Bol. Soc. Paran. Mat. (3s.) **v. 35** 1 (2017): 97–109.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v35i1.24592

On the quotients of c-spaces

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abstract: In this paper, properties of quotient maps in c-spaces are studied in detail. A method of finding quotient space of topologizable and graphical c-spaces are described.

Key Words: c-space,connectivity space, c-continuous function, quotient space, topologizable and graphical c-spaces

Contents

1. Introduction

The concept of connected sets can be found in topology, graph theory, order structure and in their fuzzy analogues. A subset A of a topological space X is said to be connected if it cannot be separated into two disjoint non empty open subsets. In this definition, separation is emphasized. When working with graphs, connectivity is defined in terms of paths.A graph G is said to be connected if for every vertices x and y there is a path in G with end vertices x and y.

Although these classical definitions have been extensively applied in image processing and analysis, these two approaches(i.e., separation by open sets and joining points by path) are incompatible [9]. For the sake of completeness of this paper, let us illustrate this with an example [9] here.

Consider the digital space $\mathbb{Z}^2$. Define two adjacency relations, the 4- adjacency and 8-adjacency, as follows.

Let (i, j) and $(i', j') \in \mathbb{Z}^2$.

2000 *Mathematics Subject Classification:* 54A05, 05C40

(i, j) is 4-adjacent to (i', j') if $|i - i'| + |j - j'| = 1$ and
(i, j) is 8-adjacent to (i', j') if $\max\{|i - i'|, |j - j'|\} = 1$.

Thus we get two graphs with $\mathbb{Z}^2$ as set of vertices and edges linking pairs of adjacent points (4 or 8-adjacency respectively). This leads to the classical notion of 4-connectivity and 8-connectivity of subsets of $\mathbb{Z}^2$.

But it is proved that there is a topology on $\mathbb{Z}^2$ for which connected sets are precisely 4-connected sets and there is no topology on $\mathbb{Z}^2$ for which connectedness is equivalent to 8-connectedness.

Further, it is proved that [2], any odd cycle C_n, $n \geq 5$ has no compatible topology. Thus there are graphs where connectivity does not arise from a topology.

Conversely there are topological spaces which are not graphical. For example, in [9] it is noted that, in $\mathbb{R}^n$ with $n \geq 1$, every pair of points is disconnected and that it is not graphical. That is, topological connectivity of $\mathbb{R}^n$ cannot arise from a graph.

In general, topological connectivity is useful for images defined over a continuous space where as graph theoretic connectivity is useful for images defined over a discrete space. Compatibility is essential as discrete images are often obtained from the discretization of the continuous scenes. Thus topological or graph theoretical approach to connectivity limits the application of connectivity or the use of connected sets in practical purposes.

In 1983, Börger R. [1] proposed an axiomatic approach to connectivity, known as the theory of *connectivity classes* or *c-structures*, which overcomes the shortfalls of the classical definitions of connectivity. Here the axioms involve the minimum properties that connected sets can have. That is, one point sets are connected and that union of connected sets having a common point is connected. A systematic study of this space was further carried out by J. Serra [12], H. J. A. M. Hejimans [4] and C. Ronse [9] etc. This space found profound applications in the areas of *Image segmentation, Image Filtering, Image coding, Digital Topology, Pattern Recognition, Mathematical Morphology etc* [4,5,10,12,13]. In this paper, some existing concepts are formalized and explored some properties of quotient spaces, which paves the foundation for our further work.

2. Preliminaries

Let X be a non empty set and $\mathcal{C}_X$ be a collection of subsets of X such that the following properties hold.

(i) $\phi \in \mathcal{C}_X$ and $\{x\} \in \mathcal{C}_X$ for every $x \in X$.

(ii) If $\{C_i : i \in I\}$ be a non empty collection of subsets in $\mathcal{C}$ with $\bigcap_{i \in I} C_i \neq \phi$, then $\bigcup_{i \in I} C_i \in \mathcal{C}_X$.

Then $\mathcal{C}_X$ is called a c-structure on X and X with a c-structure on it is called a c-space. A c-space X is said to be a *connective space* [7] if $\mathcal{C}_X$ satisfies two more conditions as given below.

(iii) Given any non empty sets $A, B \in \mathcal{C}_X$ with $A \cup B \in \mathcal{C}_X$, then there exists $x \in A \cup B$ such that $\{x\} \cup A \in \mathcal{C}_X$ and $\{x\} \cup B \in \mathcal{C}_X$.

