumptions requested in Theorem 2.2. By (3.10) and (f_0), for every $\varepsilon>0$, we get

$$F(x,\xi)\leq\varepsilon|\xi|^{p^-}+l_\varepsilon(x) \text{ for all } (x,\xi)\in\Omega\times\mathbb{R}, \tag{3.11}$$

where $l_\varepsilon\in L^1(\Omega)$. This implies that

$$\int_\Omega F(x,u)dx\leq\varepsilon c_5||u||_a^{p^-}+\int_\Omega l_\varepsilon(x)dx \text{ for all } u\in X,$$

where c_5 is a constant of Sobolev. Therefore, choosing $0 < \varepsilon < \frac{m_0}{c_5 p^+}$, from (3.11) and Proposition 2.5, we obtain

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) \geq \left(\frac{m_0}{p^+} - \varepsilon c_5\right) ||u||_a^{p^-} - \int_\Omega l_\varepsilon(x)dx.$$

for all $u \in X$ such that $||u||_a \geq 1$. So, I_λ is coercive. To apply Theorem 2.2, it suffices to verify condition (2.5). Indeed, put

$$r = \widehat{M}\left(\frac{1}{p^+}\left(\frac{\overline{c}}{k}\right)^{p^-}\right) \quad \text{and} \quad u_0(x) = \overline{d} \;\text{ for all } x \in \Omega.$$

Arguing as in the proof of Theorem 3.1 we obtain

$$\rho_2(r) \geq \frac{\int_\Omega \max_{|\xi|\leq\sigma(\overline{c})} F(x,\xi)dx - \int_\Omega F(x,\overline{d})dx}{\widehat{M}\left(\frac{1}{p^+}\left(\frac{\overline{c}}{k}\right)^{p^-}\right) - \widehat{M}\left(\frac{1}{p^-}\|a\|_1\overline{d}^{p^+}\right)}.$$

So, from our assumption it follows that $\rho_2(r) > 0$. Hence, in view of Theorem 2.2 for each $\lambda \in \overline{\Lambda}$, I_λ admits at least one local minimum $\overline{u}$ such that

$$\widehat{M}\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla\overline{u}|^{p(x)} + a(x)|\overline{u}|^{p(x)}\right)\right) > \widehat{M}\left(\frac{1}{p^+}\left(\frac{\overline{c}}{k}\right)^{p^-}\right).$$

Therefore

$$\rho(\overline{u}) > \frac{p^-}{p^+}\left(\frac{\overline{c}}{k}\right)^{p^-},$$

and our conclusion is achieved. □

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El Miloud Hssini
University Mohamed I, Faculty of Sciences, Department of Mathematics,
Oujda, Morocco
E-mail address: `hssini1975@yahoo.fr`

and

Mohammed Massar
University Mohamed I, Faculty of Sciences, Department of Mathematics,
Oujda, Morocco
E-mail address: `massarmed@hotmail.com`

and

Najib Tsouli
University Mohamed I, Faculty of Sciences, Department of Mathematics,
Oujda, Morocco
E-mail address: `tsouli@hotmail.com`