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Bertrand and Mannheim Partner D-curves on Parallel Surfaces

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abstract: In this paper we study Bertrand and Mannheim partner D-curves on parallel surface. Using the definition of parallel surfaces, first we find images of two curves lying on two different surfaces and satisfying the conditions to be Bertrand partner D-curve or Mannheim partner D-curve. Then we obtain relationships between Bertrand and Mannheim partner D-curves and their image curves.

Key Words: Parallel surface; Bertrand partner -curves; Mannheim partner -curves.

Contents

1. Introduction

In local differential geometry, the curves whose position vectors have a relationship according to curvatures have an important role. The well-known examples of such curves are Bertrand curves, involute-evolute curves and Mannheim curves which are studied by many mathematicians in different spaces [2,3,4,5,10,12,14,16]. Moreover, some new definitions of special curve pairs have been given by Kazaz and et al. They have considered the notions of Bertrand curve and Mannheim curve for the curves lying fully on regular surfaces and called these curve pairs as Bertrand partner D-curves and Mannheim partner D-curves. Using the Darboux frame of these curves, they have obtained some characterizations for these new curve pairs [6,7]. They have also studied on same subjects in the Minkowski 3-space and investigated the different conditions according to the Lorentzian casual characters of the curves and surfaces [8,9].

Moreover, analogue to the associated curves, similar relationships can be constructed between regular surfaces. For example, a surface and another surface which have constant distance with the reference surface along its surface normal have a relationship between their parametric representations. Such surfaces are called parallel surface [1]. By this definition, it is convenient to carry the points of a surface to the points of another surface. Since the curves are set of points,

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then the curves lying fully on a reference surface can be carry to a parallel surface of reference surface. By considering this fact, Önder and Kızıltuğ have defined and studied Bertrand and Mannheim partner D-curves on parallel surface in the Minkowski 3-space [13].

In this study, we consider the images of Bertrand and Mannheim partner D-curves on parallel surface in the Euclidean 3-space E^3. First, we obtain the image curves of these curves on parallel surface. Then we investigate the relationships between reference curves and their images.

2. Preliminaries

Let $S = S(u, v)$ be an oriented surface in the 3-dimensional Euclidean space E^3 and let consider a curve $\alpha(s)$ lying on S fully. Since the curve $\alpha(s)$ is also in space, there exists a Frenet frame $\{T, N, B\}$ at each points of the curve, where T is unit tangent vector, N is principal normal vector and B is binormal vector, respectively. The Frenet equations of the curve $\alpha(s)$ is given by

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned}$$

where κ and τ are curvature and torsion of the curve $\alpha(s)$, respectively [11].

Since the curve $\alpha(s)$ lies on the surface S, there exists another frame along the curve $\alpha(s)$. This new frame is called Darboux frame and denoted by $\{T, Y, Z\}$ where T is the unit tangent of the curve, Z is the unit normal of the surface S along the curve $\alpha(s)$ and Y is a unit vector given by $Y = Z \times T$. This frame gives us an opportunity to investigate the properties of the curve according to the surface. Since the unit tangent T is common in both Frenet frame and Darboux frame, the vectors N, B, Y and Z lie on the same plane. So that the relations between these frames can be given as follows

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where φ is the angle between the vectors Y and N. The derivative formulae of the Darboux frame are

$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_r \\ -k_n & -t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix},$$

where k_g, k_n and t_r are called the geodesic curvature, the normal curvature and the geodesic torsion of $\alpha(s)$, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve [11].

The relations between geodesic curvature, normal curvature, geodesic torsion and κ, τ are given as follows

$$k_g = \kappa\cos\varphi, k_n = \kappa\sin\varphi, t_r = \tau + \frac{d\varphi}{ds}.$$

Furthermore, the geodesic curvature k_g and geodesic torsion t_r of the curve $\alpha(s)$ can be calculated as follows

$$k_g = \left\langle \frac{dx}{ds}, \ \frac{d^2x}{ds^2} \times Z \right\rangle, t_r = \left\langle \frac{dx}{ds}, \ Z \times \frac{dn}{ds} \right\rangle.$$

In the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface S the followings are well-known

i) $\alpha(s)$ is a geodesic curve $\Leftrightarrow k_g = 0$,

ii) $\alpha(s)$ is an asymptotic line $\Leftrightarrow k_n = 0$,

iii) $\alpha(s)$ is a principal line $\Leftrightarrow t_r = 0$.