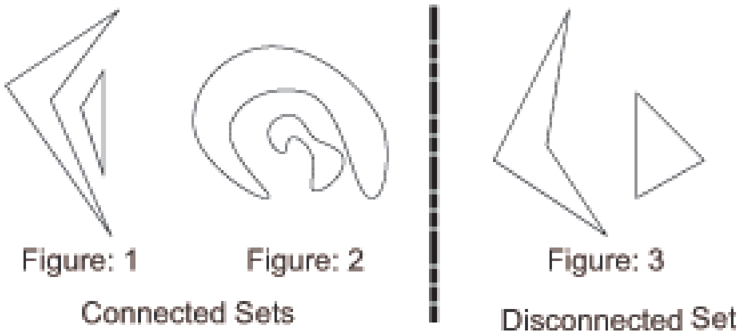
(iv) If $A, B, C_i \in \mathcal{C}_X$ are disjoint and $A \cup B \cup \bigcup_{i \in I} C_i \in \mathcal{C}_X$, then there exists $J \subseteq I$ such that $A \cup \bigcup_{j \in J} C_j \in \mathcal{C}_X$ and $B \cup \bigcup_{i \in I-J} C_i \in \mathcal{C}_X$.

A c-structure satisfies above two conditions are called a *connective structure* or *connectology* on X. Other terminologies used for c-spaces are *connectivity space* [1,12] and *integral connectivity space* [3]. In his paper [1], Reinhard Börger have the same definition for the connectivity space except that empty set is connected. Another terminology used for a c-structure is *connectivity class* [4,5,12] of X. We like to follow the terminology used in [7]. Elements of a c-structure are called *connected sets*.

$\mathcal{D}_X = \{\phi\} \cup \{\{x\} : x \in X\}$ is a c-structure on X, called the *discrete c-structure* and the space $(X, \mathcal{D}_X)$ is called the *discrete c-space*. Also, $\mathcal{I}_X = \mathcal{P}(X)$, the power set of X is a c-structure on X, called the *indiscrete c-structure* and the corresponding space is called an *indiscrete c-space*. A c-space $(Y, \mathcal{C}_Y)$ is said to be a *sub c-space* of the c-space $(X, \mathcal{C}_X)$ if $Y \subseteq X$ and $\mathcal{C}_Y = \{A \in \mathcal{C}_X : A \subseteq Y\}$. Let $\mathcal{B} \subseteq \mathcal{P}(X)$, the power set of X. Then the smallest c-structure on X containing $\mathcal{B}$ is called the c-structure *generated by* $\mathcal{B}$ and is denoted by $< \mathcal{B} >$. The c-space (X, C_X) is denoted by X if there is no ambiguity.

Let X and Y be two c-spaces and $f : X \to Y$ be a function. f is called *c-continuous* [7] or *catenuous* [7] or a *connectivity morphism* [4,12] or a connectivity map [3], if it maps connected sets of X to connected sets of Y. In [7], it is proved that the restriction of a c-continuous function and composition of two c-continuous functions are c-continuous. Also, a bijection f is said to be a *c-isomorphism* or *catenomorphism* if both f and f^{-1} are c-continuous. In [11], it is proved that if $f : X \to Y$ is a bijection, then f is a c-isomorphism if and only if $f(\mathcal{C}_A) = \mathcal{C}_B$.

Let $\{X_i : i \in I\}$ be a family of c-spaces and $\{f_i : X \to X_i : i \in I\}$ be a family of functions defined on a set X. Let $\mathcal{C}$ be a collection of subsets of X such that $A \in \mathcal{C}$ if and only if $f_i(A) \in \mathcal{C}_{X_i}$ for every i. Then $\mathcal{C}$ is a c-structure on X and is called the strong c-structure generated by the given family of functions [11] and is denoted by $< \{f_i : i \in I\} >_{_S}$. The c-structure on the product space $\prod_{i \in I} X_i$ is the strong c-structure generated by the family of projection functions $\{\pi_i : i \in I\}$. Hence the connected subsets of $\mathbb{R}^n$ are precisely those subsets of $\mathbb{R}^n$ whose projections are intervals. For example, the following figures represents some connected subsets and a disconnected subset of $\mathbb{R}^2$.



A chain of connected sets [7] is a map C from a finite index set I to $\mathcal{C}_X$ such that each C_i intersects its successor C_{i+1}, where $C_i = C(i)$ for $i \in I$.

Proposition 2.1. *[1,7]*
Let $\mathcal{B}$ be a collection of subsets of a set X and let $< \mathcal{B} >$ be a c-structure on X generated by this collection(That is, the smallest c-structure on X containing $\mathcal{B}$). Then the non trivial connected sets in $< \mathcal{B} >$ are characterized by the condition that any two points of such a connected set C can be joined by a finite chain of basic connected sets(i.e. elements of $\mathcal{B}$) in C. That is, for all $x, y \in C$, we can find elements B_i, $i = 0$ to n in $\mathcal{B}$ such that $B_i \subseteq C$, $B_i \cap B_{i+1} \neq \phi$ for $i = 0$ to $n-1$ and $x \in B_0$, $y \in B_n$.