(See [11] for details).

Definition 2.1. [6,7] *Let S and S_1 be oriented surfaces in the 3-dimensional Euclidean space E^3 and let consider the arc-length parameter curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1, respectively. Denote the Darboux frames of $\alpha(s)$ and $\alpha_1(s_1)$ by $\{T, Y, Z\}$ and $\{T_1, Y_1, Z_1\}$, respectively. If there exists a corresponding relationship between the curves $\alpha(s)$ and $\alpha_1(s_1)$ such that, at the corresponding points of the curves, direction of the Darboux frame element Y of $\alpha(s)$ coincides with direction of the Darboux frame element Y_1 of $\alpha_1(s_1)$, i.e., the vectors Y and Y_1 lie on a line, then $\alpha(s)$ is called a Bertrand D-curve, and $\alpha_1(s_1)$ is a Bertrand partner D-curve of $\alpha(s)$. Then, the pair $\{\alpha, \alpha_1\}$ is said to be a Bertrand D-pair [7].*

If there exists a corresponding relationship between the curves $\alpha(s)$ and $\alpha_1(s_1)$ such that, at the corresponding points of the curves, direction of the Darboux frame element Y of $\alpha(s)$ coincides with direction of the Darboux frame element Z_1 of $\alpha_1(s_1)$, i.e., the vectors Y and Z_1 lie on a line, then $\alpha(s)$ is called a Mannheim D-curve, and $\alpha_1(s_1)$ is a Mannheim partner D-curve of $\alpha(s)$. Then, the pair $\{\alpha, \alpha_1\}$ is said to be a Mannheim D-pair [6].

Definition 2.2. [1] *Let S be an oriented surface in the Euclidean 3-space E^3 with unit normal Z. For any constant r in R^3, let S_r is given by $S_r = \{f(p) = p + rZ_p : p \in S\}$. Then $f(p) = p + rZ_p$ defines a new surface S_r. The map f is called the natural map on S into S_r, and if f is univalent, then S_r is a parallel surface of S with unit normal $Z_{f(p)} = Z_p$ for all p on S.*

The relationships between the geodesic curvatures, normal curvatures and geodesic torsions of two curves lying on a surface and on its parallel surfaces, respectively, have been introduced in [15]. In this paper, we consider the Bertrand partner D-curves and Mannheim partner D-curves on parallel surfaces.

3. Bertrand Partner D-Curves on Parallel Surfaces

In this section, we deal with the notion of Bertrand partner D-curves by considering parallel surface.

Let S and S_1 be oriented surfaces in the 3-dimensional Euclidean space E^3 and let consider arc-length parameter Bertrand partner D-curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1, respectively. Denote the Darboux frames and invariants of $\alpha(s)$

and $\alpha_1(s_1)$ by $\{T, Y, Z\}$, k_g, k_n, t_r and $\{T_1, Y_1, Z_1\}$, k_{g_1}, k_{n_1}, t_{r_1}, respectively. Then we have

$$\alpha_1(s_1) = \alpha(s_1) - \lambda Y_1(s_1). \tag{3.1}$$

(See [7]). For the oriented surfaces S_r and S_{r_1} assume that the surface pairs (S, S_r) and (S_1, S_{r_1}) are parallel surfaces. Then from (3.1) the images of the curves $\alpha(s)$ and $\alpha_1(s_1)$ on the surfaces S_r and S_{r_1} are given by

$$\beta(s_\beta) = \alpha(s) + rZ, \tag{3.2}$$

$$\beta_1(s_{\beta_1}) = \alpha(s_1) - \lambda Y_1(s_1) + r_1 Z_1, \tag{3.3}$$

respectively. Denote the Darboux frames of $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ by $\{T^*, Y^*, Z^*\}$ and $\{T_1^*, Y_1^*, Z_1^*\}$ respectively. Differentiating (3.2) with respect to s we have

$$\frac{d\beta}{ds} = \frac{d\beta}{ds_\beta}\frac{ds_\beta}{ds} = \alpha'(s) + rZ'. \tag{3.4}$$