3. Weak c-structure Generated by a Family of Functions

Quotient spaces plays a vital role in all branches of mathematics. Though not stated in a formal language, the smallest c-structure on a set which makes a family of functions c-continuous can be found in [1] and for a single function in [3]. Here we are stating it formally as weak c-structure generated by a family of functions and quotient spaces are introduced as a particular case of this.

Formalizing the concepts found in [1,3], we state the following definitions.

Definition 3.1. *Weak c-structure generated by a family of functions*
Let X be any set and $\{X_i : i \in I\}$ be a family of c-spaces. Consider the family of functions $\{f_i : X_i \to X : i \in I\}$. Then the weak c-structure generated by $\{f_i\}_{i \in I}$ is the smallest c-structure on X which make each function f_i c-continuous and is denoted by $< \{f_i : i \in I\} >_W$

Definition 3.2. *Quotient c-space*
Let X and Y be any two c-spaces. Let $f : X \to Y$ be an onto function. Then f is said to be a quotient map or Y is said to be a quotient space of X with respect to f if $\mathcal{C}_Y$ is the weak c-structure on Y generated by $\{f\}$.

Restating a result in [1] in our terminology yields the following theorem.

Theorem 3.3. *Let $\{X_i : i \in I\}$ be a family of c-spaces. Let $\{f_i : X_i \to X : i \in I\}$ be a family of functions. Let $\mathcal{C}_X$ be the c-structure on X generated by the family $\{f_i(C) : C \in \mathcal{C}_{X_i}, i \in I\}$. Then $\mathcal{C}_X$ is the weak c-structure on X generated by the family $\{f_i : X_i \to X\}_{i\in I}$.*

3.1. Examples of Quotient Spaces

Here we are listing some examples of quotient spaces.

1. Consider the c-spaces $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ where $X = \{1, 2, 3\}$, $\mathcal{C}_X = \mathcal{D}_X \cup \{\{1, 2\}, \{2, 3\}, X\}$, $Y = \{a, b\}$ and $\mathcal{C}_Y = \mathcal{P}(Y)$. Define $f : X \to Y$ by $1 \mapsto a$, $2 \mapsto a$ and $3 \mapsto b$. As $\mathcal{C}_Y =< \{f\} >_W$, Y is a quotient space of X and f.

2. Consider the c-space $(X, \mathcal{C}_X)$, where $X = \{1, 2, 3, 4\}$ and $\mathcal{C}_X = \mathcal{D}_X \cup \{\{1, 2\}, \{3, 4\}\}$. Let $Y = \{a, b, c\}$ and define $f : X \to Y$ by $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto b$ and $4 \mapsto c$. Then $< \{f\} >_W =< \{\{a, b\}, \{b, c\}\} >= \mathcal{D}_Y \cup \{\{a, b\}, \{b, c\}, \{a, b, c\}\}$. Y with this c-structure is the quotient space of X with respect to f.

Is $Y = \{(x, y) : x^2 + y^2 = 1\}$ (as a sub c-space of $\mathbb{R}^2$) a quotient space of $[0, 2\pi]$ (as a sub c-space of $\mathbb{R}$)? What is the quotient space induced on $\mathbb{R}$ by the equivalence relation defined by $x \sim y$ if and only if $x - y \in \mathbb{Q}$ for $x, y \in \mathbb{R}$? This type of nice questions will be answered in a forthcoming section.

4. Properties of Quotient Spaces

In [1], it is proved that in the category of c-spaces, quotient maps are hereditary. That is, if $f : X \to Y$ is a quotient map, then for every $B \subseteq Y$, $f_{/f^{-1}(B)} : f^{-1}(B) \to B$ is a quotient map. We may note that in the category of topological spaces, quotient maps are not hereditary. Here some other properties of quotient spaces are discussed.

In the next proposition, we enquires how a quotient map is related to a c-isomorphism.

Proposition 4.1. *Let $f : X \to Y$ be an injective quotient function. Then f is a c-isomorphism.*

Proof: First we claim that $A \in \mathcal{C}_X$ if and only if $f(A) \in \mathcal{C}_Y$.

Let $A \in \mathcal{C}_X$. Since f is c-continuous, $f(A) \in \mathcal{C}_Y$.