By considering Darboux derivative formulae, from (3.4) it follows

$$T^*\frac{ds_\beta}{ds} = (1 - rk_n)T - rt_rY, \tag{3.5}$$

which gives us

$$\frac{ds_\beta}{ds} = \sqrt{(1 - rk_n)^2 + (rt_r)^2}. \tag{3.6}$$

From (3.5)and (3.6) we have

$$T^* = \frac{1}{\sqrt{(1 - rk_n)^2 + (rt_r)^2}}\left[(1 - rk_n)T - rt_rY\right]. \tag{3.7}$$

Since $Y^* = Z \times T^*$, from (3.7) it is obtained that

$$Y^* = \frac{1}{\sqrt{(1 - rk_n)^2 + (rt_r)^2}}\left[rt_rT + (1 - rk_n)Y\right]. \tag{3.8}$$

Then we have the following theorem.

Theorem 3.1. *Let the pair (S, S_r) be a parallel surface pair, $\alpha(s)$ be a curve lying fully on S and the curve $\beta(s_\beta)$ lying fully on S_r be the image curve of $\alpha(s)$. Then the relationships between the Darboux frames of $\alpha(s)$ and $\beta(s_\beta)$ are given as follows*

$$\begin{bmatrix} T^* \\ Y^* \\ Z^* \end{bmatrix} = \begin{bmatrix} \frac{1-rk_n}{\sqrt{(1-rk_n)^2+(rt_r)^2}} & \frac{-rt_r}{\sqrt{(1-rk_n)^2+(rt_r)^2}} & 0 \\ \frac{rt_r}{\sqrt{(1-rk_n)^2+(rt_r)^2}} & \frac{1-rk_n}{\sqrt{(1-rk_n)^2+(rt_r)^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} T \\ Y \\ Z \end{bmatrix}. \tag{3.9}$$

Similarly, from the differentiation of (3) with respect to s_1 it follows

$$\frac{d\beta_1}{ds_1} = \frac{d\beta_1}{ds_{\beta_1}}\frac{ds_{\beta_1}}{ds_1} = \frac{d\alpha}{ds}\frac{ds}{ds_1} - \lambda Y_1'(s_1) + r_1 Z_1'. \tag{3.10}$$

Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves we have

$$T\frac{ds}{ds_1} = (1-\lambda k_{g_1})T_1 + \lambda t_{r_1} Z_1, \tag{3.11}$$

(See [7]). Then substituting (3.11) in (3.10) it follows

$$T_1^*\frac{ds_{\beta_1}}{ds_1} = (1-r_1 k_{n_1})\,T_1 - r_1 t_{r_1} Y_1, \tag{3.12}$$

which gives us

$$\frac{ds_{\beta_1}}{ds_1} = \sqrt{(1-r_1 k_{n_1})^2 + (r_1 t_{r_1})^2}. \tag{3.13}$$

Thus (3.12) becomes

$$T_1^* = \frac{1}{\sqrt{(1-r_1 k_{n_1})^2 + (r_1 t_{r_1})^2}}\left[T_1\,(1-r_1 k_{n_1}) - r_1 t_{r_1} Y_1\right]. \tag{3.14}$$

Since $Y_1^* = Z_1 \times T_1^*$, from (3.14) we have

$$Y_1^* = \frac{1}{\sqrt{(1-r_1 k_{n_1})^2 + (r_1 t_{r_1})^2}}\left[r_1 t_{r_1} T_1 + (1-r_1 k_{n_1})\,Y_1\right]. \tag{3.15}$$

Then we have the following theorem.

Theorem 3.2. *Let the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1 respectively, be Bertrand Partner D-curves and the pair (S_1, S_{r_1}) be a parallel surface pair. If the curve $\beta_1(s_{\beta_1})$ lying fully on S_{r_1} is the image curve of $\alpha_1(s_1)$ under the naturel mapping f, then the relationships between the Darboux frames of $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are given by the equalities (3.14) and (3.15).*