Conversely let $f(A) \in \mathcal{C}_Y$, where $\mathcal{C}_Y =< \mathcal{B} >$, with $\mathcal{B} = \{f(C) : C \in \mathcal{C}_X\}$. Using the Proposition 2.1, $f(A) = \underset{x\in f(A)}{\cup} E_{x_0x}$, where $x_0 \in f(A)$ and $E_{x_0x} = \overset{n_x}{\underset{i=1}{\cup}} A_{x_i}$ such that $x_0 \in A_{x_1}$, $x \in A_{x_{n_x}}$, $A_{x_i} \in \mathcal{B}$ for each i with $A_{x_i} \subset f(A)$ and $A_{x_i} \cap A_{x_{i+1}} \neq \phi$ for $1 \leq i \leq n_x - 1$. Clearly E_{x_0x} is a connected set in $f(A)$ for each $x \in f(A)$. Then there exists $B_y \subseteq A$ such that $f(B_y) = E_{x_0x}$ such that $f(y) = x$.

Claim: B_y is connected in X for each $y \in A$.

Using Principle of Induction, it suffices to prove the result for $n_x = 2$. Let $f(B_y) = E_{x_0 x} = A_1 \cup A_2$ with $A_1, A_2 \in \mathcal{B}$ with $x_0 \in A_1$, $x \in A_2$, $A_1, A_2 \subseteq f(A)$ and $A_1 \cap A_2 \neq \phi$. As $\mathcal{C}_Y =< \mathcal{B} >$, there exists $C_1, C_2 \in \mathcal{C}_X$ such that $f(C_1) = A_1$ and $f(C_2) = A_2$. Since $A_1 \cap A_2 \neq \phi$ and since f is injective, $C_1 \cap C_2 \neq \phi$, so that $C_1 \cup C_2 \in \mathcal{C}_X$. Clearly, $f(C_1 \cup C_2) = A_1 \cup A_2 = f(B_y)$. Since f is injective, $B_y = C_1 \cup C_2$, so that B_y is connected in X. Hence the claim.

It can be noted that $f(\underset{y}{\cup} B_y) = f(A)$. Since f is injective, $A = \underset{y}{\cup} B_y$. As $\underset{y}{\cap} B_y \neq \phi$, and since each B_y is connected, $\underset{y}{\cup} B_y$ is connected in X and hence A is connected in X. Thus

$$A \in \mathcal{C}_X \text{ if and only if } f(A) \in \mathcal{C}_Y \tag{4.1}$$

Since f is a quotient function, it is onto. Thus f is a bijective function. Now to show that f is a c-isomorphism, it is enough to show that $f(\mathcal{C}_X) = \mathcal{C}_Y$.

Let $U \in f(\mathcal{C}_X)$. Then there exists $A \in \mathcal{C}_X$ such that $f(A) = U$. Since $A \in \mathcal{C}_X$, by equation (4.1), we have $f(A) \in \mathcal{C}_Y$. That is, $U \in \mathcal{C}_Y$. Thus

$$f(\mathcal{C}_X) \subseteq \mathcal{C}_Y \tag{4.2}$$

Conversely let $V \in \mathcal{C}_Y$. f being onto, there exists $A \subset X$ such that $f(A) = V$. That is, $f(A) \in \mathcal{C}_Y$. By equation (4.1), we have $A \in \mathcal{C}_X$. Then $f(A) \in f(\mathcal{C}_X)$, so that $V \in f(\mathcal{C}_X)$. Thus

$$\mathcal{C}_Y \subseteq f(\mathcal{C}_X) \tag{4.3}$$

From equations (4.2) and (4.3), we have $f(\mathcal{C}_X) = \mathcal{C}_Y$. □

Let $f_i : X_i \to Y_i$ be quotient maps for $i = 1$ to n, $X = \prod_{i=1}^{n} X_i$, $Y = \prod_{i=1}^{n} Y_i$ and $f : X \to Y$, where $f = \prod_{i=1}^{n} f_i$. It is of natural interest to find how the product structure on Y and the quotient structure on Y with respect to f are related to?

Theorem 4.2. *Let $f_i : X_i \to Y_i$ be quotient maps for $i = 1$ to n, $Y = \prod_{i=1}^{n} Y_i$ and $f = \prod_{i=1}^{n} f_i$. Define $\mathcal{C}_P$ and $\mathcal{C}_Q$ as the product c-structure and c-structure generated by f on Y respectevely. Then for each $K \in \mathcal{C}_P$, there exists $K_1 \in \mathcal{C}_Q$ such that $K \subset K_1$.*

Proof: It suffices to prove the theorem for $n = 2$. Let $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$. Then X and Y are c-spaces under the product c-structures $\mathcal{C}_X$ and $\mathcal{C}_P$ in order. Define $f : X \to Y$ by $f(x) = (f_1 \times f_2)(x) = (f_1(x_1), f_2(x_2))$, where $x = (x_1, x_2) \in X$. Obviously f is c-continuous. Now, by Theorem 3.3, $\mathcal{C}_Q =< \{f(C) : C \in \mathcal{C}_X\} >$. Let $K \in \mathcal{C}_P$. Then by definition $\pi_1(K) \in \mathcal{C}_{Y_1}$ and $\pi_2(K) \in \mathcal{C}_{Y_2}$. Let

$$K_1 = \pi_1(K) \times \pi_2(K)$$

Consider any two points (u, v) and (r, s) of K_1. Then $u, r \in \pi_1(K)$ and $v, s \in \pi_2(K)$. Since $f_1 : X_1 \to Y_1$ is a quotient map, by Theorem 3.3 and by Proposition 2.1, there exists a finite sequence of connected sets C_1, C_2,...C_n in X_1 such that $u \in f_1(C_1)$, $r \in f_1(C_n)$, $f_1(C_i) \cap f_1(C_{i+1}) \neq \phi$ for $i = 1$ to $n-1$ and $f_1(C_i) \subseteq \pi_1(K)$ for $i = 1$ to n. Similarly, there exists a finite sequence of connected sets D_1, D_2,...,D_m in X_2 such that $v \in f_2(D_1)$, $s \in f_2(D_m)$, $f_2(D_i) \cap f_2(D_{i+1}) \neq \phi$ for $i = 1$ to $m-1$ and $f_2(D_i) \subseteq \pi_2(K)$ for $i = 1$ to m.

Without loss of generality, we may let $m \leq n$. Define $D_j = D_m$ for $1+m \leq j \leq n$. Consider the finite sequence of connected sets $\{C_i \times D_i : i = 1 \text{ to } n\}$ in X. Now, $(u, v) \in f_1(C_1) \times f_2(D_1) = f(C_1 \times D_1)$ and $(r, s) \in f_1(C_n) \times f_2(D_n) = f(C_n \times D_n)$. Also,

$$\begin{aligned} f(C_i \times D_i) \cap f(C_{i+1} \times D_{i+1}) &= (f_1(C_i) \times f_2(D_i)) \cap (f_1(C_{i+1}) \times f_2(D_{i+1})) \\ &= (f_1(C_i) \cap f_1(C_{i+1})) \times (f_2(D_i) \cap f_2(D_{i+1})) \\ &\neq \phi \quad \text{for} \quad i = 1 \text{ to } n-1 \end{aligned}$$

For $i = 1$ to n,

$$\begin{aligned} f(C_i \times D_i) &= f_1(C_i) \times f_2(D_i) \\ &\subseteq \pi_1(K) \times \pi_2(K) \\ &= K_1 \end{aligned}$$

By Proposition 2.1, we have $K_1 \in \mathcal{C}_Q$. Obviously $K \subset \pi_1(K) \times \pi_2(K) = K_1$.

□

It can be noted that restriction of a quotient map need not be a quotient map.The following proposition answers when restriction of a quotient map become a quotient map.

Proposition 4.3. *Let X and Y be two c-spaces and $f : X \to Y$ be a quotient map. Let $A \subset X$ be such that f is injective outside A (That is, $f^{-1}(f(x)) = \{x\}$ for each $x \in A^c$). Then the restriction $f_{/A} : A \to f(A)$ is again a quotient map.*

Proof: Obviously $g = f_{/A} : A \to f(A)$ is c-continuous. To show g is a quotient map, it is enough to show that g is onto from $\mathcal{C}_A$ to $\mathcal{C}_{f(A)}$.

Let $K \in \mathcal{C}_{f(A)}$. Then K is connected in Y. Since f is a quotient map, there exists a connected set C in X such that $f(C) = K$. Since f is injective outside A, it follows that $C \subset A$, so that $C \in \mathcal{C}_A$. Further $g(C) = K$. Since K is arbitrary, g is onto from $\mathcal{C}_A$ to $\mathcal{C}_{f(A)}$. □

As a consequence of this proposition, we have the following theorem.

Theorem 4.4. *Let X be a c-space. Then any quotient of any sub c-space of X can be obtained as a sub c-space of a quotient of X.*

Proof: Let A be a sub c-space of X and let $g : A \to B$ be a quotient map. We want to get B as a sub c-space of a quotient of X.

Consider the set $Y = B \cup (X - A)$. Define $f : X \to Y$ by

$$f(x) = \begin{cases} g(x) & ; \quad x \in A \\ x & ; \quad x \in X - A \end{cases}$$

Give quotient c-structure on Y. Then obviously $f : X \to Y$ is quotient map and is injective outside A. Then by the Proposition 4.3, $f_{/A} : A \to f(A) = B$ (as a sub c-space of Y) is a quotient map.