Moreover, since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves we have

$$\frac{ds}{ds_1} = \frac{\lambda k_{g_1} - 1}{\lambda t_{r_1}}. \tag{3.16}$$

Then from (3.6), (3.13) and (3.16) we have the following corollary

Corollary 3.3. *Let the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1 respectively, be Bertrand Partner D-curves and the pairs (S, S_r) and (S_1, S_{r_1}) be parallel surface pairs. Then the relationship between arc length parameters s_{β_1} and s_β is given by*

$$s_{\beta_1} = \int \sqrt{\frac{(1-r_1 k_{n_1})^2 + (r_1 t_{r_1})^2}{(1-r k_n)^2 + (r t_r)^2}}\,\frac{(\lambda k_{g_1} - 1)}{\lambda t_{r_1}}\,ds_\beta. \tag{3.17}$$

After these computations, we can give the following characterizations. Here in after, we assume that the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on surfaces S and S_1 respectively, are Bertrand Partner D-curves, the pairs (S, S_r) and (S_1, S_{r_1}) are parallel surface pairs, the curves $\beta(s_\beta)$ and $\beta_1\left(s_{\beta_1}\right)$ are image curves of the curves $\alpha(s)$ and $\alpha_1(s_1)$ on S_r and S_{r1}, respectively.

Theorem 3.4. *$\alpha(s)$ is a principal line on S if and only if $\alpha(s)$ and $\beta(s_\beta)$ are Bertrand partner D-curves.*

Proof: Let $\alpha(s)$ be a principal line on S. Then we have $t_r = 0$ and from (3.8) it follows $Y^* = \pm Y$ i.e., $\alpha(s)$ and $\beta(s_\beta)$ are Bertrand D-curves.

Conversely, if $\alpha(s)$ and $\beta(s_\beta)$ are Bertrand D-curves, then from (3.8) we have $t_r = 0$, i.e., $\alpha(s)$ is a principal line on S. □

Theorem 3.5. *$\alpha_1(s_1)$ is a principal line on S_1 if and only if $\alpha(s)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand partner D-curves.*

Proof: Let $\alpha_1(s_1)$ be a principal line on S_1 i.e., $t_{r_1} = 0$. Then from (3.15) we have $Y_1^* = \pm Y$. It means $\alpha(s)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand D-curves. Conversely, if $\alpha(s)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand D-curves, then from (3.15) we have $t_{r_1} = 0$. Then $\alpha_1(s_1)$ is a principal line on S_1. □

Theorem 3.6. *$\alpha(s)$ is a principal line on S if and only if $\alpha_1(s_1)$ and $\beta(s_\beta)$ are Bertrand partner D-curves.*

Proof: Let $\alpha(s)$ be a principal line on S. Then we have $t_r = 0$ and from (3.8) we have $Y^* = \pm Y$. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves we have $Y = Y_1$. Then we obtain $Y^* = \pm Y_1$. Conversely, if $\alpha_1(s_1)$ and $\beta(s_\beta)$ are Bertrand partner D-curves, by considering the condition that $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves, from (3.8) it is obtained that $t_r = 0$. Then $\alpha(s)$ be a principal line on S. □

Theorem 3.7. *$\alpha_1(s_1)$ is a principal line on S_1 if and only if $\alpha_1(s_1)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand partner D-curves.*

Proof: $\alpha_1(s_1)$ is a principal line on S_1, i.e., $t_{r_1} = 0$, then (3.15) gives us $Y_1^* = \pm Y_1$. Then $\alpha_1(s_1)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand partner D-curves. Conversely, if $\alpha_1(s_1)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand partner D-curves, then from (3.15) we have $t_{r_1} = 0$, i.e., $\alpha_1(s_1)$ is a principal line on S_1. □

Theorem 3.8. *$\alpha(s)$ and $\alpha_1(s_1)$ are principal lines on S and S_1, respectively if and only if $\beta(s_\beta)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand partner D-curves.*

Proof: If $\alpha(s)$ and $\alpha_1(s_1)$ are principal lines on S and S_1, respectively, then $t_r = 0$, $t_{r_1} = 0$. In this case from (3.8) and (3.15) we have $Y^* = \pm Y$ and $Y_1^* = \pm Y_1$, respectively. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves, $Y = Y_1$. From the last two equalities we obtain that $Y^* = \pm Y_1^*$ which means that $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.Let now $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ be Bertrand partner D-curves. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves, from (3.8) we have

$$Y_1 = \frac{1}{1 - rk_n}\left[\left(\sqrt{(1-rk_n)^2 + (rt_r)^2}\right)Y^* - rt_rT\right]. \tag{3.18}$$