Since $f_{/A} = g$, as a sub c-space of Y, B is a quotient space of X with respect to g. Hence the theorem. □

5. Quotient Space Verses Equivalence Classes

In [1], it is defined that a surjective function $f : X \to Y$ is a quotient map if c-structure on Y is the smallest c-structure which make f c-continuous. In [3], a quotient space of X is defined as a partition set X^* of X equipped with the smallest c-structure such that the projection map $p : X \to X^*$ defined by $x \mapsto [x]$, the equivalence class of X containing x, is c-continuous. From the properties of Topological Categories, it follows that these two notions are equivalent [3].

In [7], a quotient space Y of a connective space X is defined using a partition of the connected sets as follows. A nonempty subset C of Y is said to be connected if its union in X is connected. Since all connective spaces are c-spaces, this implicitly defines a quotient space of a c-space which is a connective space.

Remark 5.1. *Then concept of quotient space in c-spaces [3,1] is a generalization of the concept of quotient space found in* [7]*, as defined by J. Muscat and D. Buhagiar.*

Proof:

Let Y be a quotient space of the c-space X, where $Y = \{P_i : i \in I\}$, be a partition of X by connected sets. Let $\mathcal{C}_Q$ be the quotient c-structure on Y, as defined by J. Muscat. Let $\mathcal{C}_Y$ be another c-structure on Y such that the surjective function $p : X \to Y$ defined by $x \to [x]$ is c-continuous.

Let $C \in \mathcal{C}_Q$. Then $C = \{P_i : i \in J\}$ for some $J \subseteq I$ such that $K = \bigcup_{j \in J} P_j \in \mathcal{C}_X$. Obviously $p(K) \in \mathcal{C}_Y$. But

$$\begin{aligned} p(K) &= \bigcup_j p(P_j) \\ &= \bigcup_j \{P_j\} \\ &= C \end{aligned}$$

Consequently $\mathcal{C}_Q \subseteq \mathcal{C}_Y$. Thus $\mathcal{C}_Q$ is the smallest c-structure on Y such that p is c-continuous. □

6. Quotient Space of Topologizable c-spaces

In this section, quotient space of a class of c-spaces are dealt with, called topologizable c-spaces. Let (X, τ) be a topological space. Then the collection of all connected sub sets of (X, τ) will form a c-structure on X, called the associated c-structure. X with this associated c-structure is called the associated c-space of (X, τ). The following definition can be seen in [8] with the terminology *Topological c-space*. But we feel that the terminology *Topologizable c-space* is better.

Definition 6.1. [8] *Topologizable c-space*

A c-space $(X, \mathcal{C}_X)$ is said to be topologizable if there exists a topology τ on X such that the associated c-space of (X, τ) is $(X, \mathcal{C}_X)$.

For examples and non examples, we may refer to [8]. To answer the questions which we have raised at the end of the Section 3.1, we need the following theorem.

Theorem 6.2. *Let X' be a topological space such that X be its associated c-space. Let Y' be a topological space such that $f : X' \to Y'$ be a quotient map. If Y is the associated c-space of Y', then $f : X \to Y$ is a quotient map.*

Proof: As continuous image of a connected set is connected and since $f : X' \to Y'$ is continuous, $f : X \to Y$ is c-continuous. Being a quotient space, Y' has the largest topology which make f continuous. We know that as topology become larger, the number of connected sets become smaller. Hence the associated c-space Y of Y' has minimum number of connected sets. Thus Y has minimum number of connected sets which make f c-continuous. Hence Y is a quotient space of X with respect to f. □

Corollary 6.3. *Let X be a c-space and Y be its quotient space.*

1. *If X is discrete, then Y is also discrete.*
2. *If X is indiscrete, then Y is also indiscrete.*

Proof: Proof directly follows from the fact that both discrete and indiscrete c-spaces are topologizable and that quotient spaces of discrete and indiscrete topological spaces are discrete and indiscrete respectively.

□

Remark 6.4. *From the above theorem, it can be concluded that quotient space of a topologizable c-space is topologizable.*

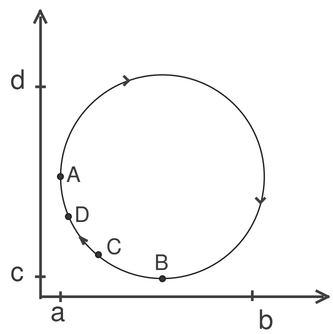
Now we consider the questions raised at the end of the Section 3.1.

1. Define an equivalence relation $\sim$ on $\mathbb{R}$ as $x \sim y$ if and only if $x - y \in \mathbb{Q}$ for $x, y \in \mathbb{R}$. Then the quotient space $\mathbb{R}_{/\sim}$ is indiscrete.

   Proof: We know that as a c-space, $\mathbb{R}$ is topologizable. In topology [6], it is proved that, the quotient space $\mathbb{R}_{/\sim}$ is indiscrete. Hence all subsets of $\mathbb{R}_{/\sim}$ are connected and hence the associated c-space is indiscrete. By Theorem 6.2, our result follows. □

2. Consider the c-spaces $[0, 2\pi]$ and the unit circle S^1 as a sub c-space of $\mathbb{R}$ and as a sub c-space of $\mathbb{R}^2$ in order. Define $f : [0, 2\pi] \to S^1$ by $f(x) = (cos(x), sin(x))$. Then S^1 is a not quotient space of $[0, 2\pi]$ with respect to f.

Proof: It is obvious that, as a topological space, the only connected sets in S^1 are the arcs of S^1. Now, as a sub c-space of $\mathbb{R}^2$, the connected subsets of S^1 are precisely those subsets C of S^1 whose projections $\pi_i(C)$, $i = 1, 2$ are intervals. Thus the connected subsets of S^1 includes all arcs and some non arcs. For example, consider the arcs $\widehat{AB}$ and $\widehat{CD}$ of the unit circle as below.



Consider a subset C of S^1 as $C = \widehat{AB} \cup \widehat{CD}$. As $\pi_1(C) = [a, b]$ and $\pi_2(C) = [c, d]$ and since each of which is connected in $\mathbb{R}$, C is a connected subset of S^1 as sub c-space of $\mathbb{R}^2$.

Considering $[0, 2\pi]$ and S^1 as topological spaces, we know that $f : [0, 2\pi] \to S^1$ is a quotient map. Since $[0, 2\pi]$ is topologizable, as a function from a c-space, $f : [0, 2\pi] \to S^1$ is c-continuous. But from the above discussion, it is clear that S^1 contains more connected sets than the minimum number of connected sets required for the c-continuity of f. Hence f is not a quotient map. $\square$

3. It can verified that $\mathcal{C}_{S^1} = \mathcal{D}_{S^1} \cup \{\widehat{AB} : \widehat{AB}$ is an arc in $S^1\}$ is a c-structure on S^1. Then $f : [0, 2\pi] \to (S^1, \mathcal{C}_{S^1})$ defined by $f(x) = (cos(x), sin(x))$ is a quotient map.

   Proof: The c-spaces $[0, 2\pi]$ and $(S^1, \mathcal{C}_{S^1})$ are the associated c-spaces of the topological spaces $[0, 2\pi]$ and S^1 in order. As a topological space, S^1 is a quotient space of $[0, 2\pi]$ with respect to f. Hence by Theorem 6.2, $f : [0, 2\pi] \to (S^1, \mathcal{C}_{S^1})$ is a quotient map. $\square$

4. On $\mathbb{R}$, define $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Then the quotient space $\mathbb{R}_{/\sim}$ is a circle.

   Proof: Consider the equivalence classes in terms of real numbers t with $0 \leq t \leq 1$. That is,
$$[t] = \{t + n : n \in \mathbb{Z}\}$$

Since every real number belongs to such a class, we reduced the problem from statement about $\mathbb{R}$ to the statement about $[0,1]$. Realizing that $[0]=[1]$, the quotient space of $\mathbb{R}_{/\sim}$ is a c-isomorphic to the quotient space of $[0,1]$ obtained by identifying 0 and 1. As $[0,1]$ is c-isomorphic to $[0,2\pi]$, by the above example, $\mathbb{R}_{/\sim}$ is c-isomorphic to the circle $(S^1, \mathcal{C}_{S^1})$. □

7. Quotient space of Graphical c-spaces

Similar to the definition of the associated c-space of a topological space, we can define the associated c-space of a graph G, where set of points are the collection of vertices of G and c-structure is nothing but the collection of all connected(in fact, path connected) subsets of G. Analogously, a c-space is said to be graphical if there exists graph G whose connected sets are precisely the connected sets of the given c-space. For example, both discrete and indiscrete c-spaces are graphical. In the next theorem, we describe a method for finding a quotient space of a c-space, where the associated graph is connected. We know that quotient graph of a connected graph is connected.

Before stating the theorem, let us first define what a quotient graph is. Let $G=(V,E)$ be a graph. For a given partition $\mathcal{P}$ of V, the quotient graph G^* is defined as a graph with vertex set $\mathcal{P}$ such that for any two vertices B, C in $\mathcal{P}$, B and C are adjacent if and only if there exists vertices $u \in B$ and $v \in C$ which are adjacent in G. A partition is said to be connected if each member of the partition is connected as a sub graph of G.