Substituting (3.18) in (3.15) gives

$$Y_1^* = \frac{1}{\sqrt{(1-r_1k_{n_1})^2 + (r_1t_{r_1})^2}}\left[r_1t_{r_1}T_1 + \frac{1-r_1k_{n_1}}{1-rk_n}((\sqrt{(1-rk_n)^2+(rt_r)^2})Y^* - rt_rT)\right]. \tag{3.19}$$

Since we assume $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves, from (3.19) it is obtained that $t_r = 0$, $t_{r_1} = 0$, i.e., $\alpha(s)$ and $\alpha_1(s_1)$ are principal lines on S and S_1, respectively. □

Theorem 3.9. *$\alpha(s)$ is both geodesic curve and principal line on S if and only if $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.*

Proof: Let $\alpha(s)$ be both geodesic curve and principal line on S, i.e., $k_g = t_r = 0$. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand partner D-curves we have

$$t_{r1} = (k_g \sin\theta + t_r \cos\theta)\frac{ds}{ds_1}, \tag{3.20}$$

$$k_{g_1} = ((1+\lambda k_g)\cos\theta - \lambda t_r \sin\theta)(k_g + \lambda k_g^2 + \lambda t_r^2)\left(\frac{ds}{ds_1}\right)^3. \tag{3.21}$$

(See [7]). Under the condition $k_g = t_r = 0$, from (3.20) and (3.21) we have $t_{r_1} = k_{g_1} = 0$, i.e., $\alpha_1(s_1)$ is also a geodesic and a principal line on S_1. Then from Theorem 3.7, we have that $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ Bertrand partner D-curves.Conversely, if $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves, from Theorem (10), $\alpha(s)$ and $\alpha_1(s_1)$ are principal lines on S and S_1, respectively, i.e., $t_r = 0$, $t_{r_1} = 0$. Then for the non-trivial case $\theta \neq 0$, from (3.20) we have $k_g = 0$. Then $\alpha(s)$ is both geodesic curve and principal line on S. □

Theorem 3.10. *If $\alpha(s)$ is both geodesic curve and principal line on S then $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.*

Proof: Let $\alpha(s)$ be both geodesic curve and principal line on S. Then from (3.20) and (3.21) we have $t_{r_1} = k_{g_1} = 0$. Then from (3.15) we have $Y_1^* = \pm Y_1$, i.e., $\alpha_1(s_1)$ and $\beta_1(s_{\beta_1})$ are Bertrand D-curves. □

Theorem 3.11. *If $\alpha(s)$ is both geodesic curve and principal line on S then $\alpha(s)$ and $\beta_1(s_{\beta_1})$ are Bertrand partner D-curves.*

Proof: Let $\alpha(s)$ be both geodesic curve and principal line on S. Then using (3.20) and (3.21) from (3.15) we have $Y_1^* = \pm Y_1$. Since $\alpha(s)$ and $\alpha_1(s_1)$ are Bertrand patner D-curves, $Y = Y_1$, and so we have $Y_1^* = \pm Y$. □

4. Mannheim Partner D-Curves on Parallel Surfaces

In this section, we deal with the notions of Mannheim partner D-curves by considering parallel surface.

Let S and S_1 be oriented surfaces in three-dimensional Euclidean space E^3 and let consider the arc-length parameter Mannheim partner D-curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1, respectively. Denote the Darboux frames and invariants of $\alpha(s)$ and $\alpha_1(s_1)$ by $\{T, Y, Z\}$, $k_g,\ k_n,\ t_r$ and $\{T_1, Y_1, Z_1\}$, $k_{g_1},\ k_{n_1},\ t_{r_1}$, respectively. Then from the definition of Mannheim partner D-curves we have

$$\alpha_1(s_1) = \alpha(s_1) - \lambda Z_1(s_1), \tag{4.1}$$

(See [6]). For the oriented surfaces S_r and S_{r_1} assume that surface pairs (S, S_r) and (S_1, S_{r_1}) are parallel surfaces. Then from (4.1) the images of the curves $\alpha(s)$ and $\alpha_1(s_1)$ on the surfaces S_r and S_{r_1} are given by