Theorem 7.1. *Let G be a finite connected graph and X be its associated c-space. Let G^* be a quotient graph of G corresponding to a connected partition of the vertex set of G. Then the associated c-space X^* of G^* is a quotient space of X.*

Proof: Consider a partition $\mathcal{P}=\{P_i : i = 1 \text{ to } n\}$ of the vertex set of G such that each P_i is connected. Let $f : G \to G^*$ be the quotient map defined by $f(x) = P_i$, the partition containing x. Obviously f maps connected sets of G to connected sets of G^* and hence is c-continuous from X to X^*. To prove X^* is a quotient space of X with respect to f, it is enough to prove that $f(\mathcal{C}_X) = \mathcal{C}_{X^*}$.

Let $C^* = \{P_i : i \in J \subseteq I\}$ be a connected set in X^*. With out loss of generality, we may assume that $J = \{1, 2, 3 \cdots m\}$, where $m \leq n$. We know that in G^*, P_i is adjacent to P_j if there exists vertices $v_i \in P_i$ and $v_j \in P_j$ such that v_i is adjacent to v_j. Since C^* is connected in G^*, for each P_i, choose vertices v_i from P_i such that v_i is adjacent to $v_j \in P_j$ for some j. Let K be the collection of these vertices. We will construct a connected set C of G, an element of $\mathcal{C}_X$, such that $f(C) = C^*$.

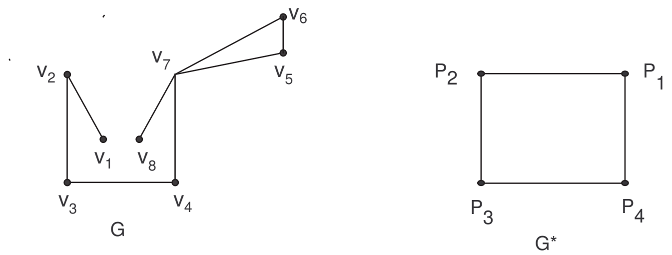
Fix a vertex u_0 in K. Let u be any vertex in K. Without loss of generality, we may assume that $u_0 \in P_1$ and $u \in P_s$ with $1 \leq s \leq m$. Since C^* is connected in G^*, there exists a path in G^* joining P_1 and P_s,(say), P_1, P_2, $\cdots$, P_s. Since P_1 and P_2 are adjacent, there exists vertices $v_{1_1} \in P_1$ and $v_{2_1} \in P_2$ such that V_{1_1} and v_{2_1} are adjacent. Let this edge be e_{12}. Since P_1 is connected, there exists a path in G, say, q_1 from u_0 to v_{1_1}. Since P_2 and P_3 are adjacent, there exists a vertex

v_{2_2} in P_2 and v_{3_1} in P_3 such that they are adjacent. Let this edge be e_{2_3}. As P_2 is connected in G, there exists a path q_2 in G from v_{2_1} to v_{2_2}. Similarly proceeding, there exists an edge $e_{(s-1)s} = v_{(s-1)_2}v_{s_1}$ between P_{s-1} and P_s and a path q_s from v_{s_1} to u.

Let $Q_{u_0u} = u_0q_1v_{1_1}e_{12}v_{2_1}q_2v_{2_2}e_{23}\cdots v_{(s-1)_2}e_{(s-1)s}v_{s_1}q_su$.

Clearly Q_{u_0u} is a path joining u_0 and u in G. Let $C = \bigcup_{u\in K} Q_{u_0u}$. Clearly C is a connected sub graph of G and hence a member of $\mathcal{C}_X$. Also, $f(C) = C^*$. Hence the result. □

Remark 7.2. *The above method will not work for all partition of vertex set of G. That is, above methods fails if at least one P_i is disconnected. For example, consider the associated graph G of a graphical c-space X and its quotient graph G^* corresponds to the partition $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ of vertex set of G, where $P_1 = \{v_5, v_6, v_7\}$, $P_2 = \{v_1, v_2, v_8\}$, $P_3 = \{v_3\}$ and $P_4 = \{v_4\}$ as below.*



Let f be the corresponding quotient map. But it can be noted that there is no connected set in G, whose f image is $\{P_1, P_2, P_3\}$, a connected set in G^*. Hence f is not a quotient map between the associated c-spaces.

Acknowledgments

The author is very much indebted to his supervising guide Dr. Ramachandran P.T., Department of Mathematics, University of Calicut for his valuable guidance and encouragement throughout this work. The financial support from the U.G.C, Govt. of India is greatly acknowledged.

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