$$\beta(s_\beta) = \alpha(s) + rZ, \tag{4.2}$$

$$\beta_1(s_{\beta_1}) = \alpha(s_1) - \lambda Z_1(s_1) + r_1 Z_1, \tag{4.3}$$

respectively. Denote the Darboux frames of $\beta(s_\beta)$ and $\beta_1(s_{\beta_1})$ by $\{T^*, Y^*, Z^*\}$ and $\{T_1^*, Y_1^*, Z_1^*\}$, respectively. Differentiating (4.3) with respect to s_1 we have

$$\frac{d\beta_1}{ds_1} = \frac{d\beta_1}{ds_{\beta_1}}\frac{ds_{\beta_1}}{ds_1} = \alpha'(s_1)\frac{ds}{ds_1} - \lambda Z_1' + r_1 Z_1'. \tag{4.4}$$

Similarly, differentiating (4.3) with respect to s_1 it follows

$$T\frac{ds}{ds_1} = (1 - \lambda k_{n_1})T_1 - \lambda t_{r_1} Y_1. \tag{4.5}$$

Then substituting (4.5) in (4.4) it follows

$$T_1^*\frac{ds_{\beta_1}}{ds_1} = (1 - k_{n_1}(r_1 + \lambda))\,T_1 - (t_{r_1}(r_1 - 2\lambda))\,Y_1, \tag{4.6}$$

which gives us

$$\frac{ds_{\beta_1}}{ds_1} = \sqrt{(1 - k_{n_1}(r_1 + \lambda))^2 + (t_{r_1}(r_1 - 2\lambda))^2}. \tag{4.7}$$

Thus (4.6) becomes

$$T_1^* = \frac{1}{\sqrt{(1-k_{n_1}(r_1+\lambda))^2+(t_{r_1}(r_1-2\lambda))^2}}((1-k_{n_1}(r_1+\lambda))T_1-(t_{r_1}(r_1-2\lambda))Y_1)\ . \tag{4.8}$$

Since $Y_1^* = Z_1 \times T_1^*$, from (4.8) we have

$$Y_1^* = \frac{1}{\sqrt{(1-k_{n_1}(r_1+\lambda))^2+(t_{r_1}(r_1-2\lambda))^2}}((t_{r_1}(r_1-2\lambda))T_1+(1-k_{n_1}(r_1+\lambda))Y_1)\ . \tag{4.9}$$

Then we have the following theorem.

Theorem 4.1. *Let the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1 respectively, be Mannheim partner D-curves and the pair (S_1, S_{r_1}) be a parallel surface pair. If the curve $\beta_1\left(s_{\beta_1}\right)$ lying fully on S_{r_1} is the image curve of $\alpha_1(s_1)$, then the relationships between the Darboux frames of $\alpha_1(s_1)$ and $\beta_1\left(s_{\beta_1}\right)$ are given by the equalities (4.8) and (4.9).*

Moreover, since $\alpha(s)$ and $\alpha_1(s_1)$ are Mannheim partner D-curves we have

$$\frac{ds}{ds_1} = \frac{\lambda k_{n_1} - 1}{\lambda t_{r_1}}. \tag{4.10}$$

Then from (3.6), (4.7) and (4.10) we have the following corollary.

Corollary 4.2. *Let the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on S and S_1 respectively, be Mannheim Partner D-curves and the pairs (S, S_r) and (S_1, S_{r_1}) be parallel surface pairs. Then the relationship between arc length parameters s_{β_1} and s_β is given by*

$$s_{\beta_1} = \int \sqrt{\frac{\left(1-k_{n_1}\left(r_1+\lambda\right)\right)^2+\left(t_{r_1}\left(r_1-2\lambda\right)\right)^2}{\left(1-rk_n\right)^2+\left(rt_r\right)^2}}\,\frac{\lambda t_{r_1}}{\lambda k_{n_1}-1}\,ds_\beta. \tag{4.11}$$

After these computations, we can give the following characterizations. Here in after we assume that the curves $\alpha(s)$ and $\alpha_1(s_1)$ lying fully on surfaces S and S_1 respectively, are Mannheim Partner D-curves, the pairs (S, S_r) and (S_1, S_{r_1}) are parallel surface pairs, the curves $\beta\left(s_\beta\right)$ and $\beta_1\left(s_{\beta_1}\right)$ are image curves of the curves $\alpha(s)$ and $\alpha_1(s_1)$ on S_r and S_{r1}, respectively.

Theorem 4.3. *$\alpha_1(s_1)$ is a principal line on S_1 or $r_1 = 2\lambda$ if and only if $\alpha_1(s_1)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand partner D-curves.*

Proof: Let $\alpha_1(s_1)$ be a principal line on S_1 or let $r_1 = 2\lambda$. Then from (4.9) we have $Y_1^* = \pm Y_1$, i.e., $\alpha_1(s_1)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand partner D-curves. Conversely, if $\alpha_1(s_1)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand partner D-curves, then from (4.9) we have that $\alpha_1(s_1)$ be a principal line on S_1 or $r_1 = 2\lambda$. □

Theorem 4.4. *$\alpha(s)$ and $\beta_1\left(s_{\beta_1}\right)$ are Mannheim partner D-curves if and only if $\tan\theta = \frac{t_{r_1}(r_1-2\lambda)}{1-k_{n_1}(r_1+\lambda)}$ holds, where θ is the angle between unit tangents T_1 and T.*

Proof: Since $\alpha(s)$ and $\alpha_1(s_1)$ are Mannheim partner D-curves, we have $T_1 = \cos\theta T + \sin\theta Z,\ Y_1 = \sin\theta T - \cos\theta Z$ [6]. Then from (4.9) we obtain

$$Y_1^* = \frac{1}{\sqrt{(1-k_{n_1}(r_1+\lambda))^2+(t_{r_1}(r_1-2\lambda))^2}}\{[\cos\theta(t_{r_1}(r_1-2\lambda))+\sin\theta(1-k_{n_1}(r_1+\lambda))]\,T + [\sin\theta(t_{r_1}(r_1-2\lambda))-\cos\theta(1-k_{n_1}(r_1+\lambda))]\,Z\} \tag{4.12}$$

From (4.12) it is clear that $\alpha(s)$ and $\beta_1\left(s_{\beta_1}\right)$ are Mannheim partner D-curves if and only if $\tan\theta = \frac{t_{r1}(r_1-2\lambda)}{1-k_{n1}(r_1+\lambda)}$ holds. □

Corollary 4.5. *. $\beta\left(s_\beta\right)$ and $\beta_1\left(s_{\beta_1}\right)$ are Mannheim partner D-curves if and only if $\tan\theta = \frac{t_{r_1}(r_1-2\lambda)}{1-k_{n_1}(r_1+\lambda)}$ holds, where θ is the angle between unit tangents T_1 and T.*

Proof: Since $\alpha(s)$ and its image curve $\beta\left(s_\beta\right)$ have same unit normal direction Z, from Theorem 4.3, we obtain that $\beta\left(s_\beta\right)$ and $\beta_1\left(s_{\beta_1}\right)$ are Mannheim partner D-curves if and only if $\tan\theta = \frac{t_{r_1}(r_1-2\lambda)}{1-k_{n_1}(r_1+\lambda)}$ holds. □

Corollary 4.6. *$\alpha_1(s_1)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand partner D-curves if and only if $\tan\theta = \frac{t_{r_1}(r_1-2\lambda)}{1-k_{n_1}(r_1+\lambda)}$ holds, where θ is the angle between unit tangents T_1 and T.*

Proof: Since $\alpha(s)$ and $\alpha_1(s_1)$ are Mannheim partner D-curves, we have $Z = \pm Y_1$. Then from (4.12) it is clear that $\alpha_1(s_1)$ and $\beta_1\left(s_{\beta_1}\right)$ are Bertrand partner D-curves if and only if $\tan\theta = \frac{t_{r_1}(r_1-2\lambda)}{1-k_{n_1}(r_1+\lambda)}$ holds. □

Conclusion 4.1. *Associated curves are important subjects of curve theory. These curves are defined as curve pairs for which some relationships between their curvatures are satisfied. Generally, these curve pairs are studied in the space. Of course, a curve can be thought as a subset of a regular surface. Then, associated curves can be considered on surfaces. This paper gives some types of such curves. By considering Darboux frames of the curves, Bertrand partner D-curves and Mannheim partner D-curves are studied on parallel surfaces. The relationships between reference curves and their image curves are obtained and discussed.*

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